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A Unifying Model for Matching Situations

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Abstract

We present a unifying framework for transferable utility coalitional games that are derived from a non-negative matrix in which every entry represents the value obtained by combining the corresponding row and column. We assume that every row and every column is associated with a player, and that every player is associated with at most one row and at most one column. The instances arising from this framework are called *matching games*, and they encompass assignment games and permutation games as two polar cases. We show that the core of a matching game is always nonempty by proving that the set of matching games coincides with the set of permutation games. Then we focus on two separate problems.

First, we exploit the wide range of situations comprised in our framework to investigate the relationship between matching games with different player sets but defined by the same underlying matrix. We show that the core is not only immune to the merging of a row player and a column player, but also to the reverse manipulation, i.e., to the splitting of a player into a row player and a column player. Other common solution concepts fail to be either merging-proof or splitting-proof in general.

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Second, we focus on permutation games only and we analyze the set of all matrices that define permutation games with the same core. In contrast to assignment games, we show that there can be multiple matrices whose entries cannot be raised without modifying the core of the corresponding permutation game and that, for small instances, every such matrix defines an exact game.

Keywords: *matching situations, permutation games, assignment games*

JEL Code: *C71*

1 Introduction

We present a unifying framework for transferable utility (TU) coalitional games that are derived from a non-negative matrix in which every entry represents the value obtained from combining the corresponding row and column. We assume that every row and every column are associated with a player, and that every player is associated with at most one row and at most one column. The instances arising from this framework are called *matching situations* and they embody two models known from the literature of TU games.

First, in the special case that every player is associated with exactly one row or one column only, the model boils down to the assignment problem (Shapley and Shubik, 1972). The assignment problem is typically used to model a two-sided market of heterogeneous goods in which every player associated with a row demands one good and every player associated with a column offers one good. Second, if every player is associated with exactly one column and exactly one row, the model represents a permutation problem (Tijs et al., 1984). This model is used to describe a situation where every player owns a machine and a job, and costs can be reduced by processing one's job on the machine of another player. For every matching situation with the same underlying matrix, the optimization problem is the same: find a matching of rows and columns that results in the maximum total value.

We analyze matching situations from a game-theoretic perspective. On the one hand, assignment games, the TU games originating from assignment problems, have been studied extensively – see, e.g., Núñez and Rafels (2002) and Martínez-Albéniz

et al. (2011). On the other hand, the relation between assignment games and permutation games is studied by Curiel and Tijs (1986) and Quint (1996). As they show, every assignment game can also be obtained from a permutation situation with a different underlying matrix. For matching games, the games obtained from matching situations, we show that the core is always non-empty by proving that the set of matching games coincides with the set of permutation games. In doing so we generalize the mentioned results from Curiel and Tijs (1986) and Quint (1996) to our unifying setting.

After that, the paper consists of two separate parts. In the first part, we analyze the relation between matching situations based on the same underlying matrix, but with different player sets. More specifically, we associate to every assignment situation a number of matching situations with the same underlying matrix up to a certain relabeling of the rows and columns. Tijs et al. (1984) and Quint (1996) showed that if the associated matching situation is a permutation situation, the core of the permutation game coincides with a certain transformation of the core of the assignment game. We generalize this result to our setting and, moreover, we show that all extreme points of the core of any such matching game can be obtained by means of the same transformation from the set of extreme points of the core of the assignment game. Furthermore, these results still hold if the initial matching situation is not an assignment situation but an arbitrary matching situation.

The generality of our model enables us to investigate vertical merges, i.e., merges between a row player and a column player. A general analysis of manipulations in the player set of the aforementioned type is to the best of our knowledge absent to date within the framework of assignment games. We note that if a row player and a column player of an assignment situation merge into a single player, the resulting situation is no longer an assignment situation. From the perspective of the study of manipulations in the player set, the results in the above paragraph indicate that the core and the set of extreme core allocations of a matching game are anonymous solution concepts that are immune to manipulations such as the merge of a row player and a column player. Then, we show that in general the core, unlike the set of extreme core allocations, is also immune to the reverse manipulation, i.e., the splitting of a player into a row player and a column player. Most of the common

solution concepts fail to be either merging-proof or splitting-proof in general.

In the second part, we provide an addition to the literature on permutation situations by focusing on the structure of the set of matrices that lead to permutation games with the same core. For assignment games, the structure of the corresponding set is studied by Núñez and Rafels (2002), who show that within this set there exists a unique matrix for which no entry can be raised without changing the core. For permutation games, there can be more than one such matrix. Interestingly, we show that for permutation games with at most four players every such matrix leads to an exact game, whereas for assignment games this is not guaranteed.

The paper is organized as follows. Section 2 introduces the general framework of matching situations and games, and shows that matching games have a non-empty core. Section 3 analyzes the relation between matching situations with different player set that are defined by the same underlying matrix. Section 4 studies the set of permutation situations leading to games with the same core.

2 A unifying model

In this section we introduce matching situations and games, and we show that for every matching situation the corresponding matching game has a nonempty core.

A *matching situation* is a triple (N^1, N^2, A) , where N^1 and N^2 are two finite nonempty player sets (of possibly different cardinality) and A is a mapping that assigns a non-negative number to every pair composed exactly of one player of N^1 and one player of N^2 . As A can be cast as a non-negative, $N^1 \times N^2$ matrix, we introduce $\mathcal{M}_{N^1 \times N^2}^+$, the set of all non-negative $N^1 \times N^2$ matrices. For $i \in N^1$ and $j \in N^2$, the entry a_{ij} of the matrix $A \in \mathcal{M}_{N^1 \times N^2}^+$ represents the value created when agents i and j are paired. We allow for the player sets N^1 and N^2 to overlap, so a player might be associated with both a row and a column of the matrix A . From a combinatorial perspective, the question is how to maximize aggregate benefits, i.e., how to find two subsets, $T^1 \subseteq N^1$ and $T^2 \subseteq N^2$, with $|T^1| = |T^2|$, and a bijection

(called *matching*) $\mu_{T^1 T^2} : T^1 \rightarrow T^2$ such that

$$\sum_{(i,j) \in \mu_{T^1 T^2}} a_{ij} = \sum_{i \in T^1} a_{i\mu_{T^1 T^2}(i)} \quad (1)$$

is maximized. In this paper, we consider matching situations from a cooperative game-theoretic perspective and thus we analyze the problem of dividing aggregate benefits among the players on the basis of coalitional considerations.

Given $S^1 \subseteq N^1$ and $S^2 \subseteq N^2$, we denote the *set of feasible matchings between S^1 and S^2* by

$$M(S^1, S^2) = \{\mu_{T^1 T^2} : T^1 \rightarrow T^2 \mid T^1 \subseteq S^1, T^2 \subseteq S^2, \mu_{T^1 T^2} \text{ is a bijection}\}.$$

Note that, for a given $\mu_{T^1 T^2} \in M(S^1, S^2)$, the rows of players in $S^1 \setminus T^1$ are not matched to a column and the columns of players in $S^2 \setminus T^2$ are not matched to a row. Given a matching situation (N^1, N^2, A) , the *set of optimal matchings for S^1 and S^2* is given by

$$M_A^*(S^1, S^2) = \arg \max_{\mu_{T^1 T^2}^* \in M(S^1, S^2)} \left[\sum_{(i,j) \in \mu_{T^1 T^2}^*} a_{ij} \right].$$

Note that, since A has non-negative entries, $M_A^*(S^1, S^2)$ is nonempty if both $|S^1| > 0$ and $|S^2| > 0$. When no possibility of confusion may arise, we omit the subsets T^1 and T^2 , and denote an optimal matching of (N^1, N^2, A) simply by μ , instead of $\mu_{T^1 T^2}$.

A *transferable utility coalitional game* (simply *game* henceforth) is a pair (N, v) , where N is the set of players and v , the characteristic function, assigns a number to any subset of N , with the condition that $v(\emptyset) = 0$. Let (N^1, N^2, A) be a matching situation. Then, the associated *matching game* (N, v_A) is the game with set of players $N = N^1 \cup N^2$ and characteristic function v_A defined, for every $\emptyset \neq S \subseteq N$, by

$$v_A(S) = \sum_{(i,j) \in \mu^*} a_{ij} \quad (2)$$

for any $\mu^* \in M_A^*(S \cap N^1, S \cap N^2)$. That is, v_A captures how best can each coalition of agents do by matching agents of the coalition among themselves.

Example 2.1 Let $N^1 = \{1, 2\}$, $N^2 = \{2, 3\}$ and let $A \in \mathcal{M}_{N^1 \times N^2}^+$ be given by

$$A = \begin{matrix} & \begin{matrix} 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \end{matrix}.$$

The unique optimal matching $\mu_{N^1 N^2}^* \in M_A^*(N^1, N^2)$ is such that $\mu_{N^1 N^2}^*(1) = 2$ and $\mu_{N^1 N^2}^*(2) = 3$, or, equivalently, $\mu_{N^1 N^2}^* = \{(1, 2), (2, 3)\}$. The matching game (N, v_A) is given by $N = \{1, 2, 3\}$ and

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	
$v_A(S)$	0	2	0	2	2	4	5	$\triangleleft$

Matching games constitute the TU version of the exchange economies introduced by Quinzii (1984), with the modification that we do not impose finite supply of money (utility). Two special classes of matching situations have specifically been analyzed in the literature to date. Regarding the player set of these special cases, these two situations form the polar cases: for assignment situations N^1 and N^2 are disjoint, whereas for permutation situations N^1 coincides with N^2 .

Definition 2.2 (Shapley and Shubik, 1972) Let (N^1, N^2, A) be a matching situation. Then (N^1, N^2, A) is called an *assignment situation* if $N^1 \cap N^2 = \emptyset$.

Definition 2.3 (Tijs et al., 1984) Let (N^1, N^2, A) be a matching situation. Then (N^1, N^2, A) is called a *permutation situation* if $N^1 = N^2$.

If the matching situation (N^1, N^2, A) is an assignment (permutation) situation, we refer to (N, v_A) as the corresponding *assignment (permutation) game*. Next we show that every matching game has a nonempty core.

Theorem 2.4 Let (N^1, N^2, A) be a matching situation. Then, $C(v_A) \neq \emptyset$.

Proof: Given (N^1, N^2, A) , it will be useful to define a corresponding permutation situation. Indeed, let (N, N, B) be the permutation situation where $N = N^1 \cup N^2$ and $B \in \mathcal{M}_{N \times N}^+$ is defined for every $i, j \in N$ by

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \in N^1 \text{ and } j \in N^2, \\ 0 & \text{if } i \in N^2 \setminus N^1 \text{ or } j \in N^1 \setminus N^2. \end{cases}$$

We show that $v_A(S) = v_B(S)$ for each $\emptyset \neq S \subseteq N$. For notational convenience, we denote throughout the proof $S^1 = S \cap N^1$ and $S^2 = S \cap N^2$. First, we show that $v_B(S) \leq v_A(S)$. Indeed, let $\mu^* \in M_B^*(S, S)$. Denote $T^1 = \{i \in S^1 \mid \exists j \in S^2 \text{ such that } (i, j) \in \mu^*\} \subseteq S^1$ and $T^2 = \mu^*(T^1) \subseteq S^2$. Next, define $\mu_{T^1 T^2} \in M(S^1, S^2)$ by $\mu_{T^1 T^2}(i) = \mu^*(i)$ for every $i \in T^1$. Then,

$$v_B(S) = \sum_{(i,j) \in \mu^*} b_{ij} = \sum_{i \in T^1} b_{i\mu^*(i)} = \sum_{i \in T^1} a_{i\mu_{T^1 T^2}(i)} \leq v_A(S),$$

where the second equality follows from the fact that $b_{ij} = 0$ for every $(i, j) \in \mu^*$ if $i \in S \setminus T^1$. Second, we show that $v_A(S) \leq v_B(S)$. Take $\mu_{T^1 T^2}^* \in M_A^*(S^1, S^2)$, i.e., $v_A(S) = \sum_{i \in T^1} a_{i\mu_{T^1 T^2}^*(i)}$ for $T^1 \subseteq S^1$ and $T^2 = \mu_{T^1 T^2}^*(T^1) \subseteq S^2$. Next, define $\mu \in M(S, S)$ so that $\mu(i) = \mu_{T^1 T^2}^*(i)$ for every $i \in T^1 \subseteq S$ and the players in $S \setminus T^1$ are either not matched or matched arbitrarily to players in $S \setminus T^2$. Then,

$$v_A(S) = \sum_{i \in T^1} a_{i\mu_{T^1 T^2}^*(i)} = \sum_{i \in T^1} b_{i\mu(i)} \leq \sum_{i \in S} b_{i\mu(i)} \leq v_B(S),$$

where the second equality follows by construction of B and the observation that $\mu_{T^1 T^2}^*(i) \in T^2$ for every $i \in T^1$.

Hence, $v_A(S) = v_B(S)$ for every $S \subseteq N$. Therefore, any matching game is in fact a permutation game (defined by a different matrix). Since permutation games always have a nonempty core (Tijs et al., 1984), it follows that $C(v_A) \neq \emptyset$. $\square$

We want to stress that the proof of Theorem 2.4 demonstrates a stronger result, namely that the set of permutation games and the set of matching games is the same.

3 Matching situations and vertical merges

In this section, we consider an arbitrary assignment situation (N^1, N^2, A) as the status quo, and we study the matching situations that emerge after relabelings of the sets N^1 and N^2 . Our analysis encompasses those matching situations that arise from the initial assignment situation in which a pair of players, one row player and one column player, collude into a unique player (both players are relabeled as the same player). The resulting matching situation after the merge is in general neither an assignment nor a permutation situation. We note that a common framework that includes assignment and permutation situations – and all possible intermediate situations – is needed to analyze the so-called *vertical merges* – as opposed to *horizontal merges* in which agents of the same side of the assignment game collude.¹ The unifying framework introduced in the previous section, namely matching situations, enables us to address this issue.

To this purpose, we start by analyzing the relation between the cores of an assignment situation and an arbitrary matching situation both defined by the same underlying matrix up to a certain relabeling of rows and columns. For finite player sets S and S' such that $|S| = |S'|$, let us denote by $\Pi(S, S')$ the set of bijections $\sigma : S \rightarrow S'$. Given an assignment situation (N^1, N^2, A) , let $\bar{N}^1$ and $\bar{N}^2$ be such that $|\bar{N}^1| = |N^1|$ and $|\bar{N}^2| = |N^2|$. Then, for each $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$, the matching situation $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$ is defined by

$$a_{ij}^{\sigma^1, \sigma^2} = a_{\sigma^1(i)\sigma^2(j)} \text{ for every } (i, j) \in \bar{N}^1 \times \bar{N}^2. \quad (3)$$

Note that the matrix A that defines the initial matching situation, (N^1, N^2, A) , and the matrix A^{σ^1, σ^2} underlying the matching situation $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$ are equal as $|N^1| \times |N^2|$ matrices up to a relabeling of rows and columns.

Since the set of players of (N^1, N^2, A) is different than that of $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$, we next define a mapping that assigns to any payoff vector of (N^1, N^2, A) a payoff vector of $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$.

¹To the best of our knowledge, the merging of players from different sides of an assignment situation has only been analyzed before in Tejada and Álvarez-Mozos (2012), where a connection between multisided assignment games and bankruptcy games is exploited to find the unique allocation that, among other properties, is immune to a type of vertical merge.

Definition 3.1 Let (N^1, N^2, A) be an assignment situation. Let $\bar{N}^1$ and $\bar{N}^2$ be such that $|N^1| = |\bar{N}^1|$ and $|N^2| = |\bar{N}^2|$. Then, for each $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$, the mapping $m^{\sigma^1, \sigma^2} : \mathbb{R}_+^{N^1 \cup N^2} \rightarrow \mathbb{R}_+^{\bar{N}^1 \cup \bar{N}^2}$ is defined, for every $x \in \mathbb{R}_+^{N^1 \cup N^2}$ and $i \in \bar{N}^1 \cup \bar{N}^2$, by

$$m_i^{\sigma^1, \sigma^2}(x) = \begin{cases} x_{\sigma^1(i)} & \text{if } i \in \bar{N}^1 \setminus \bar{N}^2, \\ x_{\sigma^2(i)} & \text{if } i \in \bar{N}^2 \setminus \bar{N}^1, \\ x_{\sigma^1(i)} + x_{\sigma^2(i)} & \text{if } i \in \bar{N}^1 \cap \bar{N}^2. \end{cases}$$

The following example illustrates the above definitions.

Example 3.2 Let $N^1 = \{1, 2\}$, $N^2 = \{3, 4\}$ and let $A \in \mathcal{M}_{N^1 \times N^2}^+$ be given by

$$A : \begin{matrix} & \begin{matrix} 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \end{matrix}.$$

Let $\bar{N}^1 = \{1, 2\}$, $\bar{N}^2 = \{2, 3\}$, and define $\sigma^1 \in \Pi(\bar{N}^1, N^1)$, $\sigma^2 \in \Pi(\bar{N}^2, N^2)$ such that $\sigma^1(1) = 1$, $\sigma^1(2) = 2$, $\sigma^2(2) = 3$, $\sigma^2(3) = 4$. The matching situation $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$ is defined by:

$$A^{\sigma^1, \sigma^2} : \begin{matrix} & \begin{matrix} 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \end{matrix}.$$

Note that the initial player 4 has changed her identity to 3, whereas the initial players 2 (row player) and 3 (column player) have colluded into just one player, 2. The mapping m^{σ^1, σ^2} is defined by $m^{\sigma^1, \sigma^2}(x_1, x_2, x_3, x_4) = (x_1, x_2 + x_3, x_4)$. One readily checks that

$$C(v_{A^{\sigma^1, \sigma^2}}) = \text{conv}\{(0, 2, 3), (0, 3, 2), (1, 2, 2), (1, 3, 1)\} = m^{\sigma^1, \sigma^2}(C(v_A)).$$

◁

The latter observation in Example 3.2 suggests that $C(v_{A^{\sigma^1, \sigma^2}}) = m^{\sigma^1, \sigma^2}(C(v_A))$ might hold in general. This is indeed the case, as the following result demonstrates.²

²Theorem 3.3 generalizes the results from Tijs et al. (1984) and Quint (1996) to arbitrary matching situations.

Theorem 3.3 Let (N^1, N^2, A) be an assignment situation. Let also $\bar{N}^1$ and $\bar{N}^2$ be such that $|N^1| = |\bar{N}^1|$ and $|N^2| = |\bar{N}^2|$. Then, for each $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$, it holds that

$$C(v_{A^{\sigma^1, \sigma^2}}) = m^{\sigma^1, \sigma^2}(C(v_A)).$$

Proof: We assume without loss of generality that (N^1, N^2, A) is square, i.e., $|N^1| = |N^2|$.³ First, let $x \in C(v_A) \subseteq \mathbb{R}_+^{N^1 \cup N^2}$ and $\bar{S} \subseteq \bar{N}^1 \cup \bar{N}^2$, and denote

$$S = \{i \in N^1 \cup N^2 \mid \exists j \in \bar{S} \text{ s.t. } \sigma^1(j) = i \text{ or } \sigma^2(j) = i\}.$$

Then,

$$m^{\sigma^1, \sigma^2}(x)(\bar{S}) = x(S) \geq v_A(S) = v_{A^{\sigma^1, \sigma^2}}(\bar{S}), \quad (4)$$

where the first equality follows from Definition 3.1, the inequality holds since $x \in C(v_A)$, and the last equality is due to (2) and (3). Moreover, the chain of equalities and inequalities in (4) reduces to a chain of equalities if $\bar{S} = \bar{N}^1 \cup \bar{N}^2$. Second, let $\bar{x} \in C(v_{A^{\sigma^1, \sigma^2}})$ and consider the assignment situation $(N^1, N^2, \bar{A})$, where $\bar{A}$ is defined, for each $\sigma^1(i) \in N^1$ and $\sigma^2(j) \in N^2$, as follows:

$$\bar{a}_{\sigma^1(i)\sigma^2(j)} = \begin{cases} \bar{x}_i + \bar{x}_j & \text{if } i \in \bar{N}^1 \setminus \bar{N}^2 \text{ and } j \in \bar{N}^2 \setminus \bar{N}^1, \\ \bar{x}_i & \text{if } i = j \in \bar{N}^1 \cap \bar{N}^2, \\ a_{ij} & \text{otherwise.} \end{cases} \quad (5)$$

Note that, by assumptions, we have $|\bar{N}^1 \setminus \bar{N}^2| = |\bar{N}^2 \setminus \bar{N}^1|$. Then, for each $\bar{\mu} \in M(\bar{N}^1 \setminus \bar{N}^2, \bar{N}^2 \setminus \bar{N}^1)$, let $\mu^*(\bar{\mu}) \in M(N^1, N^2)$ be defined by

$$\mu^*(\bar{\mu}) = \{(\sigma^1(i), \sigma^2(i))\}_{i \in \bar{N}^1 \cap \bar{N}^2} \cup \left[\bigcup_{(i,j) \in \bar{\mu}} \{(\sigma^1(i), \sigma^2(j))\} \right]. \quad (6)$$

We claim that, for each $\mu' \in M(N^1, N^2)$, we have

$$\sum_{(k,l) \in \mu^*(\bar{\mu})} \bar{a}_{kl} \geq \sum_{(k,l) \in \mu'} \bar{a}_{kl}. \quad (7)$$

³We can add to (N^1, N^2, A) dummy players whose associated entries in the matrix are all zero, and then assume that they are not relabeled by σ^1 and σ^2 , so that they are also dummy players in $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$.

Indeed, observe that, due to (5), we can write

$$\begin{aligned}
\sum_{(k,l) \in \mu'} \bar{a}_{kl} &= \sum_{(\sigma^1(i), \sigma^2(j)) \in \mu'} \bar{a}_{\sigma^1(i)\sigma^2(j)} = \sum_{\substack{(\sigma^1(i), \sigma^2(j)) \in \mu' \\ i \in \bar{N}^1 \setminus \bar{N}^2 \\ j \in \bar{N}^2 \setminus \bar{N}^1}} \bar{a}_{\sigma^1(i)\sigma^2(j)} + \sum_{\substack{(\sigma^1(i), \sigma^2(j)) \in \mu' \\ i=j \in \bar{N}^1 \cap \bar{N}^2}} \bar{a}_{\sigma^1(i)\sigma^2(j)} \\
&+ \sum_{\substack{(\sigma^1(i), \sigma^2(j)) \in \mu' \\ i \in \bar{N}^1 \setminus \bar{N}^2 \\ j \in \bar{N}^1 \cap \bar{N}^2}} \bar{a}_{\sigma^1(i)\sigma^2(j)} + \sum_{\substack{(\sigma^1(i), \sigma^2(j)) \in \mu' \\ i \in \bar{N}^1 \cap \bar{N}^2 \\ j \in \bar{N}^2 \setminus \bar{N}^1}} \bar{a}_{\sigma^1(i)\sigma^2(j)} = \bar{x}(T) + \sum_{(i,j) \in \bar{\mu}''} a_{ij}, \quad (8)
\end{aligned}$$

where

$$\begin{aligned}
T &= \{ \{i, j\} \mid (\sigma^1(i), \sigma^2(j)) \in \mu', i \in \bar{N}^1 \setminus \bar{N}^2, j \in \bar{N}^2 \setminus \bar{N}^1 \} \\
&\cup \{ \{i\} \mid (\sigma^1(i), \sigma^2(j)) \in \mu', i = j \in \bar{N}^1 \cap \bar{N}^2 \},
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mu}'' &= \{ (i, j) \mid (\sigma^1(i), \sigma^2(j)) \in \mu', i \in \bar{N}^1 \setminus \bar{N}^2, j \in \bar{N}^1 \cap \bar{N}^2 \} \\
&\cup \{ (i, j) \mid (\sigma^1(i), \sigma^2(j)) \in \mu', i \in \bar{N}^1 \cap \bar{N}^2, j \in \bar{N}^2 \setminus \bar{N}^1 \}.
\end{aligned}$$

Note that $T \subseteq \bar{N}^1 \cup \bar{N}^2$ and that $\bar{\mu}''$ is a matching in $M(\bar{N}^1 \setminus T, \bar{N}^2 \setminus T)$. Thus,

$$\begin{aligned}
\sum_{(k,l) \in \mu^*(\bar{\mu})} \bar{a}_{kl} &= \bar{x}(\bar{N}^1 \cup \bar{N}^2) \geq \bar{x}(T) + v_{A^{\sigma^1, \sigma^2}}(\bar{N}^1 \cup \bar{N}^2 \setminus T) \\
&\geq \bar{x}(T) + \sum_{(i,j) \in \bar{\mu}''} a_{ij} = \sum_{(k,l) \in \mu'} \bar{a}_{kl},
\end{aligned}$$

where the first inequality holds since $\bar{x} \in C(v_{A^{\sigma^1, \sigma^2}})$. As a consequence, (7) holds, as we claimed. Therefore, we have shown that, for each $\mu' \in M(N^1, N^2)$,

$$\mu^*(\bar{\mu}) \in M_A^*(N^1, N^2) \quad (9)$$

Moreover, since $\bar{x} \in C(v_{A^{\sigma^1, \sigma^2}})$, it follows that $\bar{A} \geq A$ and $v_A(N^1 \cup N^2) = v_{\bar{A}}(N^1 \cup N^2)$, which implies that $C(v_{\bar{A}}) \subseteq C(v_A)$. Lastly, due to Theorem 2.4, we can take $x' \in C(v_{\bar{A}})$. On the one hand, $x' \in C(v_A)$. On the other hand, it is well-known that, given an arbitrary assignment game, the core condition associated with a pair that belongs to at least one optimal matching of the assignment game is always tight for any core allocation. Since (9) holds for any arbitrary matching $\bar{\mu}$ between $\bar{N}^1 \setminus \bar{N}^2$ and $\bar{N}^2 \setminus \bar{N}^1$, we derive from the way $\mu^*(\bar{\mu})$ is defined that the following core

conditions must hold tight for the assignment game $(N^1, N^2, \bar{A})$ and $x' \in C(v_{\bar{A}})$:

$$x'_{\sigma^1(i)} + x'_{\sigma^2(i)} = \bar{a}_{\sigma^1(i)\sigma^2(i)} = \bar{x}_i \quad \text{if } i \in \bar{N}^1 \cap \bar{N}^2, \quad (10)$$

$$x'_{\sigma^1(i)} + x'_{\sigma^2(j)} = \bar{a}_{\sigma^1(i)\sigma^2(j)} = \bar{x}_i + \bar{x}_j \quad \text{if } i \in \bar{N}^1 \setminus \bar{N}^2 \text{ and } j \in \bar{N}^2 \setminus \bar{N}^1. \quad (11)$$

We note that (11) impose $|\bar{N}^1 \setminus \bar{N}^2| \times |\bar{N}^2 \setminus \bar{N}^1|$ conditions, from which we derive that $x'_{\sigma^1(i)} = x_i$ for all $i \in \bar{N}^1 \setminus \bar{N}^2$ and $x'_{\sigma^2(j)} = x_j$ for all $j \in \bar{N}^2 \setminus \bar{N}^1$. Together with (10), we have shown that $m^{\sigma^1, \sigma^2}(x') = \bar{x}$, with $x' \in C(v_A)$. $\square$

Note that in Theorem 3.3 we have assumed that the initial situation (N^1, N^2, A) corresponds to an assignment situation. Nevertheless, this assumption has only been made for the sake of the presentation of the results. Indeed, the result in Theorem 3.3 can be generalized to encompass the case where (N^1, N^2, A) corresponds to an arbitrary matching (not necessary assignment) situation, provided that we consider appropriate $\bar{N}^1, \bar{N}^2, \sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$.

Corollary 3.4 *Let (N^1, N^2, A) be a matching situation. Let $\bar{N}^1$ and $\bar{N}^2$ be such that $|N^1| = |\bar{N}^1|$, $|N^2| = |\bar{N}^2|$ and $|\bar{N}^1 \cup \bar{N}^2| \leq |N^1 \cup N^2|$. Then, for each $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$ such that $\sigma^1(i) = \sigma^2(i)$ implies $i \in \bar{N}^1 \cap \bar{N}^2$, it holds that*

$$C(v_{A^{\sigma^1, \sigma^2}}) = m^{\sigma^1, \sigma^2}(C(v_A)).$$

Proof: Given (N^1, N^2, A) , $\bar{N}^1, \bar{N}^2, \sigma^1$, and σ^2 , the matching situation $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$ and the mapping $m^{\sigma^1, \sigma^2} : \mathbb{R}_+^{N^1 \cup N^2} \rightarrow \mathbb{R}_+^{\bar{N}^1 \cup \bar{N}^2}$ can be analogously defined as in (3) and Definition 3.1 respectively. Additionally, define the assignment situation

$$(\hat{N}^1, \hat{N}^2, \hat{A}), \quad (12)$$

where $|\hat{N}^1| = |N^1|$, $|\hat{N}^2| = |N^2|$ and $\hat{N}^1 \cap \hat{N}^2 = \emptyset$, and there are $\hat{\sigma}^1 \in \Pi(N^1, \hat{N}^1)$ and $\hat{\sigma}^2 \in \Pi(N^2, \hat{N}^2)$ such that $\hat{a}_{\hat{\sigma}^1(i)\hat{\sigma}^2(j)} = a_{ij}$ for all $(i, j) \in N^1 \times N^2$. By construction, it holds that the initial matching situation (N^1, N^2, A) is precisely the matching situation $(N^1, N^2, \hat{A}^{\hat{\sigma}^1, \hat{\sigma}^2})$ obtained from the assignment situation $(\hat{N}^1, \hat{N}^2, \hat{A})$ via $\hat{\sigma}^1$ and $\hat{\sigma}^2$. From the observation that, for $k \in \{1, 2\}$, it holds that $|\hat{N}^k| = |\bar{N}^k|$ and $\sigma^k \circ \hat{\sigma}^k \in \Pi(\bar{N}^k, \hat{N}^k)$, it additionally follows that the matching situation $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$ coincides with the matching situation $(\bar{N}^1, \bar{N}^2, \hat{A}^{\sigma^1 \circ \hat{\sigma}^1, \sigma^2 \circ \hat{\sigma}^2})$

obtained from the assignment situation $(\hat{N}^1, \hat{N}^2, \hat{A})$ via $\sigma^1 \circ \hat{\sigma}^1$ and $\sigma^1 \circ \hat{\sigma}^1$. Then, the result in the corollary follows from the above and the observation that

$$m^{\sigma^1 \circ \hat{\sigma}^1, \sigma^2 \circ \hat{\sigma}^2} = m^{\sigma^1, \sigma^2} \circ m^{\hat{\sigma}^1, \hat{\sigma}^2}.$$

□

Assume for the moment that $\bar{N}^1 = (N^1 \setminus \{i\}) \cup \{k\}$ and $\bar{N}^2 = (N^2 \setminus \{j\}) \cup \{k\}$ for given $i \in N^1$, $j \in N^2$ and $k \notin N^1 \cup N^2$. Furthermore, assume that σ^1 and σ^2 are such that $\sigma^1(l) = l$ for all $l \in \bar{N}^1 \setminus \{k\}$, $\sigma^2(l) = l$ for all $l \in \bar{N}^2 \setminus \{k\}$, $\sigma^1(k) = i$ and $\sigma^2(k) = j$. That is, σ^1 and σ^2 reflect the situation in which $i \in N^1$ and $j \in N^2$ merge into one new player k , and the remaining players maintain their identity. Then, Corollary 3.4 implies that the core of a matching game is *merging-proof*, meaning that for every core allocation $\bar{x}$ of the matching game $(N^1, N^2, A^{\sigma^1, \sigma^2})$ there is a core allocation x of the initial matching game (N^1, N^2, A) such that for the merged player $k \in \bar{N}^1 \cap \bar{N}^2$ that results after the merge of $i \in N^1$ and $j \in N^2$, it holds that $\bar{x}_k = x_i + x_j$. That is, it is not immediately beneficial for a row player and a column player to merge into a single player as they can obtain the same aggregate payoff if they act as two independent players. Actually, Corollary 3.4 also implies that the converse is true, i.e., that the core of a matching game is *splitting-proof*, in the sense for every core allocation x of the initial matching game (N^1, N^2, A) there is a core allocation $\bar{x}$ of the matching game $(N^1, N^2, A^{\sigma^1, \sigma^2})$ such that, if every player $k \in \bar{N}^1 \cap \bar{N}^2$ splits into two different players $i \in N^1$ and $j \in N^2$, it holds that $\bar{x}_k = x_i + x_j$.

A *solution concept for matching situations* is a mapping, φ , that assigns to any matching situation (N^1, N^2, A) a non-empty set of allocations. The core is therefore a solution concept. If the subset assigned by φ is always a singleton, we say that φ is a *point-valued* solution concept. Given the result in Corollary 3.4, from which we derive that the core is a merging-proof and splitting-proof solution concept for matching situations, it is natural to wonder whether, apart from the core, there are other commonly used point-valued solution concepts of cooperative game theory that are merging-proof and splitting-proof. The answer is negative for the nucleolus, the Shapley value, the τ -value and the core center.⁴

⁴For example, in the case of the nucleolus a counterexample can be found in Example 1 in

In the remaining part of this section we focus on the set of the extreme core allocations for matching games, and we show that they are merging-proof but not splitting-proof. More generally, we consider the relation between the extreme points of the core of the assignment game (N, v_A) and the extreme points of the core of the associated matching game $(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2})$. Given a game (N, v) we denote by $\text{ext}(C(v))$ the set of extreme elements of the core $C(v)$.

Theorem 3.5 Let (N^1, N^2, A) be an assignment situation. Let also $\bar{N}^1$ and $\bar{N}^2$ be such that $|N^1| = |\bar{N}^1|$ and $|N^2| = |\bar{N}^2|$. Then, for each $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$, it holds that

$$\text{ext}(C(v_{A^{\sigma^1, \sigma^2}})) \subseteq m^{\sigma^1, \sigma^2}(\text{ext}(C(v_A))).$$

Proof: Let $\bar{x} \in \text{ext}(C(v_{A^{\sigma^1, \sigma^2}}))$ and let $P^1 = C(v_A)$, which is a polytope. Let also $S^1 = \{x \in P^1 \mid m^{\sigma^1, \sigma^2}(x) = \bar{x}\}$. By Theorem 3.3, S^1 is nonempty. If P^1 is a singleton, the result in the theorem trivially holds. Hence, we assume that $\dim(P^1) > 0$. We show below that there is a finite chain $P^1 \supset P^2 \supset \dots \supset P^t$, such that P^2 is a facet of P^1 , P^3 is a facet of P^2 and so on, where $\dim(P^t) = 0$ and $S^k = \{x \in P^k \mid m^{\sigma^1, \sigma^2}(x) = \bar{x}\} \neq \emptyset$ for all $k \in \{1, \dots, t\}$. This finishes the proof, as the only element $x \in P^t$ is an extreme point of $C(v_{A^{\sigma^1, \sigma^2}})$ and $m^{\sigma^1, \sigma^2}(x) = \bar{x}$.

In order to show the existence of P^2 and S^2 , it is sufficient to prove that S^1 cannot be contained in the interior of P^1 , which we denote by $\text{int}(P^1)$. Indeed, $S^1 \not\subseteq \text{int}(P^1)$ implies that there must be one facet of P^1 , let us say P^2 , such that $S^2 = S^1 \cap P^2 = (m^{\sigma^1, \sigma^2})^{-1}(\bar{x}) \cap P^2 \neq \emptyset$. Since P^2 is a facet of P^1 , $\text{ext}(P^2) \subset \text{ext}(P^1) = \text{ext}(C(v_A))$ and $\dim(P^2) < \dim(P^1)$. If $\dim(P^2) = 0$, i.e., $P^2 = \{x\}$, the proof is complete since $\bar{x} = m^{\sigma^1, \sigma^2}(x)$ and $x \in \text{ext}\{C(v_A)\}$. Otherwise, i.e., when $\dim(P^2) > 0$, repeating the argument for $S^1 \not\subseteq \text{int}(P^1)$ proves that $S^2 \not\subseteq \text{int}(P^2)$, which implies that there exists a facet P^3 of P^2 such that $S^3 = S^2 \cap P^3 = (m^{\sigma^1, \sigma^2})^{-1}(\bar{x}) \cap P^3 \neq \emptyset$. Iterating these arguments a finite number of times, there must be a facet $P^t = \{x\}$ of P^{t-1} ,

Solymosi et al. (2005). Moreover, it can be proved that a point-valued solution concept, φ , is both merging-proof and splitting-proof is equivalent to the following statement: given an assignment situation (N^1, N^2, A) , $\bar{N}^1$ such that $|N^1| = |\bar{N}^1|$, $\bar{N}^2$ such that $|N^2| = |\bar{N}^2|$, $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$, it holds that $\varphi(\bar{N}^1, \bar{N}^2, A^{\sigma^1, \sigma^2}) = m^{\sigma^1, \sigma^2}(\mathcal{F}^{N^1, N^2}(A))$ for some correspondence $\mathcal{F}^{N^1, N^2} : \mathcal{M}_{N^1 \times N^2}^+ \rightrightarrows C(v_A)$.

for some $t > 2$, such that $S^t = S^{t-1} \cap P^t = (m^{\sigma^1, \sigma^2})^{-1}(\bar{x}) \cap P^t \neq \emptyset$ and $\dim(P^t) = 0$. That is, $\bar{x} = m^{\sigma^1, \sigma^2}(p)$ and $x \in \text{ext}\{C(v_A)\}$, since $\text{ext}(P^t) \subset \text{ext}(P^{t-1}) \subset \dots \subset \text{ext}(P^1) = \text{ext}(C(v_A))$.

Lastly, it remains to show that $S^1 \not\subseteq \text{int}(P^1)$. We prove it via contradiction. Hence, assume that $S^1 \subseteq \text{int}(P^1)$ and let p^1 be any extreme point of P^1 . In particular, note that $p^1 \notin \text{int}(P^1)$. On the one hand, since m^{σ^1, σ^2} is a continuous function, S^1 is a (nonempty) compact set. Hence, there is $x^1 \in S^1$ that is the closest point to p^1 within the set S^1 with respect to the Euclidean distance, i.e., $d(p^1, x) \geq d(p^1, x^1) = \delta > 0$ for any $x \in S^1$. On the other hand, since $x^1 \in \text{int}(P^1)$, there is $\varepsilon > 0$ such that $B_\varepsilon(x^1) \subseteq \text{int}(P^1)$. Let $x' \in B_\delta(p^1) \cap B_\varepsilon(x^1)$ and $x'' \in P^1$ such that $x^1 = \frac{1}{2}x' + \frac{1}{2}x''$. By linearity of m^{σ^1, σ^2} it follows that

$$\bar{x} = m^{\sigma^1, \sigma^2}(x^1) = \frac{1}{2}m^{\sigma^1, \sigma^2}(x') + \frac{1}{2}m^{\sigma^1, \sigma^2}(x''),$$

where, by Theorem 3.3, $m^{\sigma^1, \sigma^2}(x') \in C(v_{A^{\sigma^1, \sigma^2}})$ and $m^{\sigma^1, \sigma^2}(x'') \in C(v_{A^{\sigma^1, \sigma^2}})$. Since $\bar{x} \in \text{ext}(C(v_{A^{\sigma^1, \sigma^2}}))$, this implies that $m^{\sigma^1, \sigma^2}(x') = m^{\sigma^1, \sigma^2}(x'') = \bar{x}$, so $x' \in S^1$ which contradicts $x' \in B_\delta(p^1)$. As a consequence, our supposition was incorrect, i.e., it must be the case that $S^1 \not\subseteq \text{int}(P^1)$. $\square$

The above theorem can be generalized to account for initial arbitrary matching situations (N^1, N^2, A) in the spirit of Corollary 3.4. As a consequence, considered as a solution concept the set of extreme core allocations of the associated matching games is merging-proof. It is important to point out that, unlike for the core, the reverse inclusion of Theorem 3.5 does not hold in general, as the example below demonstrates. Hence, for the set of extreme core allocations splitting-proofness is not satisfied in general.

Example 3.6 Let (N^1, N^2, A) be an assignment situation, where

$$A: \begin{matrix} & 3 & 4 \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \end{matrix}.$$

It can be checked that

$$C(v_A) = \text{conv}\{(0, 0, 2, 2), (0, 1, 1, 2), (1, 2, 0, 1), (2, 0, 2, 0), (2, 2, 0, 0)\}.$$

Let $\bar{N} = \{1, 2\}$, and let $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$ be such that $\sigma^1(1) = 1$, $\sigma^1(2) = 2$, $\sigma^2(3) = 1$, $\sigma^2(4) = 2$. Then $C(v_{A^{\sigma^1, \sigma^2}}) = \text{conv}\{(1, 3), (4, 0)\}$. Since $(0, 0, 2, 2) \in \text{ext}(C(v_A))$ but $m^{\sigma^1 \sigma^2}(0, 0, 2, 2) = (2, 2) \notin \text{ext}(C(v_{A^{\sigma^1, \sigma^2}}))$, we can conclude that $\text{ext}\{C(v_{A^{\sigma^1, \sigma^2}})\} \not\supseteq m^{\sigma^1 \sigma^2}(\text{ext}(C(v_A)))$. $\triangleleft$

Nevertheless, it is worth mentioning that with every square assignment situation we can associate a permutation situation such that the reverse inclusion holds.

Proposition 3.7 *Let (N^1, N^2, A) be an assignment situation such that $|N^1| = |N^2|$, and let $\bar{N}$ be such that $|\bar{N}| = |N^1|$. Then, there exist $\sigma^1 \in \Pi(\bar{N}, N^1)$ and $\sigma^2 \in \Pi(\bar{N}, N^2)$ such that*

$$\text{ext}(C(v_{A^{\sigma^1, \sigma^2}})) = m^{\sigma^1 \sigma^2}(\text{ext}(C(v_A))).$$

Proof: Let $\mu^* \in M_A^*(N^1, N^2)$. Let $\sigma^1 \in \Pi(\bar{N}, N^1)$ and $\sigma^2 \in \Pi(\bar{N}, N^2)$ be such that $\sigma^2(i) = \mu^*(\sigma^1(i))$ for every $i \in \bar{N}$. It is readily checked that $v_{A^{\sigma^1, \sigma^2}}$ is an additive game, so $\text{ext}(C(v_{A^{\sigma^1, \sigma^2}})) = C(v_{A^{\sigma^1, \sigma^2}}) = \{(a_{ii}^{\sigma^1, \sigma^2})_{i \in \bar{N}}\}$. Let $x \in \text{ext}(C(v_A))$. Then, $x_i + x_j = a_{ij}$ for all $(i, j) \in \mu^*$. Hence, for every $i \in \bar{N}$ it holds that

$$m_i^{\sigma^1 \sigma^2}(x) = x_{\sigma^1(i)} + x_{\sigma^2(i)} = a_{\sigma^1(i)\sigma^2(i)} = a_{ii}^{\sigma^1, \sigma^2}.$$

Therefore, $m^{\sigma^1 \sigma^2}(x) \in \text{ext}(C(v_{A^{\sigma^1, \sigma^2}}))$, which implies

$$m^{\sigma^1 \sigma^2}(\text{ext}(C(v_A))) \subseteq \text{ext}(C(v_{A^{\sigma^1, \sigma^2}})).$$

Theorem 3.5 completes the proof. $\square$

4 Permutation situations and games

In this section, the focus is on permutation situations only. We analyze the set of undominated matrices for permutation situations: given a matching situation and its associated game, the set of undominated matrices is composed of all matrices leading to matching games with the same core for which we cannot raise any entry without changing the core. A game (N, v) is *exact* (Schmeidler, 1972) if, for each

$S \subseteq N$, there is $x \in C(v)$ such that $x(S) = v(S)$. Núñez and Rafels (2002) show that the set of undominated matrices for assignment situations is a singleton and that the game that such matrix defines is in general not exact but *buyer-seller exact*, i.e., exactness holds only for coalitions of size 2 which are composed of a row player and a column player. We note that even though assignment games are particular instances of permutation games, two facts make the problem that we analyze here different from the problem analyzed in Núñez and Rafels (2002): first, every assignment game is a permutation game defined by a different, extended matrix which includes more entries (which can therefore be raised) than the original matrix; second, the core of a permutation game is obtained after a certain collusion of the core of the assignment game defined by the same matrix, so a change in the core of the assignment game might not necessarily lead to a change in the core of the permutation game. Our results reveal that for permutation situations with at most four players, every undominated matrix defines an exact permutation game.

Formally, we introduce two sets of matrices.

Definition 4.1 Let (N^1, N^2, A) be a matching situation. Then, the set of matrices generating $C(v_A)$ is given by $G(A) = \{B \in \mathcal{M}_{N^1 \times N^2}^+ \mid C(v_A) = C(v_B)\}$.

For $A, B \in \mathcal{M}_{N^1 \times N^2}^+$, we denote $A \geq B$ if $a_{ij} \geq b_{ij}$ for every $i \in N^1, j \in N^2$.

Definition 4.2 Let (N^1, N^2, A) be a matching situation. Then, the set of *undominated matrices* generating $C(v_A)$ is given by $U(A) = \{B \in G(A) \mid \nexists C \in G(A) \setminus \{B\} \text{ such that } C \geq B\}$.

We illustrate the above definitions with the following example.

Example 4.3 Let (N, N, A) and $(N, N, B(\alpha, \beta))$ be permutation situations, where, for every $\alpha, \beta \in \mathbb{R}_+$,

$$A = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix} \text{ and } B(\alpha, \beta) = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \end{matrix},$$

Then $C(v_A) = \{(1, 1)\}$, $G(A) = \{B(\alpha, \beta) \mid \alpha + \beta \leq 2 \text{ and } \alpha, \beta \geq 0\}$ and $U(A) = \{B(\alpha, \beta) \mid \alpha + \beta = 2 \text{ and } \alpha, \beta \geq 0\}$. $\triangleleft$

It is easy to check that in the case of assignment games there does not need to exist a matrix $B \in G(A)$ such that $(N^1 \cup N^2, v_B)$ is exact. For every permutation situation (N, N, A) with at most four players, however, we can show that not only there exists a matrix $B \in U(A)$ such that (N, v_B) is exact, but this holds for every matrix $B \in U(A)$. To prove this result it is useful to introduce further notation. For every permutation situation (N, N, A) and every $i, j \in N$, define

$$\mathcal{S}_A^{ij} = \left\{ S \subseteq N \mid v_A(S) = \min_{x \in C(v_A)} x(S), \text{ and } \mu^*(i) = j \text{ for some } \mu^* \in M_A^*(S, S) \right\}.$$

That is, if $S \in \mathcal{S}_A^{ij}$, then raising the entry a_{ij} would raise $v_A(S)$ and change the core of (N, v_A) .

Theorem 4.4 Let (N, N, A) be a permutation situation with $|N| \leq 4$. Then (N, v_B) is an exact game for all $B \in U(A)$.

Proof: See Appendix. $\square$

It remains an open problem if the result of Theorem 4.4 can be extended to permutation games with at least 5 players.

Appendix

We say that (N, v_B) is *cyclic* if there exists an optimal matching of (N, v_B) which is a cycle of length n .

Proof: [of Theorem 4.4] Suppose that (N, v_B) is not exact for some $B \in U(A)$. Choose $S^* \subsetneq N$ such that $v_B(S^*) < \min_{x \in C(v_B)} x(S^*)$ and set $N = \{1, \dots, n\}$. We start by distinguishing two cases.

Case 1: $|S^*| = 1$.

Define $B^r \in \mathcal{M}_{N^1 \times N^2}^+$ by

$$\begin{aligned} b_{ij}^r &= b_{ij} & \text{for all } i \in N, j \in N \setminus \{i\}, \\ b_{ii}^r &= \min_{x \in C(v_B)} x_i & \text{for all } i \in N. \end{aligned}$$

Note that $B^r \geq B$ and $B^r \neq B$, as $b_{ii} \leq v_B(\{i\})$ for every $i \in N$ with at least one strict inequality. We show that $C(v_{B^r}) = C(v_B)$, which contradicts $B \in U(A)$. First, we show $C(v_{B^r}) \subseteq C(v_B)$. Trivially,

$$v_{B^r}(S) \geq v_B(S) \text{ for all } S \subseteq N. \quad (13)$$

Let $\mu \in \mathcal{M}_{B^r}^*(N, N)$ and take $S, T \subseteq N$ such that $S \cup T = N$, $\mu(i) = i$ for every $i \in S$ and $\mu(i) \neq i$ for every $i \in T$.⁵ Then

$$\begin{aligned} v_{B^r}(N) &= \sum_{i \in S} b_{i\mu(i)}^r + \sum_{i \in T} b_{i\mu(i)}^r \leq \sum_{i \in S} \min_{x \in C(v_B)} x_i + v_B(T) \\ &\leq \min_{x \in C(v_B)} x(S) + \min_{x \in C(v_B)} x(T) \leq \min_{x \in C(v_B)} x(N) = v_B(N) \leq v_{B^r}(N). \end{aligned}$$

Therefore, $v_{B^r}(N) = v_B(N)$, which, together with (13), implies that $C(v_{B^r}) \subseteq C(v_B)$.

Second, we show that $C(v_B) \subseteq C(v_{B^r})$. Let $x \in C(v_B)$, $S \subseteq N$ and $\mu \in \mathcal{M}_{B^r}^*(S, S)$. Let $R, T \subseteq N$ such that $R \cup T = S$, $\mu(i) = i$ for every $i \in R$ and $\mu(i) \neq i$ for every $i \in T$.⁶ Then,

$$\begin{aligned} x(S) &= x(T) + \sum_{i \in R} x_i \geq v_B(T) + \sum_{i \in R} x_i \geq v_B(T) + \sum_{i \in T} \min_{x \in C(v_B)} x_i \\ &= v_B(T) + \sum_{i \in R} b_{i\mu(i)}^r = v_{B^r}(S), \end{aligned}$$

where the first inequality holds since $x \in C(v_B)$ and the last equality follows from the way S and T are constructed.

Case 2: $|S^*| = n - 1$.

Note that there are two possibilities: either (N, v_B) is cyclic or (N, v_B) is not cyclic.

Case 2.I: (N, v_B) is cyclic.

⁵Since $B^r \geq 0$, we can assume w.l.o.g that μ leaves no agent unassigned.

⁶Again, we assume w.l.o.g that μ leaves no agent unassigned.

Choose an optimal matching of (N, v_B) , say $\mu^* \in \mathcal{M}_B^*(N, N)$, which is a cycle of length n . We assume w.l.o.g. that $\mu^* = \{(1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}$ and $S^* = N \setminus \{1\}$. Let $S^{n2} \in \mathcal{S}_A^{n2} \neq \emptyset$ and $x^* \in C(v_B)$ such that $x^*(S^{n2}) = \min_{x \in C(v_B)} x(S^{n2}) = b_{n2} + \sum_{(i,j) \in \mu \setminus \{(n,2)\}} b_{ij}$, with $\mu \in \mathcal{M}_B^*(S^{n2}, S^{n2})$ such that $(n, 2) \in \mu$. Clearly

$$\begin{aligned} b_{12} + b_{23} + \dots + b_{n-1n} + b_{n1} &= x^*(N), \\ b_{n2} + \sum_{(i,j) \in \mu \setminus \{(n,2)\}} b_{ij} &= x^*(S^{n2}). \end{aligned}$$

Summing the two above equations and rearranging terms we obtain

$$\begin{aligned} x^*(N) + x^*(S^{n2}) &= b_{23} + \dots + b_{n-1n} + b_{n2} + b_{12} + b_{n1} + \sum_{(i,j) \in \mu \setminus \{(n,2)\}} b_{ij} \\ &\leq v_B(S^*) + b_{12} + b_{n1} + \sum_{(i,j) \in \mu \setminus \{(n,2)\}} b_{ij} \\ &< x^*(S^*) + x^*(S^{n2}) + x_1^* = x^*(N) + x^*(S^{n2}), \end{aligned} \quad (14)$$

where the strict inequality follows from the following two observations: first, S^* is such that $v_B(S^*) < \min_{x \in C(v_B)} x(S^*) \leq x^*(S^*)$; second, the remaining terms in the second line of (14) can be rearranged in a way that there exist $S^1, \dots, S^t \subseteq S^{n2} \cup \{1\}$ and $\mu^1 \in \mathcal{M}(S^1, S^1), \dots, \mu^t \in \mathcal{M}(S^t, S^t)$ such that

$$\bigcup_{k=1}^t \mu^k = \{(1, 2), (n, 1)\} \cup (\mu \setminus \{(n, 2)\}),$$

and thus

$$b_{12} + b_{n1} + \sum_{(i,j) \in \mu \setminus \{(n,2)\}} b_{ij} = \sum_{k=1}^t \sum_{(i,j) \in \mu^k} b_{ij} \leq \sum_{k=1}^t v_B(S^k) \leq \sum_{k=1}^t x^*(S^k),$$

which establishes a contradiction.

Case 2.II: (N, v_B) is not cyclic.

In this case, we have that, for some $\emptyset \neq T \subsetneq N$,

$$v_B(N) = v_B(T) + v_B(N \setminus T). \quad (15)$$

In particular, $x(T) = v_B(T)$ and $x(N \setminus T) = v_B(N \setminus T)$ for all $x \in C(v_B)$. Assume w.l.o.g. that $S^* = N \setminus \{1\}$ and $1 \in T$, and let T be minimal w.r.t. inclusion such

that Eq. (15) holds and $1 \in T$. First, if $|T| = 1$, we immediately obtain that $\min_{x \in C(v_B)} x(S^*) = v_B(S^*)$, which is contradiction. Second, let $|T| = 2$. Assume w.l.o.g. that $T = \{1, 2\}$, and consider $x^* \in C(v_B)$ such that $x_2^* = b_{22}$. We note that the existence of x^* is guaranteed by Case 1. Then,

$$\begin{aligned} x^*(S^*) &= x_2^* + x^*(N \setminus \{1, 2\}) = b_{22} + v_B(N \setminus \{1, 2\}) \leq v_B(S^*) \\ &< \min_{x \in C(v_B)} x(S^*) \leq x^*(S^*), \end{aligned}$$

which is again a contradiction. Third, and last, let $|T| > 2$. By minimality of T , there are two different players $i, j \in N \setminus \{1\}$ such that, for some $\mu \in \mathcal{M}_B^*(T, T)$,

$$v_B(T) = b_{1i} + b_{j1} + \sum_{(k,l) \in \mu \setminus \{(1,i), (k,l)\}} b_{kl}. \quad (16)$$

Let $S^{ji} \in \mathcal{S}_A^{ji} \neq \emptyset$ and $x^* \in C(v_B)$ such that

$$x^*(S^{ji}) = \min_{x \in C(v_B)} x(S^{ji}) = b_{ji} + \sum_{(k,l) \in \mu \setminus \{(j,i)\}} b_{kl}, \quad (17)$$

with $\mu' \in \mathcal{M}_B^*(S^{ji}, S^{ji})$ such that $(j, i) \in \mu'$. It follows that

$$\begin{aligned} v_B(N \setminus T) &= x^*(N \setminus T), \\ b_{1i} + b_{j1} + \sum_{(k,l) \in \mu \setminus \{(1,i), (k,l)\}} b_{kl} &= x^*(T), \\ b_{ji} + \sum_{(k,l) \in \mu' \setminus \{(j,i)\}} b_{kl} &= x^*(S^{ji}). \end{aligned}$$

Summing the above equations and rearranging terms we obtain

$$\begin{aligned} x^*(N) + x^*(S^{ji}) &= x^*(T) + x^*(N \setminus T) + x^*(S^{ji}) \\ &= b_{ji} + \sum_{(k,l) \in \mu \setminus \{(1,i), (k,l)\}} b_{kl} + v_B(N \setminus T) + b_{1i} + b_{j1} + \sum_{(k,l) \in \mu' \setminus \{(j,i)\}} b_{kl} \\ &< x^*(S^*) + v_B(S^{ji} \cup \{1\}) \leq x^*(N) + x^*(S^{ji}), \end{aligned}$$

where the strict inequality follows from the assumption that S^* is such that $v_B(S^*) < \min_{x \in C(v_B)} x(S^*)$. Hence, we have reached a contradiction.

It follows from all the above considerations that (N, v_B) is an exact game for $|N| \leq 3$, so to establish the result of the theorem it is enough to analyze the following case.

Case 3: $|N| = 4$ and $|S^*| = 2$.

Again, we distinguish between two cases.

Case 3.I: (N, v_B) is cyclic.

Assume w.l.o.g. that $\mu^* = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. If the two players in S^* are consecutive in μ^* , say $S^* = \{1, 2\}$, we obtain

$$\begin{aligned} x^*(N) + x^*(S^{21}) &= b_{12} + b_{23} + b_{34} + b_{41} + b_{21} + \sum_{(i,j) \in \mu \setminus \{(2,1)\}} b_{ij} \\ &< x^*(S^*) + b_{23} + b_{34} + b_{41} + \sum_{(i,j) \in \mu \setminus \{(2,1)\}} b_{ij} \leq x^*(N) + x^*(S^{21}), \end{aligned}$$

where $S^{21} \in \mathcal{S}_A^{21} \neq \emptyset$ and $x^* \in C(v_B)$ are such that $x^*(S^{21}) = \min_{x \in C(v_B)} x(S^{21}) = b_{21} + \sum_{(i,j) \in \mu \setminus \{(2,1)\}} b_{ij}$, with $\mu \in \mathcal{M}_B^*(S^{21}, S^{21})$ such that $(2, 1) \in \mu$. This establishes a contradiction.

If the two players in S^* are not consecutive in μ^* , w.l.o.g., set $S^* = \{1, 3\}$. We distinguish between two subcases.

Case 3.I.A: $\mathcal{S}_A^{13} \cap \mathcal{S}_A^{31} \neq \emptyset$.

Let $S \in \mathcal{S}_A^{13} \cap \mathcal{S}_A^{31}$ and $x^* \in C(v_B)$ be such that $x^*(S) = \min_{x \in C(v_B)} x(S)$. Then, there are $\mu^{13}, \mu^{31} \in \mathcal{M}(S, S)$ such that

$$\begin{aligned} b_{13} + \sum_{(i,j) \in \mu^{13} \setminus \{(1,3)\}} b_{ij} &= x^*(S), \\ b_{31} + \sum_{(i,j) \in \mu^{31} \setminus \{(3,1)\}} b_{ij} &= x^*(S). \end{aligned}$$

Summing the above equations and rearranging terms we obtain

$$2x^*(S) = b_{13} + b_{31} + \sum_{(i,j) \in \mu^{13} \setminus \{(1,3)\}} b_{ij} + \sum_{(i,j) \in \mu^{31} \setminus \{(3,1)\}} b_{ij} < 2x^*(S),$$

where the strict inequality follows from the assumption $b_{13} + b_{31} \leq v_B(S^*) < \min_{x \in C(v_B)} x(S^*)$. Again, we have established a contradiction.

Case 3.I.B: $\mathcal{S}_A^{13} \cap \mathcal{S}_A^{31} = \emptyset$.

We claim and prove that $\{1, 3, 4\} \in \mathcal{S}_A^{13}$. The proof that $\{1, 2, 3\} \in \mathcal{S}_A^{31}$ proceeds along similar lines. Hence, let $S^{13} \in \mathcal{S}_A^{13}$. Then, for $x^* \in C(v_B)$ such that $x^*(S^{13}) = \min_{x \in C(v_B)} x(S^{13})$ and some $\mu^{13} \in \mathcal{M}(S^{13}, S^{13})$ it holds that

$$\begin{aligned} b_{12} + b_{23} + b_{34} + b_{41} &= x^*(N), \\ b_{13} + \sum_{(i,j) \in \mu^{13} \setminus \{(1,3)\}} b_{ij} &= x^*(S^{13}). \end{aligned}$$

Rearranging terms and using a similar argument as in Eq. (14) we obtain

$$\begin{aligned} x^*(N) + x^*(S^{13}) &= b_{13} + b_{34} + b_{41} + b_{12} + b_{23} + \sum_{(i,j) \in \mu \setminus \{(1,3)\}} b_{ij} \\ &\leq x^*(\{1, 3, 4\}) + x_2^* + x^*(S^{13}), \end{aligned}$$

which implies that $\{1, 3, 4\} \in \mathcal{S}_A^{13}$, as we claimed.

In the following, we prove that there is $x^{***} \in C(v_B)$ such that

$$x^{***}(S^{13}) = \min_{x \in C(v_B)} x(S^{13}) \quad \text{and} \quad x^{***}(S^{31}) = \min_{x \in C(v_B)} x(S^{31}). \quad (18)$$

Note that if (18) holds, we obtain

$$\begin{aligned} x^{***}(S^{13}) + x^{***}(S^{31}) &= b_{13} + b_{34} + b_{41} + b_{31} + b_{12} + b_{23} \leq v_B(S^*) + v_B(N) \\ &< x^{***}(S^{13}) + x^{***}(S^{31}), \end{aligned}$$

a contradiction.

To prove (18), let $x^* \in C(v_B)$ be such that $x^*(S^{13}) = \min_{x \in C(v_B)} x(S^{13})$, where $S^{13} = \{1, 3, 4\}$. Then, define $x^\varepsilon = x^* + (-\varepsilon, 0, 0, \varepsilon)$ and increase $\varepsilon \geq 0$ up to ε^* until some core condition, $x(S') \geq v_B(S')$ with $1 \in S'$ and $4 \notin S'$, is tight. By construction, we can assume that either $S' = \{1\}$ or $S' = \{1, 2\}$ – note that by assumption it cannot be that $S' = \{1, 3\}$, while $S' = \{1, 2, 3\}$ immediately implies (18) with $x^{***} = x^{\varepsilon^*}$. Next, let $x^{**} = x^{\varepsilon^*}$, reformulate $x^\varepsilon = x^{**} + (0, 0, -\varepsilon, \varepsilon)$ and increase $\varepsilon \geq 0$ up to ε^{**} until some core condition, $x(S'') \geq v_B(S'')$ with $3 \in S''$ and $4 \notin S''$, is tight. In an analogous fashion, we can deduce that either $S'' = \{3\}$ or $S'' = \{2, 3\}$.

Now define $x^{***} = x^{\varepsilon^{**}}$. By construction, it follows that $x^{***}(S^{13}) = x^*(S^{13})$ and $x^{***} \in C(v_B)$. There are four possibilities.

First, if $S' = \{1\}$ and $S'' = \{3\}$ we have that

$$x_1^{***} + x_3^{***} = \min_{x \in C(v_B)} x_1 + \min_{x \in C(v_B)} x_3 = b_{11} + b_{33} \leq v_B(S^*) < \min_{x \in C(v_B)} x(S^*) \leq x^{***}(S^*),$$

which is a contradiction.

Second, if $S' = \{1, 2\}$ and $S'' = \{3\}$ we have that

$$\begin{aligned} x_1^{***} + x_2^{***} + x_3^{***} &= \min_{x \in C(v_B)} x(\{1, 2\}) + \min_{x \in C(v_B)} x_3 = v_B(\{1, 2\}) + v_B(\{3\}) \\ &\leq v_B(\{1, 2, 3\}) \leq \min_{x \in C(v_B)} x(\{1, 2, 3\}) \leq x^{***}(\{1, 2, 3\}), \end{aligned}$$

where the second equality follows from the fact that players 1 and 2 are consecutive in μ^* . Thus, we established (18).

Third, if $S' = \{1\}$ and $S'' = \{2, 3\}$ similar arguments as in the previous second case establish (18).

Fourth, and last, assume that $S' = \{1, 2\}$ and $S'' = \{2, 3\}$. Then,

$$\begin{aligned} x_1^{***} + x_3^{***} &= v_B(\{1, 2\}) + v_B(\{2, 3\}) - 2x_2^{***} \\ &= v_B(\{1, 2\}) + v_B(\{2, 3\}) - 2(v_B(N) - v_B(\{1, 3, 4\})) \\ &= b_{12} + b_{21} + b_{23} + b_{32} - 2(b_{12} + b_{23} + b_{34} + b_{41} - (b_{13} + b_{34} + b_{41})) \\ &= (b_{13} + b_{32} + b_{21}) - (b_{31} + b_{23} + b_{12}) + b_{13} + b_{31} \leq b_{13} + b_{31} \\ &\leq v_B(S^*) < \min_{x \in C(v_B)} x(S^*) \leq x^{***}(S^*) = x_1^{***} + x_3^{***}, \end{aligned} \tag{19}$$

which is a contradiction. Here, the first equality follows from $S' = \{1, 2\}$ and $S'' = \{2, 3\}$, the second equality follows from $x^*(\{1, 3, 4\}) = v_B(\{1, 3, 4\})$, the third equality holds since players 1 and 2, and players 2 and 3 are consecutive in μ^* , and the first inequality holds since $S^{31} = \{1, 2, 3\}$.

Case 3.II: (N, v_B) is not cyclic.

Assume w.l.o.g. that $S^* = \{1, 3\}$. Let $\emptyset \neq T \subsetneq N$ such that

$$v_B(N) = v_B(T) + v_B(N \setminus T). \tag{20}$$

If $\mathcal{S}_A^{13} \cap \mathcal{S}_A^{31} \neq \emptyset$, we can repeat the steps in Case 3.I.A to obtain a contradiction. Hence, assume that $\mathcal{S}_A^{13} \cap \mathcal{S}_A^{31} = \emptyset$, and let $S^{13} \in \mathcal{S}_A^{13}$ and $S^{31} \in \mathcal{S}_A^{31}$ be minimal

w.r.t. inclusion in the respective sets. Since (N, v_B) is not cyclic, we can assume w.l.o.g. that $S^{13} = \{1, 3, 4\}$ and $S^{31} = \{1, 2, 3\}$. Additionally, assume w.l.o.g. that $1 \in T$, and let T be minimal w.r.t. inclusion such that Eq. (20) holds and $1 \in T$. We distinguish some subcases.

First, if $T = \{1\}$, we have $b_{11} = x_1$ for all $x \in C(v_B)$. Given $x^* \in \arg \min_{x \in C(v_B)} x_3$, we obtain

$$x^*(S^*) = x_1^* + x_3^* = b_{11} + b_{33} \leq v_B(S^*) < x^*(S^*),$$

which is a contradiction.

Second, if $T = \{1, 3\}$, we trivially obtain a contradiction with the fact that $v_B(S^*) < \min_{x \in C(v_B)} x(S^*)$.

Third, if $T = \{1, 3, 4\} = S^{13}$, we have $x(S^{13}) = v_B(S^{13})$ for all $x \in C(v_B)$. Since $S^{13} \in \mathcal{S}_A^{13}$, it must be necessarily the case that $v(S^{13}) = b_{13} + b_{34} + b_{41}$. Therefore,

$$x^*(S^{13}) + x^*(S^{31}) = b_{13} + b_{34} + b_{41} + b_{31} + b_{23} + b_{12} < x^*(S^{13}) + x^*(S^{31}),$$

where $x^* \in \arg \min_{x \in C(v_B)} x^*(S^{31})$. Again, we found a contradiction.

Fourth, the case $T = \{1, 2, 3\}$ leads to a contradiction similar to the above third case.

Fifth, let $T = \{1, 2, 4\}$. Then, $x_3 = b_{33}$ for all $x \in C(v_B)$, so

$$x^*(S^*) = x_1^* + x_3^* = b_{11} + b_{33} \leq v_B(S^*) < x^*(S^*),$$

where $x^* \in \arg \min_{x \in C(v_B)} x_1$, establishing a contradiction.

Sixth, let $T = \{1, 2\}$. If $\mathcal{S}_A^{23} \cap \mathcal{S}_A^{32} \neq \emptyset$, we can repeat the steps in Case 3.I.A to obtain a contradiction. Therefore, assume that $\mathcal{S}_A^{23} \cap \mathcal{S}_A^{32} = \emptyset$, and let $S^{23} \in \mathcal{S}_A^{23}$ and $S^{32} \in \mathcal{S}_A^{32}$ be minimal w.r.t. inclusion in the respective sets. In particular, we necessarily have $|S^{23}| > 2$ and $|S^{32}| > 2$. Since (N, v) is not cyclic, we have two possibilities: $S^{23} = \{2, 3, 4\}$ and $S^{32} = \{1, 2, 3\}$, or $S^{23} = \{1, 2, 3\}$ and $S^{32} = \{2, 3, 4\}$. If $S^{23} = \{2, 3, 4\}$ and $S^{32} = \{1, 2, 3\}$, we obtain

$$2x^*(\{1, 2, 3\}) = x^*(S^{31}) + x^*(S^{32}) = b_{31} + b_{12} + b_{23} + b_{32} + b_{21} + b_{13} < 2x^*(\{1, 2, 3\}),$$

where the strict inequality follows from the fact that $S^* = \{1, 3\}$ and $v_B(S^*) < \min_{x \in C(v_B)} x(S^*)$, establishing a contradiction. If $S^{23} = \{1, 2, 3\}$ and $S^{32} = \{2, 3, 4\}$,

we claim that $v_B(\{2, 3\}) = b_{23} + b_{32}$, so we can repeat the steps in Case 3.I.B to obtain a contradiction, with the only modification that the third equality in (19) now is an inequality. To prove that $v_B(\{2, 3\}) = b_{23} + b_{32}$ note that

$$b_{31} + b_{12} + b_{23} = \min_{x \in C_v(B)} x(S^{23}) = \min_{x \in C_v(B)} x_3 + v_B(\{1, 2\}) = b_{33} + v_B(\{1, 2\}) \quad (21)$$

and

$$b_{32} + b_{24} + b_{43} = \min_{x \in C_v(B)} x(S^{32}) = \min_{x \in C_v(B)} x_2 + v_B(\{3, 4\}) = b_{22} + v_B(\{3, 4\}). \quad (22)$$

If we sum (21) and (22) we obtain

$$b_{23} + b_{32} + b_{31} + b_{12} + b_{24} + b_{43} = b_{22} + b_{33} + v_B(N).$$

Since (N, v_B) is not cyclic, $b_{31} + b_{12} + b_{24} + b_{43} < v_B(N)$, which implies that $b_{22} + b_{33} < b_{23} + b_{32}$.

Seventh, and last, the case $T = \{1, 4\}$ leads to a contradiction similarly to the previous case.

□

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