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Graphs inducing totally balanced and submodular Chinese postman games

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Abstract

A Chinese postman (CP) game is induced by a a weighted undirected, connected graph in which the edges are identified as players and a vertex is chosen as post-office location.

Granot and Granot (2012) characterized graphs that give rise to CP games that are balanced. This note completes this line of research by characterizing graphs that give rise to CP games that are submodular (totally balanced, respectively).

Key words: Chinese Postman games, submodularity, totally balancedness

JEL classification: C71

1 Introduction

Chinese postman (CP) games, introduced in Hamers *et al.* (1999), are defined on a weighted undirected connected graph in which a vertex is fixed, referred to as post-office, and the players reside in the edges. More precisely, the choice of the location of the post-office and the non-negative weighted (or cost) function determines a specific CP game on this graph, since the value of a coalition in a Chinese postman (CP) game is obtained by a cheapest tour that starts and ends at the post-office and visits all members of this coalition. Hence, the value of a coalition reflects the cheapest costs a coalition can be visited. Observe that the costs of the cheapest tour that visits all edges in a graph at least once is equal to the value of the grand coalition, i.e. the set that consists of all players, of the CP game that is induced by that graph. Hence, the value of the grand coalition is the result of solving the related Chinese postman (CPP) problem (cf. Mei-Ko Kwan (1962), Edmonds and Johnson (1973)).

For a cooperative game (N, c), where N is the set of players and $c : 2^N \to I\!R$ is the characteristic function, the core of a cooperative game (cf. Gillies (1953)) consists of all vectors which distributes the cost of the grand coalition among the players in such a way that no subset of the grand coalition has an incentive to deviate from the grand coalition, i.e.

$$Core(N,c) = \{x \in I\!R^N \mid \sum_{i\in S} x_i \leq c(S) \text{ for all } S \subset N, \;\; \sum_{i\in N} x_i = c(N)\}.$$

Hamers (1997) showed that CP games may not be balanced, i.e. the core is empty. However, they showed that a CP game is balanced if the corresponding graph is weakly Eulerian. Further,

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they showed that a CP game is submodular if the corresponding graph is weakly cyclic, that is, every edge in this graph is contained in at most one circuit. A game (N, c) is called submodular if its characteristic function is submodular, i.e. for all $S, T \subset N$ holds

$$c(S \cup T) + c(S \cap T) \leq c(S) + c(T).$$

A game (N, c) is called totally balanced if each subgame (T, c_T) is balanced, where $c_T(S) = c(S)$ for all $S \subseteq T$. Observe that submodular games are totally balanced (cf. Shapley (1953)), and totally balanced games are balanced.

Granot *et al.* (1999) called a graph global CP balanced (totally balanced, submodular, respectively), if for all locations of the post-office and for all non-negative weight functions the corresponding CP game is balanced (totally balanced, submodular, respectively). They showed that a graph is CP balanced if and only if the graph is weakly Eulerian, and a graph is CP submodular if and only if it is CP totally balanced if and only if it is weakly cyclic.

Granot and Hamers (2004) called a graph locally CP balanced (totally balanced, submodular, respectively), if there exists at least one location of the post-office and for all non-negative weight functions the corresponding CP game is balanced (totally balanced, submodular, respectively) They showed that the locally CP balanced (totally balanced, submodular) graphs coincide with the globally balanced (totally balanced, submodular) graphs, respectively.

Granot and Granot (2012) recently characterized super locally CP balanced graphs. A graph is called super locally CP balanced if for at least one location of the post-office and at least one positive weight function the corresponding CP game is balanced. They showed that a graph is super locally CP balanced if and only if for every odd-cardinality minimal cutset A of order larger or equal to three, there exists a one-or-two edge cutset which is closer than A to the post-office in the graph.

In this paper we characterize super locally CP totally balanced and submodular graphs, which are defined similar to super locally CP balanced graphs. A graph is super locally CP totally balanced if and only if for every minimal cutset A of order larger equal to three, there exists a one-or-two edge cutset which is closer than A to the post-office in the graph. Observe that the condition odd-cardinality on the cutset A is not required in the class of super locally CP totally balancedness graphs, in contrary to super locally CP balanced graphs, which shows that the class of super locally CP totally balanced graphs is a proper subset of the class of super locally CP balanced graphs. A graph is super locally CP submodular graph if and only if it is weakly cyclic, which is equivalent to the class of local and global CP submodular graphs.

Totally balancedness is an interesting property since for each subgame a core element can be provided and from the perspective of Population Monotonic Allocation Schemes (PMAS), introduced by Sprumont (1990), since it is a necessary condition for the existence of a PMAS. The significance of submodularity is even more important since for these games some solution concepts have nice properties. For instance, the Shapley value is the barycentre of the core (Shapley (1971)), the Aumann-Davis-Masschler bargaining set coincides with the core and the nucleolus coincides with the kernel (Maschler *et al.* (1972)) and the compromise value (Tijs (1981)) can be calculated in polynomial time.

CP games are contained in the class of OR games arising from network problems. There exists a line of research in which game theoretical properties of the OR game are characterized by properties of the underlying network (graph). For example, Herer and Penn (1995) showed that graphs which are obtained as a 1-sum of K_4 and outerplanar graphs characterize submodular Steiner-traveling salesman games. Okamoto (2003) showed that minimum vertex cover games are submodular if and only if the underlying graph is (K_3, P_3)-free, i.e., no induced subgraph is isomorphic to K_3 or P_3 and minimum coloring games are submodular if and only if the underlying graph is complete multipartite. Deng *et al.* (2000) showed that minimum coloring games are totally balanced if and only if the underlying graph is perfect. Hamers *et al.* (2011) showed that minimum coloring games have a Population Monotonic Allocation scheme if and only if the graph is $(P_4, 2K_2)$-free.

This note introduces in section 2 besides some notions from graph theory, the Chinese postman game. The characterization of super locally CP submodular and super locally CP totally balanced graphs is presented in section 3, respectively.

2 Chinese postman games

Let $G = (V(G), E(G))$ be an undirected graph where $V(G)$ and $E(G)$ denote the set of vertices and edges of G respectively. Let $v_0 \in V(G)$ denote the *p*ost-office in G. A *walk* in $G = (V(G), E(G))$ is a finite sequence of vertices and edges of the form $v_1, e_1, v_2, e_2, ..., v_k, e_k, v_{k+1}$ with $k \geq 0$, $v_1, v_2, ..., v_{k+1} \in V(G)$ and $e_1, e_2, ..., e_k \in E(G)$ such that $e_j = (v_j, v_{j+1})$ for all $j \in \{1, ..., k\}$. If $v_1 = v_{k+1}$ then the walk is referred to as *a closed walk* and if $v_1 = v_{k+1} = v_0$ as a *tour*. If all edges of a walk are different then the walk is a *path*. The graph $G = (V(G), E(G))$ is *connected* if for any two vertices in G there is a path in G between the two vertices.

Let $l : E(G) \to \mathbb{R}_{++}$ be a positive *edge-cost function*. We assume that each edge belongs to a different player. Therefore, the set of players $N(G)$ can be identified with the edge set $E(G)$, i.e. $N(G) = E(G)$.

Let $T = v_0, e_1, ...e_k, v_0$ be a tour in G. Then T is *feasible* for a coalition $S \subseteq E(G)$ if every edge of S is visited by T, i.e. $S \subseteq \{e_1, ..., e_k\}$. The total cost of T is $k(T) = \sum_{j=1}^{k} l(e_j)$. Given and edge set $E_1 \subset E(G)$ we denote by $k(E_1)$ the costs of the edges in E_1, i.e. $k(E_1) = \sum_{e \in E_1} l(e)$, and the costs of a path P equals $k(P) = \sum_{j \in E(P)} l(e)$ where $E(P)$ are the edges of path P.

Definition 2.1 *The Chinese Postman (CP) game,* $(N(G), c)$*, induced by a connected graph* $G = (V(G), E(G))$*, in which* $v_0 \in V(G)$ *is the post-office and* l *the positive edge-cost function is defined by*

$$c_{l_G}(S) = \min \{k(T) : T \text{ is a feasible tour for } S\}$$

for every $S \subseteq N(G)$*.*

From now on, for short, we will say that (G, v_0, l) induces the CP game $(N(G), c_{l_G})$.

3 Submodularity and totally balancedness

Granot and Granot (2012) completely characterize super locally CP balanced graphs. In this section we provide a complete characterization of super locally CP submodular and super locally CP totally balanced graphs. We first provide the formal definition of these graphs.

Definition 3.1 *A connected graph* $G = (V(G), E(G))$ *is a super locally CP submodular (resp. totally balanced) graph if there exists a location of the post-office* $v_0 \in V(G)$ *and a positive edge-cost function* l_G *such that the* CP *game* $(N(G), c_{l_G})$ *induced by* (G, v_0, l_G) *is submodular (resp. totally balanced).*

It turns out that the super locally CP submodular graphs are the weakly cyclic ones, that is, the same graphs obtained by employing the stronger definitions of submodularity considered by Granot *et al.* (1999) and by Granot and Hamers (2004).

Theorem 3.1 *A connected graph is a super locally CP submodular graph if and only if it is weakly cyclic.*

Proof: Since the class of globally CP submodular graphs is contained in the class of super locally CP submodular graphs, the if-part follows from Hamers (1997).

Assume for the only if-part that $G = (V(G), E(G))$ is a connected graph that is not weakly cyclic. We show that G is not super locally CP submodular. We have to show that for every choice of the location of the post-office $v_0 \in V(G)$ and every positive edge-cost function $l_G : E(G) \to \mathbb{R}_{++}$, the CP game $(N(G), c_{l_G})$ induced by (G, v_0, l_G) is not submodular.

If G is not weakly cyclic then G contains a connected subgraph $G^* = (V(G^*), E(G^*))$ such that $E(G^*) = E_1 \cup E_2 \cup E_3$ in which E_1, E_2 and E_3 are the edges as depicted in Figure 1 and let $V(G_1), V(G_2)$ and $V(G_3)$ be the vertices corresponding to E_1, E_2 and E_3, respectively. Let $w_1, w_2 \in V(G^*)$ be the two vertices of degree 3 in G^* as indicated in Figure 1.

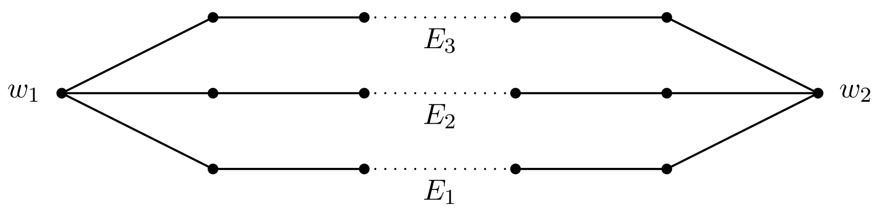

Figure 1 The subgraph G^*.

Take $v_0 \in V(G)$ as the post-office and l_G as positive edge-cost function. We denote by $P^{v_0,v}$ the set of paths from the post-office v_0 to some vertex $v \in V(G^*)$ that contains no vertex of the set $V(G^*) \setminus \{v\}$. Since G is connected, the set $P^{v_0,v}$ is non-empty. Let P_1 be a cheapest path that connects the post-office v_0 to $V(G^*)$, i.e. for all $v \in V(G^*)$ and for all $P \in P^{v_0,v}$ holds $k(P_1) \leq k(P)$. Note, if $v_0 \in V(G^*)$ we reduce P_1 to the vertex v_0, and the cost associated with P_1 is zero. Let us suppose without loss of generality that the vertex v that connects P_1 to G^* is located at $V(G_1)$. We distinguish between two cases.
Case 1: $k(P_1) < k(P)$ for all $P \in P^{v_0,w_1}$.
The assumption implies that $v \neq w_1$. Let us consider $A = E_1 \cup E_3$ and $B = E_1 \cup E_2$. Let $(N(G), c_{l_G})$ be the CP game induced by (G, v_0, l_G). Then by definition of P_1 we have

$$c_{l_G}(A) = k(E_1) + k(E_3) + 2k(P_1),$$

$$c_{l_G}(B) = k(E_1) + k(E_2) + 2k(P_1).$$

Now, we prove

$$c_{l_G}(A \cap B) > k(E_1) + 2k(P_1). \tag{1}$$

Since l_G is a positive edge-cost function, we have that $c_{l_G}(E_1) > k(E_1)$. So, if $v = v_0$ we have $k(P_1) = 0$ which immediately proves (1).

Hence, we may assume that $v \neq v_0$. Consider a feasible tour $T = v_0, e_1, v_1, e_2, ..., v_{i-1}, e_i, ..., v_{j-1}, e_j, ..., v_{k-1}, e_k, v_0$ of $A \cap B = E_1$. Without loss of generality we can assume that there exists i, j such that $e_i \in E_1$, $\{e_1, ..., e_{i-1}\} \cap E_1 = \varnothing$, $e_j \in E_1$ and $\{e_{j+1}, ..., e_k\} \cap E_1 = \varnothing$. Observe that $E_1 \subseteq \{e_i, ..., e_j\}$. Since $k(P_1)$ is the cheapest connection between v_0 and $V(G_1)$, it follows that

$$\sum_{h=1}^{i-1} l_G(e_h) + \sum_{h=j+1}^{k} l_G(e_h) \geq 2k(P_1). \tag{2}$$

If this inequality is strict for each feasible tour of $A \cap B$, then the definition of $c_{l_G}(E_1)$ implies (1). If the inequality is an equality for some feasible tour of $A \cap B$, then by the assumption in this case we can conclude that $w_1 \notin \{v_{i-1}, v_j\}$. Taking into account that $E_1 \subseteq \{e_i, ..., e_j\}$, the walk

$v_{i-1}, e_i, ..., v_{j-1}, e_j, v_j$ has more edges than E_1: those in E_1, and some outside E_1 or some edges repeated from E_1 . Hence,

$$\sum_{h=i}^{j} l_G(e_h) > k(E_1). \tag{3}$$

From (2), (3) and the definition of $c_{l_G}(E_1)$ follows (1).

We can prove in a similar way that

$$c_{l_G}(A \cup B) > k(E_1) + k(E_2) + k(E_3) + 2k(P_1).$$

Hence, we can conclude

$$c_{l_G}(A \cup B) + c_{l_G}(A \cap B) > c_{l_G}(A) + c_{l_G}(B),$$

and therefore, (N, c_{l_G}) is not submodular. Hence G is not submodular.

Case 2: There exists a $P \in P^{v_0,w_1}$ such that $k(P_1) = k(P)$.
Obviously, if $k(P_1) < k(P)$ for all $P \in P^{v_0,w_2}$, we can apply the same argument as in case 1. Hence, we can assume that there exists a $P \in P^{v_0,w_2}$ such that $k(P_1) = k(P)$. Let P_{w_1} (P_{w_2}) be a cheapest path from v_0 to w_1 (from v_0 to w_2, respectively). Since $k(P_{w_1}) = k(P_{w_2}) > 0$ and $w_1 \neq w_2$ we have that $v_0 \notin V(G^*)$. There exists $u_k \in V(G) \backslash V(G^*)$ such that $P_{w_1} = v_0, e_1, ...e_{k-1}, u_k, e^1_{k+1}, ..., e^1_{t_1}, w_1$ and $P_{w_2} = v_0, e_1, ...e_{k-1}, u_k, e^2_{k+1}, ..., e^2_{t_2}, w_2$, i.e., both path have a common path $v_0, e_1, ...e_{k-1}, u_k$, which possibly only consists of v_0. Let $\overline{E}_1 = \{e^1_{k+1}, ..., e^1_{t_1}, e^2_{k+1}, ..., e^2_{t_2}\}$ be the edge set formed by the two disjoint parts of the paths P_{w_1} and P_{w_2}. Notice that the edges in $\overline{E}_1$ join w_1 and w_2, and that $\overline{E}_1$, E_2 and E_3 are pairwise disjoint. Further, observe that $v_0, e_1, ...e_{k-1}, u_k$ is the cheapest connection to $\overline{E}_1$. Now, we can follow the arguments of case 1 by replacing E_1 by $\overline{E}_1$, which completes the proof. □

We address in the second part the characterization for super locally CP totally balanced graphs. Now we prove that the class of globally CP totally balanced graphs is a proper subset of the class of super locally CP totally balanced graphs. That is, the class of super locally CP totally balanced graphs is bigger than the class of totally balanced CP graphs according to the stronger definitions of totally balancedness by Granot *et al.* (1999) and by Granot and Hamers (2004). We also prove that it is smaller than the class of core-nonempty CP graphs obtained by Granot and Granot (2012). Before we provide the second main result we need some definitions. An egde cut set in a connected graph $G = (V(G), E(G))$ is a set of edges $A, A \subseteq E(G)$, whose removal disconnects G. An edge cut in G is called minimal if no proper subset thereof is also an edge cutset of G.

Theorem 3.2 *A connected graph $G = (V(G), E(G))$ is super locally CP totally balanced if and only if there exists a $v_0 \in V(G)$ such that for every minimal edge cutset A, $|A| \geq 3$, there exists a one-or-two-edge cutset which is closer than A to v_0 in G.*

Proof: Let $G = (V(G), E(G))$ be a super locally CP totally balanced graph, and assume in contrary, that for any location of the post-office v_0 there does not exist a one-or-two-edge cutset which is closer than A to v_0 in G. Take a $v_0 \in V(G)$ and a positive edge-cost funcion $l_G : E(G) \to \mathbb{R}_{++}$. Then G contains a connected subgraph $G^* = (V(G^*), E(G^*))$ as depicted in Figure 1 with $w_1 = v_0$. Let $(N(G), c_{l_G})$ be the CP game induced by (G, v_0, l_G). Assume that the core of $\left(E(G^*), (c_{l_G})_{E(G^*)}\right)$ is non-empty. Indeed, if $x \in \mathbb{R}^{E(G^*)}$ is in the core of $\left(E(G^*), (c_{l_G})_{E(G^*)}\right)$ then

$$\begin{aligned}
&x(E(G^*)) = c_{l_G}(E(G^*)), \\
&x(E_1 \cup E_2) \leq c_{l_G}(E_1 \cup E_2) = k(E_1) + k(E_2), \\
&x(E_1 \cup E_3) \leq c_{l_G}(E_1 \cup E_3) = k(E_1) + k(E_3), \\
&x(E_2 \cup E_3) \leq c_{l_G}(E_2 \cup E_3) = k(E_2) + k(E_3).
\end{aligned}$$

Summing the inequalities, we get $x(E(G^*)) \leq k(E_1) + k(E_2) + k(E_3)$, which contradicts the first equality since $k(E_1) + k(E_2) + k(E_3) < c_{l_G}(E(G^*))$, which contradicts the non-emptyness of the core.

Now consider the if-part. The condition on the graph G implies that each edge that is connected to the post-office v_0 is either a minimal one-edge cutset or contained in a minimal two-edge cutset. Let v_0 be connected to k_1 minimal one-edge custsets and to k_2 minimal two-edge cutsets. Then v_0 induces $k_1 + k_2$ subgraphs, say $G_1 = (V(G_1), E(G_1)), ..., G_{k_1+k_2} = (V(G_{k_1+k_2}), E(G_{k_1+k_2}))$, that only coincides in v_0. For any edge-cost function l_G it holds that the CP game $(N(G), c_{l_G})$ induced by (G, v_0, l_G) satisfies $c_{l_G}(S) = \sum_{j=1}^{k_1+k_2} c_{l_G}(S \cap E(G_i))$. Therefore, $(N(G), c_{l_G})$ is totally balanced if each subgame $(E(G_i), c_{l_G|E(G_i)}), i \in \{1, 2, ..., k_1 + k_2\}$ is totally balanced. Hence, it is sufficient to consider two subgraphs.

First, consider a subgraph $H = (V(H), E(H))$ that is formed by a single connected subgraph $H_1 = (V(H_1), E(H_1))$ and an appended path P between the post-office v_0 and $v_k \in V(H_1)$ such that $v_k \neq v_0$ and $V(H_1) \cap V(P) = \{v_k\}$. Let l_H be an edge-cost function on $E(H)$ such that the sum of the cost of the edges in P is M. Let $G^* = (V(G^*), E(G^*))$ be a subgraph of H. We have that $E(G^*) = P^* \cup (E(G^*) \cap E(H_1))$, where P^* is formed by the edges of the path P which are in $E(G^*)$.

We distinguish between two cases:

Case 1: $E(G^*) \cap E(H_1) \neq \varnothing$.

Let $m = |E(G^*) \cap E(H_1)|$ and let Q be the cost of the cheapest tour in $E(H_1)$ which contains $E(G^*) \cap E(H_1)$ taking v_k as the post-office. Consider $x \in \mathbb{R}^{E(G^*)}$ such that $x_j = \frac{2M+Q}{m}$ if $j \in E(G^*) \cap E(H_1)$ and $x_j = 0$ otherwise. Then $x(E(G^*)) = x(E(G^*) \cap E(H_1)) = 2M + Q$. Since $x_j = 0$ for all $j \in P^*$, it is sufficient to show that for each $S \subseteq E(G^*)$ such that $S \cap E(H_1) \neq \varnothing$ and $E(G^*) \cap E(H_1) \nsubseteq S$ holds $x(S) \leq (c_{l_H})_{E(G^*)}(S)$. Take $M \geq \frac{(m-1)Q}{2}$, then $x(S) \leq \frac{|S|(2M+Q)}{m} \leq \frac{(m-1)(2M+Q)}{m} \leq 2M \leq (c_{l_H})_{E(G^*)}(S)$. Hence $x \in Core(E(G^*), (c_{l_H})_{E(G^*)})$.

Case 2: $E(G^*) \cap E(H_1) = \varnothing$.

Let P be $v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k$. Let P^* be containing the edges $e_{p_1}, ..., e_{p_t}$, with $1 \leq p_1 < ... < p_t \leq k$. Define $x \in \mathbb{R}^{E(G^*)}$ by $x_{p_1} = l_H(e_1) + .. + l_H(e_{p_1})$ and $x_{p_j} = l_H(e_{p_{j-1}+1}) + ... + l_H(e_{p_j})$ for all $j \in \{2, ..., t\}$. It readily follows that $x \in Core(E(G^*), (c_{l_H})_{E(G^*)})$.

Hence, (H, v_0, l_H) induces a totally balanced CP game.

Second consider the subgraph $H = (V(H), E(H))$ formed by a single connected subgraph $H_1 = (V(H_1), E(H_1))$ and two appended edge-disjoint paths P_1 and P_2 between the post-office v_0 and $v_1, v_2 \in V(H_1)$ such that $v_0 \notin \{v_1, v_2\}$, $V(H_1) \cap V(P_1) = \{v_1\}$ and $V(H_1) \cap V(P_2) = \{v_2\}$. If we consider an edge-cost function l_H such that the sum of the cost of the edges in P_1 and in P_2 coincide, we can prove in a similar way as the case of the first subgraph that (H, v_0, l_H) induces the totally balanced CP game $(E(H), c_{l_H})$. □

In the following example we provide of a super locally CP totally balanced graph that is not super locally CP submodular, and a super locally CP balanced graph that is not super locally CP totally balanced.

Example 3.1 *Consider the graph $G = (V(G), E(G))$ as depicted in Figure 1 with the post-office $v_0 \notin \{w_1, w_2\}$. It is super locally CP totally balanced but not super locally CP submodular.*

Consider the graph $G = (V(G), E(G))$ formed by P_1, P_2, P_3, P_4 that join the post-office $v_0 \in V(G)$ and a vertex $v_1 \in V(G) \setminus \{v_0\}$, and such that any two of the four paths join only in v_0 and v_1. Then this graph is super locally CP balanced but not super locally CP totally balanced.

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