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On a Parameterized System of Nonlinear Equations with Economic Applications

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Abstract We study a parameterized system of nonlinear equations. Given a nonempty, compact, and convex set, an affine function, and a point-to-set mapping from the set to the Euclidean space containing the set, we constructively prove that, under certain (boundary) conditions on the mapping, there exists a connected set of zero points of the mapping, i.e., the origin is an element of the image for every point in the connected set, such that the connected set has a nonempty intersection with both the face at which the affine function is minimized and the face at which that function is maximized. This result generalizes and unifies several well-known existence theorems including Browder's fixed point theorem and Ky Fan's coincidence theorem. An economic application with constrained equilibria is also discussed.

Keywords Parameterized system of nonlinear equations · Fixed point · Variational inequality · Algorithm · Equilibrium

1 Introduction

The primary purpose of this paper is to study a parameterized system of nonlinear equations. Given an arbitrary nonempty, convex and compact set, an arbitrary nonzero vector, and an arbitrary point-to-set mapping (correspondence) from this set to the Euclidean space containing the set, (i) what condition should be imposed upon

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the mapping so that the mapping has a connected set of zero points, i.e., the vector of zeros is contained in the image of the mapping for all elements of the connected set, such that the connected set intersects the two boundary faces at which the affine function obtained from the nonzero vector attains its minimum and its maximum, respectively; (ii) how to compute a connected set of zero points of such a correspondence. For each parameter value between the minimum and the maximum value of the affine function, we require that there exists at least one point in the set which is a zero point of the mapping and has affine function value equal to that parameter value. We may interpret the parameter values as a homotopy parameter between 0 and 1, time between a time span, a policy instrument like a tax or wealth level, or an endogenously determined parameter, depending on the context. If we strengthen the system of nonlinear equations by requiring some differentiability, the connected set of solutions can be seen as the solution set of some two-point boundary-value dynamic problem.

Our main result establishes under mild conditions the existence of a connected set of zero points of an upper semicontinuous correspondence, linking two distinct points on the boundary of its domain. We also demonstrate that the theorem extends and unifies several powerful existence theorems such as Browder's fixed-point theorem [1] and theorems in [2–4]. Browder's fixed-point theorem is a continuum version of the celebrated Brouwer's fixed-point theorem and states that for any continuous function from the Cartesian product of a set and the interval between 0 and 1 to the set itself there exists a continuum of fixed points connecting the zero-level with the one-level. Here, the variable lying between 0 and 1 can be seen as a homotopy parameter and at each value of this homotopy parameter there exists a fixed point of the homotopy mapping. As another special case of our main Theorem 2.1, we establish the existence of a continuum of coincidences of two mappings, thereby generalizing the well-known Fan's coincidence theorem [5] to a connected set of such points. A coincidence is a point at which the images of two different mappings have a nonempty intersection. Furthermore, we give several general results on the existence of a continuum of fixed points, zero points, optima, and solutions to nonlinear variational inequalities. For recent studies on variational inequalities, we refer to [6–9] among others.

In contrast to nonconstructive (sometimes sophisticated) approaches used in the literature, such as Browder's and Fan's, one prominent feature of our approach is its constructive nature and simplicity in its arguments. To be more precise, we prove Theorem 2.1 in a constructive manner making use of a new algorithm. The algorithm differs from most of the algorithms known in the literature in that our algorithm computes a continuum of solutions, whereas the existing algorithms compute only a single solution and were originally designed to do so; see [10–15] among others. Homotopy methods, like the ones in [12] and [13], calculate a zero point of a mapping by generating a piecewise linear path of approximate zero points of a homotopy mapping on a homotopy set, starting with a unique zero point of a trivial function on the zero level and ending with a zero point of the mapping on the one level. In this way, a set of zero points of the homotopy mapping is generated connecting a trivial solution with a zero point of some given mapping. The interested reader may consult [16–18] for overviews on the subject.

In our approach, we embed the domain of the mapping into an elaborately designed full-dimensional rectangular and design a simplicial algorithm on it. With respect to a specific simplicial subdivision of the set rectangular, the algorithm generates a finite sequence of adjacent simplices, starting at an arbitrarily chosen point in the face of the rectangular at which the affine function is minimized and ending with a simplex on the face of the rectangular at which the affine function is maximized. When the mesh size of the triangulation is small enough, the sequence of simplices induces a piecewise linear path of approximate zero points of the point-to-set mapping of interest linking those two facets of the rectangular. By a limit argument we can show that there exists a connected set of zero points of the mapping linking the two distinct facets of the domain at which the affine function is minimized and maximized, respectively. Our constructive approach extends and unifies results in [2–4]. Herings, Talman, and Yang propose in [2] an algorithm for computing a connected set of solutions on a cube and they generalize in [3] this algorithm to polytopes. Talman and Yamamoto establish in [4] via a nonconstructive approach the existence of a connected set of solutions on an arbitrary convex and compact set. Our constructive method applies to any compact and convex set and to more general correspondences than those known in the literature. The formulation of our boundary condition is closely related to that in [19] where the existence problem of a single zero point rather than a connected set is considered. In another related work, Allgower and Sommese [20] study piecewise linear approximation of smooth compact fibers.

The current study is motivated by two fundamental economic problems. The first concerns a general competitive exchange economy under price rigidities. Departing from the idealistic Arrow–Debreu perfect competition model, such an economic model deals with more realistic economic environments where there may be restrictions on the prices of commodities and services. There are many economic or political reasons for price rigidities. For instance, to prevent breakdown of stock markets, ceilings and floors are imposed upon the prices of each stock; price controls are used to reduce inflation or deflation; and minimum wages are employed to protect certain group of a society. Such economic models are studied in [21–32] among many others, see [33] for a comprehensive study. The second economic problem concerns fair and efficient allocation of indivisible resources and monetary compensation among a group of agents, where the total amount of monetary compensation may vary over time. Such problems are examined in, for example, [34] and [35]. In this paper, we focus on the first economic problem as application of our theory. For this problem, the allocation of goods are state variables, the price vector and the rationing on purchase or sale are control variables, the parameter stands for the value of the total initial endowment of goods, and the objective is to maximize the welfare of every economic agent and at the same time to balance demand and supply.

The rest of the paper is organized as follows. In Sect. 2, we present basic concepts and the main existence theorem. In Sect. 3, we propose the simplicial algorithm which will be used to approximate a continuum of solutions and we prove its convergence. In Sect. 4, we give the constructive proof of the main existence theorem. In Sect. 5, we show how the main theorem implies several existence theorems of a continuum of coincidences, fixed points, zero points, optima, and solutions to nonlinear variational inequality problems. In Sect. 6, we present the economic application of price rationing. Section 7 concludes.

2 The Main Existence Theorem

Consider an arbitrary nonempty, convex and compact set X in the n-dimensional Euclidean space $\mathbb{R}^n$. Let c be an arbitrary non-zero vector in $\mathbb{R}^n$. Without loss of generality we assume that the norm of c is 1. Define

$$\begin{aligned} X^+ &:= \big\{x \in X \mid \langle c, x\rangle \geq \langle c, y\rangle \text{ for all } y \in X\big\},\\ X^- &:= \big\{x \in X \mid \langle c, x\rangle \leq \langle c, y\rangle \text{ for all } y \in X\big\},\\ t^+ &:= \langle c, x\rangle \quad \text{for any } x \in X^+,\\ t^- &:= \langle c, x\rangle \quad \text{for any } x \in X^-. \end{aligned}$$

Clearly, $t^- \leq t^+$. To avoid triviality, throughout the paper we assume that $t^- < t^+$ and thus $X^- \cap X^+ = \emptyset$. For $t \in [t^-, t^+]$ define $X(t) = \{x \in X \mid \langle c, x\rangle = t\}$. Note that for every $t \in [t^-, t^+]$ the set $X(t)$ is a nonempty, convex, and compact set in $\mathbb{R}^n$, $X(t^-) = X^-$, and $X(t^+) = X^+$.

Let Y be an arbitrary nonempty set in $\mathbb{R}^n$. For $x \in Y$, the set

$$N(Y, x) := \big\{y \in \mathbb{R}^n \mid \big\langle x - x', y\big\rangle \geq 0 \text{ for all } x' \in Y\big\}$$

denotes the *normal cone* of Y at x and its polar cone

$$T(Y, x) := \big\{z \in \mathbb{R}^n \mid \langle z, y\rangle \leq 0 \text{ for all } y \in N(Y, x)\big\}$$

denotes the *tangent cone* of Y at x. If Y is compact and convex, $N(Y, \cdot)$ is an upper semicontinuous, convex-valued, and closed-valued mapping on Y and $T(Y, \cdot)$ is a convex-valued and closed-valued mapping on Y and, for every $y \in Y$, both $N(Y, y)$ and $T(Y, y)$ are nonempty cones.

$\mathbb{N}$ denotes the set of positive integers and, for $k \in \mathbb{N}$, I_k denotes the set of the first k positive integers. The symbols 0^n, 1^n and $E(n)$ stand for the vector of zeros and ones of dimension n and the $n \times n$ identity matrix, respectively. Given a set D, $\operatorname{bd} D$ and $\operatorname{int} D$ represent the boundary and the interior of D, respectively, and $\operatorname{co}(D)$ represents the convex hull of D. For an $n \times n$ matrix R and a subset Y of $\mathbb{R}^n$, we define $RY := \{z \in \mathbb{R}^n \mid z = Ry,\ \ y \in Y\}$. Furthermore, we define $B := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1, \langle c, x\rangle = 0\}$ and the two-dimensional cone with apex

$$C(v) := \big\{y \in \mathbb{R}^n \mid y = \mu v + \beta c, \mu \geq 0, \beta \in \mathbb{R}\big\}$$

for any $v \in B$.

A topological space W is said to be *connected* iff the only subsets of W both open and closed are $\emptyset$ and W. A subset of W is called a *connected set* iff it is connected as a subspace of W. Given an element $y \in W$, the union of all connected subsets of W containing y is called the *component* of y in W, see [36].

Let ϕ be an upper semi-continuous mapping from X to the collection of nonempty convex and compact subsets of $\mathbb{R}^n$. A point $x^* \in X$ is called a *zero point* of ϕ iff $0^n \in \phi(x^*)$, a *fixed point* of ϕ iff $x^* \in \phi(x^*)$, a *coincidence* of ϕ and some other mapping ψ on X iff $\phi(x^*) \cap \psi(x^*) \neq \emptyset$, and a *stationary point* of ϕ or a solution to the *nonlinear variational inequality problem* for ϕ on X iff there exists $f^* \in \phi(x^*)$ satisfying $\langle x^* - x, f^*\rangle \geq 0$ for all $x \in X$. Note that x^* is a stationary point of ϕ if and only if $\phi(x^*) \cap N(X, x^*) \neq \emptyset$. Without any further conditions on ϕ, there may not exist any solution at all on X for some of the solution concepts, not to mention a continuum of solutions.

In this paper, we are interested in conditions on the mapping ϕ, under which there exists a connected set of solutions of ϕ in X having a nonempty intersection with both X^- and X^+. Since the intersection of X^- and X^+ is empty this implies that the connected set of solutions contains a continuum of points. A solution could be a zero point, fixed point, stationary point, or a coincidence with some other mapping on X. Now we are going to state the main existence theorem of this paper.

Theorem 2.1 *Let ϕ be an upper semicontinuous mapping from X to the collection of nonempty convex and compact subsets of $\mathbb{R}^n$. Suppose there exists an upper semicontinuous mapping π from B to the collection of nonempty convex and closed subsets of $\mathbb{R}^n$ and a nonsingular $n \times n$ matrix mapping $A(\cdot)$ being continuous on X, such that for every $x \in X$ and $v \in N(X(t), x) \cap B$, where $t = \langle c, x\rangle$, the following two conditions hold*:

1. *The set $A(x)\phi(x) \cap \pi(v) \cap C(v)$ is either empty or contains 0^n*;
2. *The set $A(x)\phi(x) \cap \pi(v) \neq \emptyset$.*

Then there exists a connected set C of zero points of ϕ in X such that $X^- \cap C \neq \emptyset$ and $X^+ \cap C \neq \emptyset$.

The theorem says that the mapping ϕ has a continuum of zero points on X connecting X^- and X^+, if there exists a continuous regular matrix mapping A on X and an upper semicontinuous, convex-valued, and closed-valued mapping π on B satisfying that, for every element v of B in the normal cone of $X(\langle c, x\rangle)$ at any point x of X, the two sets $A(x)\phi(x)$ and $\pi(v)$ intersect, but this intersection has no points in common with the two-dimensional cone $C(v)$ determined by the vectors v, c, and $-c$, unless the origin is contained in this intersection.

The matrix $A(x)$ translates the image $\phi(x)$ in a linear way, so that $A(x)\phi(x)$ has the same convexity properties as $\phi(x)$ has. Due to the regularity of the matrix $A(x)$ at any x in X, a point x^* is a zero point of ϕ if and only if x^* is a zero point of $A(x)\phi(x)$. The use of the linear mapping $A(\cdot)$ expands the cases to which our result applies. For example, consider $X = B^n$, $c = 1^n$ and the function $f : B^n \to \mathbb{R}^n$ defined by $f(x) = x - \langle c, x\rangle c/n$, where B^n is the n-dimensional unit ball. Then there is no mapping π that satisfies both conditions 1 and 2 with $A(x) = E(n)$, although $f(\beta 1^n) = 0^n$ for any feasible β, connecting X^- and X^+. However, when we take $A(x) = -E(n)$ for all $x \in B^n$, conditions 1 and 2 are satisfied for $\pi(v) = \mathbb{R}^n$ for any $v \in B$.

3 A Simplicial Algorithm

In this section, we propose a simplicial algorithm which will lead to a constructive proof of Theorem 2.1. For $t \in \mathbb{R}$, let $H(t) = \{y \in \mathbb{R}^n \mid \langle c, y\rangle = t\}$, the union of $H(t)$ over t, $t^- \leq t \leq t^+$. Let $H^- = H(t^-)$ and $H^+ = H(t^+)$. For $x \in H$, let $p(x)$ be the orthogonal projection of x on $X(t)$, where $t = \langle c, x\rangle$. Since X

is a nonempty, compact, convex set, p is a continuous function on H. Moreover, $x - p(x) \in N(X(t), p(x))$ for $x \in H(t)$, $t^- \leq t \leq t^+$. For t, $t^- \leq t \leq t^+$, the set $Q(t)$ is defined by

$$Q(t) := \{q \in H(t) \mid \|q - p(q)\|_2 \leq 1\}.$$

The union of $Q(t)$ over t, $t^- \leq t \leq t^+$, is denoted by Q.

Lemma 3.1 *The set Q is a full-dimensional, compact, convex subset of H.*

Proof Clearly, Q is a full-dimensional set in $\mathbb{R}^n$ and a subset of H. Since X is compact, Q is also compact. To prove convexity of Q, take any $q^1, q^2 \in Q$ and $0 \leq \lambda \leq 1$ and let

$$q(\lambda) = \lambda q^1 + (1 - \lambda)q^2$$

and

$$p(\lambda) = \lambda p(q^1) + (1 - \lambda)p(q^2).$$

Since X is convex, we have that $p(\lambda) \in X$. Let $t = \langle c, p(\lambda)\rangle$, i.e. $p(\lambda) \in H(t)$. Note that $\langle c, q^1\rangle = \langle c, p(q^1)\rangle$ and $\langle c, q^2\rangle = \langle c, p(q^2)\rangle$. Then we have $\langle c, q(\lambda)\rangle = t$, i.e., $q(\lambda) \in H(t)$. Moreover,

$$\|q(\lambda) - p(\lambda)\|_2 \leq \lambda \|q^1 - p(q^1)\|_2 + (1 - \lambda)\|q^2 - p(q^2)\|_2 \leq 1.$$

Therefore, $q(\lambda) \in Q$, i.e., Q is a convex set. □

For $q \in Q$, let $v(q) = q - p(q)$. By construction, $v(q) \in B$ for every $q \in Q$, $\|v(q)\|_2 = 1$ if and only if $q \in \operatorname{bd} Q$, and $v(q) = 0^n$ if and only if $q \in X$.

Lemma 3.2 *For every $q \in Q$ it holds that $N(Q(t), q) = C(v(q))$ and $N(Q(t), q) \subseteq N(X(t), p(q))$, where $t = \langle c, q\rangle$.*

Proof Since Q is full-dimensional, for $q \in \operatorname{int} Q(t)$ it holds that $N(Q(t), q) = C(0^n)$ and, hence, also $N(Q(t), q) \subseteq N(X(t), p(q))$. Take any point $q \in \operatorname{bd} Q(t)$. Since $p(q)$ is the projection of q on $X(t)$, we must have that $N(Q(t), q)$ contains $C(v(q))$. Let $B(p(q)) = \{x \in H(t) \mid \|x - p(q)\|_2 \leq 1\}$. Clearly, $B(p(q)) \subseteq Q(t)$ and $q \in \operatorname{bd} B(p(q))$ and, therefore, $N(Q(t), q) \subseteq N(B(p(q)), q)$. However, $q \in \operatorname{bd} B(p(q))$ implies $N(B(p(q)), q) = C(q - p(q))$. Since $v(q) = q - p(q)$, we obtain $N(Q(t), q) \subseteq C(v(q))$. Hence, $N(Q(t), q) = C(v(q))$. Since $q - p(q) \in N(X(t), p(q))$, it follows that $C(v(q)) \subseteq N(X(t), p(q))$. □

Let $a^1, \dots, a^{n-1}$ be an orthogonal basis for the $(n - 1)$-dimensional subspace H^0, where

$$H^0 := \{y \in \mathbb{R}^n \mid \langle c, y\rangle = 0\}.$$

Hence, we have

- $H^0 = \{y \in \mathbb{R}^n \mid y = \sum_{i=1}^{n-1} \lambda_i a^i, \lambda_i \in \mathbb{R} \text{ for all } i\}$;
- $\langle c, a^i \rangle = 0$ for all i;
- $\langle a^i, a^i \rangle = 1$ for all i;
- $\langle a^i, a^j \rangle = 0$ for all $j \neq i$.

Take any x^0 in X^- and $M > 0$. Then the $(n-1)$-dimensional cube P^- in H^- is defined by

$$P^- := \{x \in H^- \mid -M \le \langle a^i, x - x^0 \rangle \le M,\ i = 1, \dots, n-1\}.$$

and the rectangular P in H is defined by

$$P := \{x \in H \mid x = x^- + \lambda c, 0 \le \lambda \le t^+ - t^-, x^- \in P^-\}.$$

For t, $t^- \le t \le t^+$, let $P(t) = P \cap H(t)$. We can choose M so large that for every t, $t^- \le t \le t^+$, the cube $P(t)$ contains $Q(t)$ in its relative interior. For $i = 1, \dots, n-1$, define $a^{-i} = -a^i$ and let $I = \{-(n-1), \dots, -1, 1, \dots, n-1\}$. Then P can be reformulated as

$$P = \{x \in \mathbb{R}^n \mid \langle a^i, x \rangle \le M - \langle a^i, x^0 \rangle \text{ for all } i \in I, \langle c, x \rangle \le t^+, \langle -c, x \rangle \le t^-\}.$$

Let $\bar{I} = I \cup \{-n, n\}$, $a^n = c$, $a^{-n} = -c$, $b_i = M - \langle a^i, x^0 \rangle$ for every $i \in I$, $b_n = t^+$, and $b_{-n} = t^-$. Then $P^- = P(t^-)$ and P can be further rewritten as

$$P^- = \{x \in \mathbb{R}^n \mid \langle a^i, x \rangle \le b_i \text{ for all } i \in I \text{ and } \langle c, x \rangle = t^-\}$$

and

$$P = \{x \in \mathbb{R}^n \mid \langle a^i, x \rangle \le b_i \text{ for all } i \in \bar{I}\}.$$

Similarly, we may write $P^+ = P(t^+)$ as

$$P^+ = \{x \in \mathbb{R}^n \mid \langle a^i, x \rangle \le b_i \text{ for all } i \in I \text{ and } \langle c, x \rangle = t^+\}.$$

Notice that P is a simple full-dimensional polytope, and no constraints are redundant. Let $\mathcal{I}$ be the collection of subsets J of I such that $|J| \le n-1$ and $j \notin J$ whenever $-j \in J$. For each $J \in \mathcal{I}$, define

$$F(J) := \{x \in P^- \mid \langle a^i, x \rangle = b_i \text{ for all } i \in J\}.$$

Clearly, $F(J)$ is a face of P^- and $F(\emptyset) = P^-$.

Let v be any point in the relative interior of P^-. The point v will be the starting point of the algorithm to be described below. For $J \in \mathcal{I}$, let $vF(J)$ be the convex hull of the point v and $F(J)$. Now we first describe a simplicial subdivision of the polytope P which underlies the algorithm.

For a nonnegative integer t, a *t-dimensional simplex* or *t-simplex*, denoted by σ, is defined by the convex hull of $t+1$ affinely independent points $x^1, \dots, x^{t+1}$ in $\mathbb{R}^n$. We often write $\sigma = \sigma(x^1, \dots, x^{t+1})$ and call $x^1, \dots, x^{t+1}$ the vertices of σ. A $(t-1)$-simplex being the convex hull of t vertices of σ is said to be a *facet* of σ. The facet $\tau(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{t+1})$ is called the facet of $\sigma(x^1, \dots, x^{t+1})$ opposite to the vertex x^i. For k, $0 \le k \le t$, a k-simplex being the convex hull of $k+1$ vertices of σ is said to be a *k-face* or *face* of σ. A finite collection $\mathcal{T}$ of n-simplices is a *simplicial subdivision* or *triangulation* of the polytope P iff

(i) P is the union of all simplices in $\mathcal{T}$;
(ii) The intersection of any two simplices of $\mathcal{T}$ is either the empty set or a common face of both.

The *diameter* of a simplex $\sigma(x^1,\dots,x^{n+1})$ is the maximum Euclidean distance between any two points in σ and is denoted by $\mathrm{diam}(\sigma)$. The *mesh size* of a triangulation $\mathcal{T}$ is defined as

$$\mathrm{mesh}(\mathcal{T}) := \max_{\sigma\in\mathcal{T}}\bigl\{\mathrm{diam}(\sigma)\bigr\}.$$

Let $\mathcal{T}$ be a triangulation of P such that every subset $vF(J)$ of P^- is subdivided into t-simplices, where $t=n-|J|$ is the dimension of $vF(J)$. For example, we may take the V-triangulation introduced by [37] for triangulating a polytope. Since $\mathcal{T}$ is finite and P is compact, every facet τ of an n-simplex σ on P either lies on the boundary of P and is only a facet of σ or does not lie on the boundary of P and is a facet of exactly one other n-simplex in $\mathcal{T}$. Similarly, a facet of a t-simplex σ on $vF(J)$, where $t=n-|J|$, either lies on the boundary of $vF(J)$ and is only a facet of σ or does not lie on the boundary of $vF(J)$ and is a facet of exactly one other t-simplex on $vF(J)$.

Now we consider the point-to-set mapping $\bar{\phi}: P\mapsto\mathbb{R}^n$ defined by

$$\bar{\phi}(x) := \begin{cases} \{p(x)-x\}, & \text{if } x\in P\setminus Q,\\ \mathrm{co}(\{p(x)-x\}\cup[A(p(x))\phi(p(x))\cap\pi(x-p(x))]), & \text{if } x\in \mathrm{bd}\,Q,\\ A(p(x))\phi(p(x))\cap\pi(x-p(x)), & \text{if } x\in \mathrm{int}Q. \end{cases}$$

Recall that for $x\in Q$ the vector $x-p(x)$ is an element of $N(X(t),p(x))\cap B$, where $t=\langle c,x\rangle$, so that according to condition 2 of Theorem 2.1 $\bar{\phi}(x)\neq\emptyset$ for all $x\in P$. One can easily verify that $\bar{\phi}$ is an upper semicontinuous mapping with convex and compact values. Let x be a vertex of a simplex of $\mathcal{T}$, then we assign to x the vector label $f(x)$, where $f(x)$ is an arbitrarily chosen element in $\bar{\phi}(x)$. Now we extend f piecewise linearly on each simplex of $\mathcal{T}$, i.e., f is affine on each simplex of $\mathcal{T}$. We call f the *piecewise linear approximation* of $\bar{\phi}$ with respect to $\mathcal{T}$.

A row vector is called *lexicopositive* iff it is a nonzero vector and its first nonzero entry is positive and a matrix is *lexicopositive* iff all its rows are lexicopositive. A matrix is said to be *semi-lexicopositive* iff each row except possibly the last row is lexicopositive.

Definition 3.1 Let $\tau(x^1,\dots,x^t)$ be a facet of a t-simplex on $vF(J)$, where $J\in\mathcal{I}$ with $J=\{j_{t+1},\dots,j_n\}$, $t=n-|J|$. The $(n+1)\times(n+1)$ matrix

$$A_{\tau,J} := \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0\\ -f(x^1) & \cdots & -f(x^t) & a^{j_{t+1}} & \cdots & a^{j_n} & c \end{bmatrix}$$

is the *label matrix* of τ with respect to J. The simplex τ is *J-complete* if $A_{\tau,J}^{-1}$ exists and is semi-lexicopositive.

Definition 3.2 Let $\tau(x^1,\dots,x^n)$ be a facet of an n-simplex on P. The $(n+1)\times(n+1)$ matrix

$$A_{\tau} := \begin{bmatrix} 1 & 1 & \cdots & 1 & 0\\ -f(x^1) & -f(x^2) & \cdots & -f(x^n) & c \end{bmatrix}$$

is the *label matrix* of τ. The simplex τ is *complete* if A_τ^{-1} exists and is semi-lexicopositive.

Notice that if for a J-complete (complete) simplex τ we change the ordering of the first n columns of the matrix $A_{\tau,J}$ (A_τ), the inverse of the resulting matrix still exists and is semi-lexicopositive. Clearly, if, for some $J \in \mathcal{I}$, a $(t-1)$-simplex $\tau(x^1, \dots, x^t)$ is a J-complete facet of a simplex $\sigma(x^1, \dots, x^{t+1})$ on $vF(J)$, then the system of $n+1$ linear equations with $n+2$ variables

$$\sum_{i=1}^{t+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} + \sum_{j \in J} \mu_j \begin{pmatrix} 0 \\ a^j \end{pmatrix} + \beta \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \tag{1}$$

has a solution $(\lambda, \mu, \beta) = (\lambda_1, \dots, \lambda_{t+1}, (\mu_j)_{j \in J}, \beta)$ satisfying $\lambda_i \geq 0$ for all $i \in I_{t+1}$ and $\mu_j \geq 0$ for all $j \in J$, with $\lambda_{t+1} = 0$. Let x be defined by $x = \sum_{i=1}^{t+1} \lambda_i x^i$ at a solution (λ, μ, β) of (1). Then x lies in σ and $f(x) = \sum_{j \in J} \mu_j a^j + \beta c$. Similarly, if an $(n-1)$-simplex $\tau(x^1, \dots, x^n)$ is a complete facet of a simplex $\sigma(x^1, \dots, x^{n+1})$ on P, then the system of $n+1$ linear equations with $n+2$ variables

$$\sum_{i=1}^{n+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} + \beta \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \tag{2}$$

has a solution $(\lambda, \beta) = (\lambda_1, \dots, \lambda_{n+1}, \beta)$ satisfying $\lambda_i \geq 0$ for all $i \in I_{n+1}$, with $\lambda_{n+1} = 0$. Let x be defined by $x = \sum_{i=1}^{n+1} \lambda_i x^i$ at a solution (λ, β) of (2). Then x lies in σ and $f(x) = \beta c$ lies in $C(0^n)$.

Here, we recall the following result from [38] which will be used below.

Lemma 3.3 *Suppose that* $V = \{x \in \mathbb{R}^n \mid \langle c^i, x\rangle \leq d_i, i \in I_m$ *and* $\langle c^0, x\rangle = d_0\}$ *is an* $(n-1)$*-dimensional simple polytope with no redundant constraints for some* $m \geq n$. *Then, for any* $g \in \mathbb{R}^n$, *there exists a unique subset* $\{j_1, \dots, j_{n-1}\}$ *of* I_m *such that the inverse of the matrix*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ g & c^{j_1} & c^{j_2} & \cdots & c^{j_{n-1}} & c^0 \end{bmatrix}$$

exists and is semi-lexicopositive.

We now show that $\{v\}$ is a J-complete 0-simplex for a unique index set $J \in \mathcal{I}$ containing $n-1$ indices.

Lemma 3.4 *There exists a unique subset* $J \in \mathcal{I}$ *with* $|J| = n-1$ *such that* $\{v\}$ *is a* J*-complete* 0*-simplex.*

Proof Since the set

$$P^- = \left\{x \in \mathbb{R}^n \mid \left\langle a^i, x\right\rangle \leq b_i \text{ for all } i \in I \text{ and } \langle c, x\rangle = t^-\right\}$$

is an $(n-1)$-dimensional simple polytope with no redundant constraints, it follows from Lemma 3.3 that there exists a unique subset $J = \{j_1, \dots, j_{n-1}\}$ of I such that the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -f(v) & a^{j_1} & a^{j_2} & \cdots & a^{j_{n-1}} & c \end{bmatrix}$$

exists and is semi-lexicopositive. Clearly, $J \in \mathcal{I}$. This means that the 0-simplex $\{v\}$ is J-complete. Note that by definition it holds that $\{v\}$ is on $vF(J)$. □

The following lemma is well known in linear programming theory and can easily be proved. Let R be a matrix. We denote its ith row by $R_{i.}$ and its j-th column by $R_{.j}$.

Lemma 3.5 *Let $R = (R_{.1}, \ldots, R_{.n+1})$ be a nonsingular $(n+1) \times (n+1)$ matrix and let x be a vector in $\mathbb{R}^{n+1}$. Let $k \in I_{n+1}$ and $\bar{R} = (R_{.1}, \ldots, R_{.k-1}, x, R_{.k+1}, \ldots, R_{.n+1})$. Then, either $(R^{-1}x)_k = 0$ and $\bar{R}$ is singular, or $(R^{-1}x)_k \neq 0$, $\bar{R}$ is nonsingular and $\bar{R}^{-1}$ is given by*

$$\bar{R}^{-1} = \begin{bmatrix} (R^{-1})_{1.} - \frac{(R^{-1}x)_1}{(R^{-1}x)_k}(R^{-1})_{k.} \\ \vdots \\ (R^{-1})_{k-1.} - \frac{(R^{-1}x)_{k-1}}{(R^{-1}x)_k}(R^{-1})_{k.} \\ \frac{1}{(R^{-1}x)_k}(R^{-1})_{k.} \\ (R^{-1})_{k+1.} - \frac{(R^{-1}x)_{k+1}}{(R^{-1}x)_k}(R^{-1})_{k.} \\ \vdots \\ (R^{-1})_{n+1.} - \frac{(R^{-1}x)_{n+1}}{(R^{-1}x)_k}(R^{-1})_{k.} \end{bmatrix}.$$

Using this lemma, the following lemmas will be proved.

Lemma 3.6 *Let σ be a t-simplex on $vF(J)$, where $J \in \mathcal{I}$ and $t = n - |J|$. If σ has a J-complete facet τ, then, exactly one of the following three cases occurs:*

(1) *The simplex σ is a complete $(n-1)$-simplex on P^-;*
(2) *The simplex σ is a $\bar{J}$-complete simplex on $vF(\bar{J})$, where $\bar{J} = J \setminus \{j\} \in \mathcal{I}$ for precisely one index $j \in J$;*
(3) *The simplex σ has exactly one other J-complete facet.*

Proof Let x^{t+1} be the vertex of $\sigma = \sigma(x^1, \ldots, x^{t+1})$ opposite to τ, and let

$$y = A_{\tau,J}^{-1}\big(1, -f\big(x^{t+1}\big)^\top\big)^\top.$$

Notice that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. We first prove that $|K| > 0$. Since $A_{\tau,J}y = (1, -f(x^{t+1})^\top)^\top$, we have $\sum_{i=1}^{t} y_i = 1$. This implies that there exists at least one index $i \in I_t$ such that $y_i > 0$. Hence, K is nonempty.

Consider the ratio vectors $(1/y_j)(A_{\tau,J}^{-1})_{j.}$ for all $j \in K$. Choose $k \in K$ such that the kth ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau,J}^{-1}$ is regular, k is uniquely determined. Now we consider the following two cases.

(i) Let $J = \{j_{t+1}, \ldots, j_n\}$. If $k \in I_n \setminus I_t$, then let $l = j_k$ and $\bar{J} = J \setminus \{l\}$. If $\bar{J} = \emptyset$, then $t = n - 1$ and therefore σ is a complete $(n-1)$-simplex on P^-. Otherwise, $\bar{J} \in \mathcal{I}$ and σ is on $vF(\bar{J})$. Let R be the matrix obtained from $A_{\tau,J}$ by replacing its kth column by $(1, -f(x^{t+1})^\top)^\top$. It follows from Lemma 3.5 that R^{-1} exists and is semi-lexicopositive. By reordering the columns of R we get $A_{\sigma,\bar{J}}$ whose inverse exists and is semi-lexicopositive. So, σ is $\bar{J}$-complete.

(ii) If $k \in I_t$, then let $\bar{\tau}$ be the facet of σ opposite to the vertex x^k. Using Lemma 3.5, it follows from the choice of k that $A_{\bar{\tau},J}^{-1}$ exists and is semi-lexicopositive. Hence, $\bar{\tau}$ is a J-complete $(t-1)$-simplex on $vF(J)$.

It follows immediately from Lemma 3.5 that if any column other than the kth column is replaced, then the inverse of the resulting matrix is not semi-lexicopositive. □

Lemma 3.7 *Let σ be a J-complete $(t-1)$-simplex on $vF(J)$, where $J \in \mathcal{I}$ and $t = n - |J|$. If σ is on $vF(\bar{J})$, where $\bar{J} = J \cup \{l\} \in \mathcal{I}$ for some $l \in I \setminus J$, then exactly one of the following two cases occurs*:

(1) *The simplex σ is a J'-complete simplex on $vF(J')$ where $J' = \bar{J} \setminus \{h\} \in \mathcal{I}$ for precisely one index $h \in \bar{J}$, $h \neq l$;*
(2) *The simplex σ has exactly one other $\bar{J}$-complete facet.*

Proof Let $x = (0, a^{l\top})^\top$ and $y = A_{\sigma,J}^{-1} x$. Note that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. Note that $A_{\sigma,J} y = (0, a^{l\top})^\top$. We first show that K is nonempty. Suppose that $y_i = 0$ for all $i \in I_t$. Then there must exist some parameters y_i for $i = t+1, t+2, \ldots, n$, such that $a^l = \sum_{i=t+1}^n y_i a^{j_{i-t}} + y_{n+1} c$, and y_i must be nonzero for some i. This implies that the vectors c, a^l, a^j for all $j \in J$ are linearly dependent, which is not the case. Hence, there exists at least one index $i \in I_t$ such that $y_i \neq 0$. If there exists an index $j \in I_t$ such that $y_j < 0$, then there must exist an index $i \in I_t$ such that $y_i > 0$ since $\sum_{k=1}^t y_i = 0$. Hence, K is nonempty.

Consider the ratio vectors $(1/y_j)(A_{\sigma,J}^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the kth ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau,J}^{-1}$ is regular, k is uniquely determined. Now we consider the following two cases.

(1) Let $J = \{j_{t+1}, \ldots, j_n\}$. If $k \in I_n \setminus I_t$, then let $h = j_k$ and $J' = J \cup \{l\} \setminus \{h\}$. Clearly, $h \neq l$, $J' \in \mathcal{I}$ and σ is on $vF(J')$. Let R be the matrix obtained from $A_{\sigma,J}$ by replacing its kth column by x. It follows from Lemma 3.5 that R^{-1} exists and is semi-lexicopositive. It is clear that $A_{\sigma,J'} = R$. So, σ is a J'-complete $(t-1)$-simplex on $vF(J')$.
(2) If $k \in I_t$, then let τ be the facet of σ opposite to the vertex x^k. Clearly, τ is a $(t-2)$-simplex on $vF(\bar{J})$. Let R be the matrix obtained from $A_{\sigma,J}$ by replacing its kth column by x. It follows from Lemma 3.5 that R^{-1} exists and is semi-lexicopositive. By reordering the columns of R we get $A_{\tau,\bar{J}}$ whose inverse also exists and is semi-lexicopositive. So, τ is a $\bar{J}$-complete $(t-2)$-simplex on $vF(\bar{J})$.

Again it follows from Lemma 3.5 that if any other column is replaced, then the inverse of the resulting matrix is not semi-lexicopositive. □

Lemma 3.8 *Let τ be a complete $(n-1)$-simplex on $vF(\{i\})$ for some $i \in I$. Then there exists exactly one n-simplex σ on P which has τ as one of its facets and σ has exactly one other complete facet $\bar{\tau}$. Furthermore, τ has exactly one facet τ^* which is $\{i\}$-complete.*

Proof Since τ is an $(n-1)$-simplex on P^-, there is precisely one n-simplex $\sigma(x^1, \dots, x^{n+1})$ on P having τ as one of its facets. Let x^{n+1} be the vertex of σ opposite to τ, and let $y = A_\tau^{-1}(1, -f(x^{n+1})^\top)^\top$. Notice that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. We first prove $|K| > 0$. Since $A_\tau y = (1, -f(x^{t+1})^\top)^\top$, we have $\sum_{i=1}^t y_i = 1$. This implies that there exists at least one index $i \in I_t$ such that $y_i > 0$. Hence, K is nonempty.

Consider the ratio vectors $(1/y_j)(A_\tau^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the kth ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since A_τ^{-1} is regular, k is uniquely determined. Let $\bar{\tau}$ be the facet of σ opposite to the vertex x^k. Using Lemma 3.5, it follows from the choice of k that $A_{\bar{\tau}}^{-1}$ exists and is semi-lexicopositive. Hence, $\bar{\tau}$ is a complete facet of σ.

It follows immediately from Lemma 3.5 that if any column other than the kth column is replaced, then the inverse of the resulting matrix is not semi-lexicopositive.

To prove the second part of the lemma, let $x = (0, a^{i\top})^\top$ and $y = A_\tau^{-1}x$. Note that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. Note that $A_\tau y = (0, a^{i\top})^\top$. We will show that K is nonempty. Suppose that $y_i = 0$ for all $i \in I_n$. Then there must exist some parameter β such that $\beta c = a^i$. This implies that the vectors c and a^i are linearly dependent, which is not the case. Hence, there exists at least one index $i \in I_n$ such that $y_i \neq 0$. If there exists an index $j \in I_n$ such that $y_j < 0$, then there must exist an index $i \in I_n$ such that $y_i > 0$ since $\sum_{i=1}^n y_i = 0$. Hence, K is nonempty.

Consider the ratio vectors $(1/y_j)(A_\tau^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the kth ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since A_τ^{-1} is regular, k is uniquely determined. Now let τ^* be the facet of τ opposite to the vertex x^k. Let R be the matrix obtained from A_τ by replacing its kth column by x. It follows from Lemma 3.5 that R^{-1} exists and is semi-lexicopositive. By reordering the columns of R we get $A_{\tau^*,\{i\}}$ whose inverse also exists and is semi-lexicopositive. So, τ^* is an $\{i\}$-complete $(n-2)$-simplex on $vF(\{i\})$.

Again it follows from Lemma 3.5 that if any other column is replaced, then the inverse of the resulting matrix is not semi-lexicopositive. □

By repeating the first part of the proof of Lemma 3.8, one can show the following lemma.

Lemma 3.9 *If an n-simplex on P has a complete facet, then it has exactly one other facet, which is also complete.*

In the following, we will show that starting at v there exists a finite sequence of adjacent J-complete or complete simplices for varying J, $J \in \mathcal{I}$, which terminates with a complete $(n-1)$-simplex σ^+ on P^+. First we show that, for any $J \in \mathcal{I}$, a J-complete facet can not lie on the boundary of P^-.

Lemma 3.10 *If τ is a J-complete $(t-1)$-simplex on $\upsilon F(J)$, where $t = n - |J|$, then, τ does not lie on the boundary of P^-.*

Proof Suppose to the contrary that $\tau(x^1, \dots, x^t)$ is on the boundary of P^-. Then τ must be a subset of $F(J)$, so $f(x^i) = p(x^i) - x^i$ for all $i = 1, \dots, t$, and $\langle a^j \cdot x^i \rangle = b_j$ for all $j \in J$ and $i = 1, \dots, t$. Since $\langle a^j, p(x^i) \rangle < b_j$ we obtain that $\langle a^j, f(x^i) \rangle < 0$ for all $j \in J$ and all $i = 1, \dots, t$. Because τ is J-complete, we also have

$$\sum_{i=1}^{t} \lambda_i f\left(x^i\right) = \sum_{j \in J} \mu_j a^j + \beta c \tag{3}$$

for some $\lambda_i \geq 0$, $i = 1, \dots, t$, $\mu_j \geq 0$ for all $j \in J$, and $\beta \in \mathbb{R}$, with $\sum_{i=1}^{t} \lambda_i = 1$. By premultiplying Eq. (3) with any vector a^i, $i \in J$, we obtain

$$\begin{aligned} 0 &> \sum_{h=1}^{t} \lambda_h \left\langle a^i, f(x^h) \right\rangle \\ &= \sum_{j \in J} \mu_j \left\langle a^i, a^j \right\rangle + \beta \left\langle a^i, c \right\rangle \\ &= \mu_i \\ &\geq 0, \end{aligned}$$

yielding a contradiction. The assumptions imposed on the a^is and c imply the above equalities. □

Now we prove the next lemma stating that every complete simplex not on P^- or P^+ cannot lie on the boundary of P.

Lemma 3.11 *If τ is a complete $(n-1)$-simplex on* bd P, *either τ lies on P^- or τ lies on P^+.*

Proof Suppose to the contrary that $\tau(x^1, \dots, x^n)$ is a boundary facet on bd $P \setminus (\text{int}\, P^- \cup \text{int}\, P^+)$. Hence, there exists some $i \in I$ such that $\langle a^i, x^h \rangle = b_i$ for all $h = 1, \dots, n$. Then we have $f(x^h) = p(x^h) - x^h$ for all $h = 1, \dots, n$, and so $\langle a^i, f(x^h) \rangle < 0$ for all $h = 1, \dots, n$. Because τ is complete, we also have

$$\sum_{h=1}^{n} \lambda_h f\left(x^h\right) = \beta c \tag{4}$$

for some $\lambda_h \geq 0$, $h = 1, \dots, n$, and some β, with $\sum_{h=1}^{n} \lambda_h = 1$. Premultiplying Eq. (4) by the vector a^i yields

$$\begin{aligned} 0 &> \sum_{h=1}^{n} \lambda_h \left\langle a^i, f(x^h) \right\rangle \\ &= \beta \left\langle a^i, c \right\rangle \\ &= 0, \end{aligned}$$

which is impossible. We are done. □

We construct a graph $G = (N, E)$ where N denotes a set of nodes and E denotes a set of edges. A simplex σ is called *a node* iff it is an J-complete $(n - |J| - 1)$-simplex for some $J \in \mathcal{I}$ or it is a complete $(n - 1)$-simplex. Two nodes σ_1 and σ_2 are said to be *adjacent* iff both σ_1 and σ_2 are facets of an n-simplex, or iff σ_1 and σ_2 are J-complete and are facets of an $(n - |J|)$-simplex on $vF(J)$, or iff σ_1 is J-complete and σ_2 is J'-complete and σ_1 is a facet of σ_2 and σ_2 is an $(n - |J|)$-simplex on $vF(J)$, or if σ_1 is $\{j\}$-complete and σ_2 is a complete $(n - 1)$-simplex on $vF(\{j\})$ and σ_1 is a facet of σ_2. Then $\{\sigma_1, \sigma_2\}$ is *an edge* iff σ_1 and σ_2 are adjacent nodes. The degree of a node σ in G is defined to be the number of adjacent nodes and is denoted by $\deg(\sigma)$. A finite sequence of adjacent simplices in G from σ_0 to σ_l is defined as $(\sigma_0, \sigma_1, \ldots, \sigma_l)$, where $\sigma_0, \sigma_1, \ldots, \sigma_l$ are nodes in G and $e_i = \{\sigma_{i-1}, \sigma_i\}$ are edges in G for all $i \in I_l$.

Theorem 3.1 *Let $\mathcal{T}$ be the triangulation of P given above. Then there exists a finite sequence of adjacent complete and J-complete, for varying $J \in \mathcal{I}$, simplices in $\mathcal{T}$ connecting $\{v\}$ on P^- and a complete $(n - 1)$-simplex σ^+ on P^+.*

Proof From Lemma 3.4, it follows that $\{v\}$ is a J-complete 0-simplex on $vF(J)$ for some unique set $J \in \mathcal{I}$ with $|J| = n - 1$. Since $\{v\}$ is a facet on the boundary of $vF(J)$, there exists a unique 1-simplex σ on $vF(J)$ having $\{v\}$ as its facet. By Lemma 3.6, either σ is a $\bar{J}$-complete simplex on $vF(\bar{J})$ where $\bar{J} = J \setminus \{j\}$ for some unique $j \in J$, or σ has exactly one other J-complete facet τ. Hence, there exists a unique node adjacent to $\{v\}$. Thus, $\deg(\{v\}) = 1$.

Let τ be any node on P^+. Then τ is a complete $(n - 1)$-simplex on the boundary of P. This implies that there is a unique n-simplex σ in $\mathcal{T}$ having τ as its facet. By Lemma 3.9, σ has exactly one other complete facet $\bar{\tau}$, which is a node by definition. Hence, we have $\deg(\tau) = 1$.

In all other cases, we will show that $\deg(\tau) = 2$ if τ is a node. We need to address several cases. (1) If τ is a complete $(n-1)$-simplex and does not lie on P^- or on P^+, then according to Lemma 3.11 τ does not lie on the boundary of P. Hence, there exist exactly two n-simplices σ_1 and σ_2 on P sharing τ as their common facet. It follows from Lemma 3.9 that there are two nodes adjacent to τ. Thus, $\deg(\tau) = 2$. (2) If τ is a complete $(n - 1)$-simplex and lies on P^-, then it follows from Lemma 3.8 that $\deg(\tau) = 2$. (3) If τ is a J-complete $(n - |J| - 1)$-simplex on $vF(J)$ for some $J \in \mathcal{I}$, either τ does not lie on the boundary of $vF(J)$ or τ lies on the boundary of $vF(J)$. If τ does not lie on the boundary of $vF(J)$, then τ is a facet of precisely two $(n - |J|)$-simplices on $vF(J)$. It follows from Lemma 3.6 that τ is adjacent to exactly two nodes. If τ lies on the boundary of $vF(J)$, then there exists exactly one $(n - |J|)$-simplex σ on $vF(J)$ having τ as its facet. By Lemma 3.6 either σ is a $\bar{J}$-complete $(n - |\bar{J}| - 1)$-simplex on $F(\bar{J})$ for some unique $\bar{J} \in \mathcal{I}$ with $|\bar{J}| = |J| - 1$ and has no other J-complete facets, or σ has exactly one other J-complete facet. This yields one adjacent node to τ. On the other hand, since τ lies on the boundary of $vF(J)$, it follows from Lemma 3.10 that τ does not lie on the boundary of P^-. Hence, τ lies on $vF(\tilde{J})$ for some unique set $\tilde{J} \in \mathcal{I}$ with $|\tilde{J}| = |J| + 1$. By Lemma 3.7 either τ is J'-complete for some unique set $J' \in \mathcal{I}$ with $|J'| = |J|$ and $J' \neq J$, or τ has exactly one $\tilde{J}$-complete facet. It follows again that in both these cases there exists exactly one node adjacent to τ. This concludes that τ has exactly two adjacent nodes. That is, $\deg(\tau) = 2$.

As shown above, the degree of each node in the graph $G = (N, E)$ is at most two. Since the number of simplices on P is finite, the number of nodes in G is finite. Since $\deg(\{v\}) = 1$, there exists a finite sequence of adjacent nodes starting from $\{v\}$. The end node of this sequence must be a node of degree 1 and different from $\{v\}$. The only possibility is that this node is a complete simplex on P^+. □

The algorithm is such that it generates the sequence of adjacent simplices described in the theorem. From the theorem, it follows that starting at the point v, the algorithm generates a finite sequence of adjacent J-complete or complete simplices for varying $J \in \mathcal{I}$, which leads to a complete $(n-1)$-simplex σ^+ on P^+. After leaving the set P^-, the algorithm may return to P^- to generate again J-complete simplices on P^- for varying $J \in \mathcal{I}$. In this way, the algorithm may generate an (odd) number of complete $(n-1)$-simplices on P^- before it leaves P^- forever and terminates with a complete $(n-1)$-simplex σ^+ on P^+. Let σ^- be the last complete $(n-1)$-simplex generated by the algorithm on P^-. Then it is clear that from σ^- on, the algorithm generates a finite sequence of adjacent complete $(n-1)$-simplices on P from σ^- on P^- to σ^+ on P^+. We summarize this in the following corollary.

Corollary 3.1 *There exists a finite sequence of adjacent complete $(n-1)$-simplices on P connecting a complete $(n-1)$-simplex on P^- and a complete $(n-1)$-simplex on P^+.*

Given a function $g : P \mapsto \mathbb{R}^n$, a point $x \in P$ is called a *stationary point with respect to c* if $g(x)$ is an element of $N(P(t), x)$, where $t = \langle c, x\rangle$. Such a solution is also called a *parameterized stationary point* of g. From Corollary 3.1 and the system of Eqs. (2), we see that every simplex generated by the algorithm from σ^- to σ^+ contains a stationary point of the piecewise linear approximation function f with respect to the vector c. By taking the straight line segments between the parameterized stationary points of any two adjacent complete simplices, we obtain in P a piecewise linear path of parameterized stationary points of f connecting the simplices σ^- and σ^+.

Corollary 3.2 *With respect to the vector c there exists a piecewise linear path $\rho([0,1])$ in P of parameterized stationary points of the piecewise linear approximation f of $\bar{\phi}$ with respect to $\mathcal{T}$ and this path connects a point $\rho(0)$ in P^- and a point $\rho(1)$ in P^+. No point of this path is on the boundary of $P(t)$ for some t, $t^- \le t \le t^+$.*

Proof The first part is obvious. The second part follows from the proof of Lemma 3.11. □

From this corollary, it follows that for every $q \in \rho([0,1])$ it holds that $f(q) = \beta c$ for some $\beta \in \mathbb{R}$, i.e., $f(q) \in C(0^n)$. In the next section, we show by taking a sequence of triangulations of P with mesh size going to zero that there exists a connected set of zero points of ϕ in X having a nonempty intersection with both X^- and X^+.

4 A Constructive Existence Proof

By making use of the results obtained in Sect. 3, we will give a constructive proof for Theorem 2.1. To achieve this, a sequence of triangulations $\mathcal{T}^r$, $r \in \mathbb{N}$, with mesh size converging to zero is taken. According to Corollary 3.2, for every $r \in \mathbb{N}$, there exists a piecewise linear function $\rho^r : [0,1] \mapsto P$ with image set $\rho^r([0,1])$ connecting P^- and P^+ and satisfying that any $q^r \in \rho^r([0,1])$ is a parameterized stationary point of the piecewise linear approximation f^r of $\bar{\phi}$ with respect to $\mathcal{T}^r$. In the next lemma, we show by a limit argument that if the sequence $(q^r)_{r\in\mathbb{N}}$ converges to some q^*, then $p(q^*)$ is a zero point of ϕ. Recall that $p(\cdot)$ is a continuous function.

Lemma 4.1 *Let $\phi : X \mapsto \mathbb{R}^n$ be a point-to-set mapping satisfying the conditions in Theorem 2.1. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and let $\rho^r([0,1])$ be the piecewise linear path as constructed in Corollary 3.2 with respect to $\mathcal{T}^r$. Then, for every convergent sequence $(q^r)_{r\in\mathbb{N}}$ with limit q^* satisfying $q^r \in \rho^r([0,1])$ for all $r \in \mathbb{N}$, it holds that $x^* = p(q^*)$ is a zero point of ϕ in X.*

Proof Since $q^r \in P$ for all $r \in \mathbb{N}$ and P is a closed set, we have $q^* \in P$. Moreover, since the mesh size of the sequence of triangulations of P converges to zero and $\bar{\phi}$ is upper semicontinuous and both compact-valued and convex-valued, the system of Eqs. (2) at q^r will reduce in the limit for r going to infinity, after taking subsequences if necessary, to

$$f^* = \beta^* c$$

for some $\beta^* \in \mathbb{R}$ and some $f^* \in \bar{\phi}(q^*)$. Let $t^* = \langle c, q^* \rangle$ and $v^* = q^* - p(q^*)$. Clearly, $v^* \in N(X(t^*), p(q^*)) \cap B$ whenever $q^* \in P$. We have to consider the following cases.

(i) In case $q^* \in P \setminus Q$, we have $f^* = p(q^*) - q^* = \beta^* c$. Since $\langle c, p(q^*)\rangle = \langle c, q^* \rangle = t^*$ and so $\langle c, f^* \rangle = 0$, we obtain $\beta^* = 0$ and therefore $p(q^*) = q^*$. Since q^* is not in X, we obtain a contradiction.

(ii) In case $q^* \in \operatorname{bd} Q$, we have $f^* = \mu^*(p(q^*) - q^*) + (1 - \mu^*) f = \beta^* c$ for some $0 \leq \mu^* \leq 1$ and some $f \in A(p(q^*))\phi(p(q^*)) \cap \pi(v^*)$. For $\mu^* = 1$ this case reduces to case i). For $\mu^* < 1$, we obtain $f \in C(v^*)$. According to condition 1 of Theorem 2.1 $x^* = p(q^*)$ is a zero point of ϕ.

(iii) In case $q^* \in Q \setminus X$ and $q^* \notin \operatorname{bd} Q$, we have $f^* \in A(p(q^*))\phi(p(q^*)) \cap \pi(v^*)$ and $f^* = \beta^* c \in C(v^*)$. According to condition 1 of Theorem 2.1 $x^* = p(q^*)$ is a zero point of ϕ.

(iv) In case $q^* \in X$, we have $q^* = p(q^*)$. This implies that $f^* \in A(q^*)\phi(q^*) \cap \pi(0^n)$. Since $f^* = \beta^* c \in C(0^n)$, we obtain $A(q^*)\phi(q^*) \cap \pi(0^n) \cap C(0^n) \neq \emptyset$. According to condition 1 of Theorem 2.1, $x^* = q^*$ is a zero point of ϕ. □

Let $Z := \{x \in X \mid 0^n \in \phi(x)\}$ be the set of zero points of ϕ in X. For a nonempty, compact set $S \subset \mathbb{R}^n$, define the distance function $d_S \mathbb{R}^n \to \mathbb{R}$ by $d_S(x) := \min\{\|x - y\|_2 \mid y \in S\}$. It is well known that d_S is continuous.

Proof of Theorem 2.1 From Lemma 4.1 it immediately follows that $Z \cap X^- \neq \emptyset$ and $Z \cap X^+ \neq \emptyset$, also Z is compact. For $x \in Z$, let Z_x be the component of x in Z. We know that Z_x is connected and compact. The collection of all distinct components in Z forms a partition of Z. Define $Z^- := \bigcup_{x \in Z \cap X^-} Z_x$ and $Z^+ := Z \setminus Z^-$, and let $V^- := X^- \cup Z^-$, $V^+ := X^+ \cup Z^+$, and $V := Z \cup X^- \cup X^+$. Suppose the theorem is false. Then V^- and V^+ are nonempty disjoint compact sets. Hence, there exists $\varepsilon > 0$ such that $\min\{\|q^0 - q^1\|_2 \mid q^0 \in V^-,\ q^1 \in V^+\} \geq \varepsilon$. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and let $\rho^r : [0,1] \mapsto P$ be the corresponding continuous function with image set connecting P^- and P^+, as constructed in Corollary 3.2. Define $g^r : [0,1] \mapsto \mathbb{R}$ by

$$g^r(t) := d_{V^-}\big(p\big(\rho^r(t)\big)\big) - d_{V^+}\big(p\big(\rho^r(t)\big)\big), \quad \forall t \in [0,1].$$

Since g^r is continuous, $g^r(0) \leq -\varepsilon$, and $g^r(1) \geq \varepsilon$, there exists a point $t^r \in (0,1)$ such that $g^r(t^r) = 0$. Hence, $d_{V^-}(p(\rho^r(t^r))) = d_{V^+}(p(\rho^r(t^r))) = d_V(p(\rho^r(t^r))) \geq \frac{1}{2}\varepsilon$. Without loss of generality, we assume that $(\rho^r(t^r))_{r \in \mathbb{N}}$ converges to a point $q^* \in P$. Hence,

$$d_V\big(p\big(q^*\big)\big) = d_V\Big(\lim_{r \to \infty} p\big(\rho^r\big(t^r\big)\big)\Big) = \lim_{r \to \infty} d_V\big(p\big(\rho^r\big(t^r\big)\big)\big) \geq \frac{1}{2}\varepsilon > 0.$$

By Lemma 4.1, we have $d_Z(p(q^*)) = 0$. Because $p(q^*) \in Z \subseteq V$, it follows that $d_V(p(q^*)) = 0$, yielding a contradiction. □

From Lemma 4.1 and the proof of Theorem 2.1 above, it follows that the projection on X of any point of the piecewise linear path of parameterized stationary points of f^r being generated by the algorithm can be seen as an approximate zero point of ϕ in the sense that every convergent sequence of parameterized stationary points of f^r, for a sequence of triangulations $(\mathcal{T}^r)_{r \in \mathbb{N}}$ with mesh size converging to zero, converges to a zero point of ϕ when r goes to infinity. The approximation typically improves when the mesh size of the triangulation becomes smaller. The fact that for every triangulation in the sequence the algorithm generates a path of such points from P^- to P^+ guarantees that there exists a continuum of zero points of ϕ connecting X^- and X^+.

5 Continuum of Coincidences, Fixed Points, and Optima

In this section, we derive several results from Theorem 2.1 about the existence of a continuum of zero points, coincidences, fixed points, optima, and stationary points. Unless otherwise stated, we maintain the notations and assumptions used in Sect. 2. In particular, c is a nonzero vector in $\mathbb{R}^n$ with $\langle c, c\rangle = 1$, $B = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1, \langle c, x\rangle = 0\}$, and ϕ is an upper semicontinuous point-to-set mapping from the nonempty convex and compact set X in $\mathbb{R}^n$ to the collection of nonempty, convex, and compact subsets of $\mathbb{R}^n$.

Theorem 5.1 *Suppose that for every $x \in X$ and every $v \in N(X(t), x)$ with $\langle c, v\rangle = 0$, there is a $y \in \phi(x)$ satisfying $\langle c, y\rangle = 0$ and $\langle v, y\rangle \leq 0$, where $t = \langle c, x\rangle$. Then there exists a connected set C of zero points of ϕ in X such that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$.*

Proof Take $A(x) = E(n)$ for every $x \in X$ and $\pi(v) = \{y \in \mathbb{R}^n \mid \langle c, y\rangle = 0, \langle v, y\rangle \leq 0\}$ for $v \in B$. Clearly, π is an upper semicontinuous mapping from B to the collection of nonempty convex and closed subsets of $\mathbb{R}^n$. Since for every $v \in B$ it holds that $\pi(v) \cap C(v) = \{0^n\}$, condition 1 of Theorem 2.1 is satisfied. The condition in the theorem also implies that condition 2 of Theorem 2.1 is satisfied. Hence, according to Theorem 2.1, there exists a connected set of zero points of ϕ in X intersecting with both X^- and X^+. □

By taking $A(x) = -E(n)$ for every $x \in X$ the same result holds when for every $x \in X$ and every $v \in N(X(t), x)$ with $\langle c, v\rangle = 0$ there is a $y \in \phi(x)$ satisfying $\langle c, y\rangle = 0$ and $\langle v, y\rangle \geq 0$.

From Theorem 5.1, we immediately obtain the following results. The first result gives a sufficient condition for the existence of a continuum of coincidences and therefore generalizes Fan's coincidence theorem, see [5], in which the existence of a single coincidence is proved.

Theorem 5.2 *Let ϕ and ψ be two upper semicontinuous mappings from X to the collection of nonempty compact and convex subsets of $\mathbb{R}^n$. Suppose that for every $x \in X$ and every $d \in \mathbb{R}^n$ satisfying both $\langle c, d\rangle = 0$ and $\langle d, x\rangle = \max\{\langle d, y\rangle \mid y \in X,\ \langle c, y\rangle = \langle c, x\rangle\}$, there exist $u \in \phi(x)$ and $w \in \psi(x)$ such that $\langle c, u\rangle = \langle c, w\rangle$ and $\langle d, u\rangle \geq \langle d, w\rangle$. Then there exists a connected set C of points in X such that $\phi(x) \cap \psi(x) \neq \emptyset$ for every $x \in C$, $C \cap X^- \neq \emptyset$, and $C \cap X^+ \neq \emptyset$.*

Proof Define the mapping γ on X by $\gamma(x) = \psi(x) - \phi(x)$ for all $x \in X$. Since $\langle c, d\rangle = 0$ and $\langle d, x\rangle = \max\{\langle d, y\rangle \mid y \in X,\ \langle c, y\rangle = \langle c, x\rangle\}$ implies $d \in N(X(t), x)$, where $t = \langle c, x\rangle$, $\gamma(\cdot)$ satisfies the conditions of Theorem 5.1. Hence, there exists a connected set of zero points of γ in X intersecting with both X^- and X^+. By construction, every zero point of the mapping γ is a coincidence of the mappings ϕ and ψ. □

The next result can be seen as a generalization of Browder's fixed-point theorem for point-to-set mappings.

Theorem 5.3 *Suppose that for every $x \in X$ it holds that $\phi(x) \cap X(t) \neq \emptyset$, where $t = \langle c, x\rangle$. Then there exists a connected set of fixed points of ϕ in X intersecting with both X^- and X^+.*

Proof For any $x \in X$, take some $y \in \phi(x) \cap X(t)$, where $t = \langle c, x\rangle$. Since $y \in X(t)$, we have that $\langle c, y\rangle = \langle c, x\rangle = t$ and $\langle v, y\rangle \leq \langle v, x\rangle$ for all $v \in N(X(t), x)$. Hence, the mapping ψ on X defined by $\psi(x) = \phi(x) - \{x\}$ for all $x \in X$ satisfies the conditions of Theorem 5.1. Therefore, there exists a connected set of zero points of ψ in X intersecting with both X^- and X^+. Clearly, a zero point of ψ is a fixed point of ϕ in X. □

Notice that in this theorem we only require that $\phi(x) \cap X(t) \neq \emptyset$ for every $x \in X(t)$. The image $\phi(x)$ may contain elements outside the set $X(t)$. Clearly, when ϕ

is a fixed point mapping in the sense that for every $x \in X$ it holds that $\phi(x) \subseteq X(t)$, where $t = \langle c, x\rangle$, then Theorem 5.3 implies that there exists a continuum of fixed points connecting X^- and X^+.

Corollary 5.1 *Suppose that $\phi(x) \subseteq X(t)$ whenever $x \in X(t)$. Then there exists a connected set of fixed points of ϕ in X having a nonempty intersection with both X^- and X^+.*

Browder's fixed-point theorem for mappings is now an immediate consequence of this corollary. In [1], the continuous function case is proved and in [39] the result is extended to the upper semicontinuous point-to-set mapping case. For a mapping ϕ from $X \times [0, 1]$ to X a point $(x, t) \in X \times [0, 1]$ is called a fixed point of ϕ if $x \in \phi(x, t)$.

Corollary 5.2 *Let ϕ be an upper semicontinuous mapping from $X \times [0, 1]$ to the collection of nonempty convex and compact subsets of X, where X is a nonempty, convex and compact set in $\mathbb{R}^n$. Then there exists a connected set of fixed points of ϕ in $X \times [0, 1]$ intersecting with both $X \times \{0\}$ and $X \times \{1\}$.*

Proof Define $S = X \times [0, 1]$ and the mapping ψ on S by $\psi(s, t) = \phi(s, t) \times \{t\}$ for all $(s, t) \in S$. Take $c = (0^{n\top}, 1)^\top$. Then S and ψ satisfy the conditions of Corollary 5.1 with respect to the nonzero vector c. Hence, there exists a connected set of fixed points of ψ in S intersecting with both $S^- = X \times \{0\}$ and $S^+ = X \times \{1\}$. Clearly, a fixed point of ψ in S is a fixed point of ϕ in $X \times [0, 1]$. □

By generalizing Theorem 3.4 in [3] on polytopes, in [4] the following existence theorem on a continuum of zero points is established by using the concept of tangent cone. We will show that this result also follows from Theorem 5.1. Recall that for $x \in X$ the tangent cone of $X(t)$ at x with $t = \langle c, x\rangle$ equals the set

$$T\big(X(t), x\big) = \big\{z \in \mathbb{R}^n \mid \langle y, z\rangle \le 0 \text{ for all } y \in N\big(X(t), x\big)\big\}.$$

Corollary 5.3 *Suppose that for every $x \in X$ it holds that $\phi(x) \cap T(X(t), x) \neq \emptyset$ with $t = \langle c, x\rangle$. Then there exists a connected set of zero points of ϕ in X intersecting with both X^- and X^+.*

Proof For any $x \in X$ it holds that $T(X(t), x) \subseteq \{y \in \mathbb{R}^n \mid \langle c, y\rangle = 0, \langle v, y\rangle \le 0\}$ for every $v \in N(X(t), x)$. Hence, ϕ satisfies the conditions of Theorem 5.1. □

When in Theorem 2.1 $\pi(v) = \mathbb{R}^n$ for all $v \in B$, we may derive the following result, which is a generalization of both Theorem 3.2 in [3] on polytopes and of Theorem 3.1 in [4] on nonempty compact and convex sets.

Theorem 5.4 *Suppose there exists a continuous and nonsingular matrix map A on X such that for every $x \in X$, $A(x)\phi(x) \cap N(X(t), x)$ is either empty or contains 0^n, where $t = \langle c, x\rangle$. Then there exists a connected set of zero points of ϕ in X intersecting with both X^- and X^+.*

Proof Take $\pi(v) = \mathbb{R}^n$ for all $v \in B$. Clearly, π is an upper semicontinuous mapping from B to the collection of nonempty convex and closed subsets of $\mathbb{R}^n$. Condition 2 of Theorem 2.1 is satisfied because $\phi(x) \neq \emptyset$ for all $x \in X$. Concerning condition 1 of Theorem 2.1 take any $x \in X$ and $v \in N(X(t), x) \cap B$. From Lemma 3.2, we obtain that $C(v) \subseteq N(X(t), x)$. Hence, $A(x)\phi(x) \cap N(X(t), x) = \emptyset$ implies that $A(x)\phi(x) \cap \pi(v) \cap C(v) = \emptyset$. Moreover, since both $N(X(t), x)$ and $C(v)$ contain 0^n, we have that if $A(x)\phi(x) \cap N(X(t), x)$ contains 0^n then also $A(x)\phi(x) \cap \pi(v) \cap C(v)$ contains 0^n. □

The theorem states that if every image of a point can be transformed in a continuously linear way such that every transformed image has an empty intersection with the normal cone at that point, unless it contains the origin, then there exists a connected set of zero points. Special cases are when $A(x) = E(n)$ for all $x \in X$, $A(x) = -E(n)$ for all $x \in X$, or more generally $A(x)$ is a diagonal matrix with nonzero diagonal elements depending in a continuous way on $x \in X$.

When in Theorem 2.1 $\pi(v) = \{y \in \mathbb{R}^n \mid \langle v, y\rangle \leq 0\}$, the following result is obtained.

Theorem 5.5 *Suppose that for every $x \in X$ and every $v \in N(X(t), x) \cap B$, where $t = \langle c, x\rangle$, the following two conditions hold*:

(i) *The set $\phi(x) \cap \{y \mid \langle v, y\rangle \leq 0\} \cap C(v)$ is either empty or contains 0^n*;
(ii) *The set $\phi(x) \cap \{y \mid \langle v, y\rangle \leq 0\} \neq \emptyset$.*

Then, there exists a connected set C of zero points of ϕ in X such that $X^- \cap C \neq \emptyset$ and $X^+ \cap C \neq \emptyset$.

Proof Take $A(x) = E(n)$ for all $x \in X$ and $\pi(v) = \{y \mid \langle v, y\rangle \leq 0\}$ for all $v \in B$. Clearly, π is an upper semi-continuous mapping from B to the collection of nonempty convex and closed subsets of $\mathbb{R}^n$. Moreover, for every $x \in X$ and $v \in N(X(t), x) \cap B$ conditions (i) and (ii) imply conditions 1 and 2 of Theorem 2.1, respectively. □

Theorem 5.5 implies Theorem 4.3 in [2] on the unit cube. Let U^n be the n-dimensional cube given by

$$U^n := \left\{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \forall i \in I_n\right\},$$

where $I_n := \{1, \ldots, n\}$.

Theorem 5.6 *Let ϕ be an upper semicontinuous mapping from U^n to the collection of nonempty convex and compact subsets of $\mathbb{R}^n$. Suppose that*

(1) *For every $x \in U^n$, there exists $f \in \phi(x)$ such that, for every $j \in I_n$, $x_j = 0$ implies $f_j \geq 0$, and $x_j = 1$ implies $f_j \leq 0$;*
(2) *For every $x \in U^n$, for every $f \in \phi(x)$, there exists some $p \in \mathbb{R}^n_{++}$ such that $\langle p, f\rangle = 0$.*

Then there exists a connected set C of zero points of ϕ such that $0^n \in C$ and $1^n \in C$.

Proof First we show that 0^n and 1^n are both zero points of ϕ. Condition (1) implies that there exists $f^0 \in \phi(0^n)$ and $f^1 \in \phi(1^n)$ satisfying $f^0_j \geq 0$ and $f^1_j \leq 0$ for all $j = 1, \dots, n$. Condition (2) then implies that $f^0 = 0^n$ and $f^1 = 0^n$. Now take $c = (1, 0, \dots, 0)^\top$. Take any $x \in U^n$ and $v \in N(U^n(t), x) \cap B$, where $U^n(t) = \{x \in U^n \mid x_1 = t\}$ and $t = \sum_{j=1}^n x_j$. Clearly, $v_1 = 0$ and for $j \geq 2$ we have $v_j \leq 0$ if $x_j = 0$, $v_j \geq 0$ if $x_j = 1$, and $v_j = 0$ otherwise. According to (1), there exists $f \in \phi(x)$ such that $x_j = 0$ implies $f_j \geq 0$ and $x_j = 1$ implies $f_j \leq 0$. Hence, $\langle f, v\rangle \leq 0$ and, therefore, condition (ii) of Theorem 5.5 is satisfied. Now take any $f \in \phi(x) \cap \{y \mid \langle c, y\rangle \leq 0\}$. To show that condition (i) of Theorem 5.5 holds, suppose $f \in C(v)$. Then there exists $\beta \in \mathbb{R}$ and $\mu \geq 0$ such that $f = \beta c + \mu v$. Since $\langle f, v\rangle \leq 0$ and $\langle c, v\rangle = 0$ we obtain $0 \geq \beta\langle c, v\rangle + \mu\langle v, v\rangle = \mu\langle v, v\rangle \geq 0$, from which it follows that $\mu = 0$. Hence, $f = \beta c$, i.e., $f_1 = \beta$ and $f_j = 0$ for $j \geq 2$. According to (2), there exists $p \in \mathbb{R}^n_{++}$ satisfying $\langle f, p\rangle = 0$. Hence, $\beta p_1 = 0$, and so $\beta = 0$. The latter implies that $f = 0^n$, and so condition (i) of Theorem 5.5 is satisfied. Hence, all the conditions of the latter theorem are satisfied and there exists a connected set C of zero points of ϕ having a nonempty intersection with both $U^n(0)$ and $U^n(1)$. By letting the algorithm start with 0^n or 1^n as described in [2], it can be shown that there exists a connected set C of zero points of ϕ such that $0^n \in C$ and $1^n \in C$. □

Next, we give an application in the field of optimization theory. Let $f : X \mapsto \mathbb{R}$ be a function and let c be an arbitrary nonzero vector in $\mathbb{R}^n$. Given t, $t^- \leq t \leq t^+$, an optimal solution of the problem

$$\max f(x) \quad \text{s.t.} \quad x \in X(t)$$

is called an *optimum with respect to* c of f on X. Then we have the following result saying that there exists a continuum of optima with respect to c in case f is concave and smooth. This result seems to be the first of such kind.

Theorem 5.7 *Let $f : X \mapsto \mathbb{R}$ be a concave smooth function and let c be a nonzero vector in $\mathbb{R}^n$. Then there exists a connected set C of optima with respect to c of f on X satisfying that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$.*

Proof For $x \in X$, let $\nabla f(x)$ be the gradient of f at x. Since f is a concave smooth function, $\nabla f(x)$ is a continuous function from X to $\mathbb{R}^n$. Let $g(x)$ be the projection of the point $x + \nabla f(x)$ on $X(t)$, where $t = \langle c, x\rangle$. Then g is a continuous function from X to X satisfying $g(x) \in X(t)$ whenever $x \in X(t)$. Therefore, g satisfies the condition of Corollary 5.1. Hence, there exists a connected set of fixed points of g intersecting with both X^- and X^+. Clearly, a fixed point of g is an optimum of f with respect to c. □

Finally, we establish a general existence theorem on a continuum of solutions to the nonlinear variational inequality problem with respect to some nonzero vector; see also [4]. Given an arbitrary point-to-set mapping ϕ defined on the set X, the *variational inequality problem with respect to* c, where c is some nonzero vector in $\mathbb{R}^n$, is to find a point $x^* \in X$ and $f^* \in \phi(x^*)$ such that

$$\langle x^* - x, f^*\rangle \geq 0, \quad \forall x \in X(t^*),$$

where $t^* = \langle c, x^* \rangle$. Recall that such a solution is called a parameterized stationary point of ϕ. We give a constructive proof.

Theorem 5.8 *Let ϕ be an upper semicontinuous mapping from X to the collection of nonempty convex and compact subsets of $\mathbb{R}^n$ and let c be a nonzero vector in $\mathbb{R}^n$. Then there exists a connected set C in X of solutions to the variational inequality problem for ϕ with respect to c satisfying that $X^- \cap C \neq \emptyset$ and $X^+ \cap C \neq \emptyset$.*

Proof Take $\pi(v) = \mathbb{R}^n$ for all $v \in B$ and $A(x) = E(n)$ for all $x \in X$. Clearly, the mapping π satisfies condition 2 of Theorem 2.1. Corollary 3.2 implies that for any triangulation $\mathcal{T}$ of P there exists a piecewise linear path $\rho([0, 1])$ in P of parameterized stationary points of the piecewise linear approximation f of $\bar{\phi}$ with respect to $\mathcal{T}$ connecting P^- and P^+. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size less than or equal to $\frac{1}{r}$ and let $\rho^r([0, 1])$ be the corresponding piecewise linear path connecting P^- and P^+. Following the proof of Lemma 4.1, we obtain that for every convergent sequence $(q^r)_{r \in \mathbb{N}}$ with limit q^* satisfying $q^r \in \rho^r([0, 1])$ for all $r \in \mathbb{N}$, it holds that $x^* = p(q^*)$ is a parameterized stationary point of ϕ with respect to c. By taking the set Z as the set of parameterized stationary points of ϕ in X, it follows from the proof of Theorem 2.1 that there exists a connected set of parameterized stationary points of ϕ in X having a nonempty intersection with both X^- and X^+. □

Notice that to show that the variational inequality problem with respect to c has a continuum of solutions, no additional assumptions on ϕ are needed. Moreover, the result holds for any nonzero vector c.

6 Continuum of Constrained Equilibria

In this section, we apply Theorem 5.4 to a general exchange economy with price rigidities and show the existence of a continuum of constrained equilibria in the economy. Departing from the idealistic Arrow–Debreu perfect competition model, many economists have studied more realistic economic models where there may be restrictions on the prices of commodities and services. There are many economic or political reasons for price rigidities. For instance, to prevent breakdown of stock markets, often ceilings and floors are imposed upon the price of each stock; price controls are used to reduce inflation or deflation; and minimum wages are employed to protect certain groups of the society. Such models have been intensively investigated in the economics literature as we mentioned in the Introduction.

We first introduce some concepts and notation. For a vector $c \in \mathbb{R}^n_{++}$ with $\langle c, c \rangle = 1$, define

$$[-c, c] := \{x \in \mathbb{R}^n \mid -c_k \leq x_k \leq c_k \text{ for all } k \in I_n\}.$$

An n-dimensional compact set X in $\mathbb{R}^n$ is said to have a *frictionless boundary* if for every boundary point x of X the normal cone $N(X, x)$ of X at x is just a ray, i.e., $N(X, x) := \{y \in \mathbb{R}^n \mid y = \mu a, \mu \geq 0\}$ for some nonzero vector a. Using Theorem 5.4, we prove the following result through which we will show the existence of a continuum of constrained equilibria in an economy with price rigidities.

Theorem 6.1 *Let $X \subset \mathbb{R}^n$ be an n-dimensional compact and convex set with a frictionless boundary, let $z : X \to \mathbb{R}^n$ be a continuous function, and let c be a vector in $\mathbb{R}^n_{++}$ with $\langle c, c\rangle = 1$. Suppose that*

(1) *There exists a continuous function $p : X \to \mathbb{R}^n_{++}$ such that $\langle p(q), z(q)\rangle = 0$ for all $q \in X$;*
(2) *There exists a function $r : X \to [-c, c]$ such that $q \in \operatorname{bd} X$ implies $r(q) \in N(X, q)$ and $|r_k(q)| = c_k$ for some $k \in I_n$;*
(3) *For every $q \in \operatorname{bd} X$, $r_k(q) = -c_k$ implies $z_k(q) \geq 0$ and $r_k(q) = c_k$ implies $z_k(q) \leq 0$.*

Then there exists a connected set of zero points of z in X having a nonempty intersection with both X^- and X^+.

Proof For $q \in X$, let $D(q)$ be the $n \times n$ diagonal matrix with kth diagonal element equal to $d_k(q) = p_k(q)/c_k$ for every $k \in I_n$. We will show that the function z satisfies the conditions of Theorem 5.4 with $A(q) = D(q)$ for all $q \in X$. Since the function p is a continuous function from X to $\mathbb{R}^n_{++}$ and $c_k > 0$ for all $k \in I_n$, we have that $D(\cdot)$ is a continuous and nonsingular matrix mapping.

First, consider any $q \in \operatorname{int} X$. Let $t = \langle c, q\rangle$ and define $X(t) = \{x \in X \mid \langle c, x\rangle = t\}$. Since X is full-dimensional, it holds that $N(X(t), q) = \{y \mid y = \lambda c, \lambda \in \mathbb{R}\}$. Now suppose $D(q)z(q) \in N(X(t), q)$. This implies that there exists $\lambda \in \mathbb{R}$ such that $D(q)z(q) = \lambda c$ and, therefore, $p_k(q)z_k(q) = \lambda c_k^2$ for all $k \in I_n$. Summing up over all $k \in I_n$ yields $\lambda\langle c, c\rangle = 0$, since $\langle p(q), z(q)\rangle = 0$ by assumption. Because $c \neq 0^n$, we obtain $\lambda = 0$ and, therefore, $D(q)z(q) = 0^n$. Since $D(q)$ is a nonsingular matrix, we must have that $z(q) = 0^n$.

Next, consider any $q \in \operatorname{bd}(X)$. Again, let $t = \langle c, q\rangle$. Suppose that $D(q)z(q) \in N(X(t), q)$. We will show again that $z(q) = 0^n$. Since q is on the boundary of X, we have $r(q) \in N(X, q)$ and $N(X, q)$ is a ray. Hence,

$$N\big(X(t), q\big) = \big\{y \mid y = \mu r(q) + \lambda c,\ \mu \geq 0,\ \lambda \in \mathbb{R}\big\}.$$

Therefore, there exists $\mu \geq 0$ and $\lambda \in \mathbb{R}$ such that

$$D(q)z(q) = \mu r(q) + \lambda c.$$

Moreover, since $q \in \operatorname{bd}(X)$, there exists $j \in I_n$ such that $|r_j(q)| = c_j$. Suppose that $r_j(q) = -c_j$. Then $z_j(q) \geq 0$ and since $p_j(q) > 0$, we have

$$\mu c_j r_j(q) + \lambda c_j^2 = p_j(q)z_j(q) \geq 0.$$

Since $r_j(q) = -c_j$, we obtain

$$(-\mu + \lambda)c_j^2 \geq 0.$$

Because $c_j \neq 0$, this implies $0 \leq \mu \leq \lambda$. For $h \neq j$, we have

$$p_h(q)z_h(q) = \mu r_h(q)c_h + \lambda c_h^2.$$

Since $r_h(q) \geq -c_h$ and $\mu \leq \lambda$, we obtain

$$p_h(q)z_h(q) \geq (-\mu + \lambda)c_h^2 \geq 0.$$

Because $p_h(q) > 0$, this implies $z_h(q) \geq 0$. Consequently, $z_h(q) \geq 0$ for all $h \in I_n$. Since $\langle p(q), z(q)\rangle = 0$ and $p(q) \in \mathbb{R}^n_{++}$, it follows that $z_h(q) = 0$ for all $h \in I_n$ and so $z(q) = 0^n$. Similarly, if $r_j(q) = c_j$ for some $j \in I_n$, it can be shown that $z(q) = 0^n$.

By Theorem 5.4, there exists a connected set of zero points of z in X intersecting with both sets X^- and X^+. Note that since $c \neq 0^n$ and X is a nonempty convex and compact set, $X^- \neq \emptyset$, $X^+ \neq \emptyset$, and $X^- \cap X^+ = \emptyset$. □

The theorem will be applied to an exchange economy with price rigidities. In such an economy, there are a finite number m of consumers, a finite number of n commodities, and a set of admissible prices, denoted by P. Each consumer h, $h \in I_m$, is endowed with a strictly positive commodity bundle $w^h \in \mathbb{R}^n_{++}$ and has preferences represented by a utility function $u_h : \mathbb{R}^n_+ \to \mathbb{R}$. For simplicity, we assume that u^h is a strictly quasiconcave, strictly monotonic, and a continuous function on the set of commodity bundles $\mathbb{R}^n_+$. This assumption can be relaxed considerably. Without price rigidities each consumer h, $h \in I_m$, maximizes his utility $u_h(x^h)$ over $x^h \geq 0$ under the budget constraint $\langle p, x^h\rangle \leq \langle p, w^h\rangle$ at any price vector $p \in \mathbb{R}^n_+$. Under the assumptions made here on the preferences and endowment, the solution $d^h(p)$ to this utility maximization problem is a continuous function of p and satisfies $\langle p, d^h(p)\rangle = \langle p, w^h\rangle$ for all $p \in \mathbb{R}^n_{++}$ (Walras' law). A price vector $p^* \in \mathbb{R}^n_{++}$ is called a Walrasian equilibrium price vector if all markets clear at p^*, i.e. $z(p^*) = 0^n$, where $z(p) = \sum_{h=1}^m (d^h(p) - w^h)$ denotes the aggregate excess demand at price vector p. Let $w = \sum_{h=1}^m w^h$ denote the total endowment of all consumers. Without loss of generality we assume $\langle w, w\rangle = 1$.

The set P of admissible prices is assumed to be a nonempty, convex, and compact set contained in $\mathbb{R}^n_{++}$. When the price vectors are restricted to the set P, a Walrasian equilibrium price vector may not be admissible. In this case, to clear the markets, we need to impose rationing upon the net supply and net demand of each consumer. Let $\ell \in -\mathbb{R}^n_+$ and $u \in \mathbb{R}^n_+$ denote the (uniform) quantity constraint vectors on the net supply and net demand of the consumers, respectively. Then at an admissible price vector $p \in P$ and under the rationing scheme (ℓ, u), the constrained budget set of consumer h, $h \in I_m$, is given by

$$B^h(p, \ell, u) := \{x^h \in \mathbb{R}^n_+ \mid \langle p, x^h\rangle \leq w^h,\ l_k \leq x^h_k - w^h_k \leq u_k \text{ for all } k \in I_n\}.$$

Consumer h, $h \in I_m$, maximizes utility $u^h(x^h)$ over his constrained budget set $B^h(p, \ell, u)$, resulting in his constrained demand $d^h(p, \ell, u)$.

Under price rigidities, the notion of equilibrium is called a constrained equilibrium and defined as follows.

Definition 6.1 A constrained equilibrium consists of a price system $p^* \in P$, a rationing scheme $(\ell^*, u^*) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$, and a consumption vector $x^{*h} \in \mathbb{R}^n_+$ for each consumer h, $h \in I_m$, such that

(1) For all $h \in I_m$, $x^{*h} = d^h(p^*, \ell^*, u^*)$;
(2) $\sum_{h=1}^m x^{*h} = w$;
(3) For every $k \in I_n$, $x^{*h}_k - w^h_k = \ell^*_k$ for some $h \in I_m$ implies $x^{*i}_k - w^i_k < u^*_k$ for all $i \in I_m$, and $x^{*h}_k - w^h_k = u^*_k$ for some $h \in I_m$ implies $x^{*i}_k - w^i_k > \ell^*_k$ for all $i \in I_m$;

(4) There exists $r^* \in N(P, p^*)$ so that for every $k \in I_n$, $\ell_k^* = x_k^{*i} - w_k^i$ for some $i \in I_m$ implies $\ell_k^* = -w_k - r_k^*$, and $u_k^* = x_k^{*i} - w_k^i$ for some $i \in I_m$ implies $u_k^* = w_k - r_k^*$.

Condition (1) states that every consumer is maximizing his utility given the equilibrium prices and rationing scheme. Condition (2) is the market clearing condition. Condition (3) implies that there can be no simultaneous rationing on both sides of any market, so that all markets are frictionless. Condition (4) links the rationing scheme to the admissible price vectors. The vector r^* in the normal cone of P at p^* is a vector pointing outward to the set of admissible prices and is therefore a direction in which the prices are restricted to move. The components of r^* completely determine the level of rationing. If some consumer is being rationed on his supply of commodity k, the supply rationing level for commodity k is equal to $\ell_k^* = -w_k - r_k^* \leq 0$, whereas if some consumer is being rationed on his demand of commodity h the demand rationing level for commodity h is equal to $u_h^* = w_k - r_k^* \geq 0$.

To show that there exists a connected set of constrained equilibria, we extend the admissible set P to the set Q defined by

$$Q := \{ q \in \mathbb{R}^n \mid \|q - p\|_2 \leq 1 \text{ for some } p \in P \}.$$

Clearly, Q is a full-dimensional, compact and convex set in $\mathbb{R}^n$, contains the set P of admissible prices in its interior and the normal cone $N(Q, q)$ of Q at any point q on the boundary of Q is a ray. For any $q \in Q$, the admissible price vector $p(q) \in P$ induced by q is defined by the orthogonal projection of q on P, i.e.,

$$p(q) := \arg \min_{p \in P} \|p - q\|_2.$$

Since the set P is convex and compact, $p(q)$ is well defined and is a continuous function from Q to P. Moreover, since $P \subset \mathbb{R}^n_{++}$, it holds that $p(q) \in \mathbb{R}^n_{++}$ for all $q \in Q$.

To connect the admissible price vectors with rationing schemes, we need to introduce the function $r : Q \to \mathbb{R}^n$ defined by

$$r(q) := 0^n \quad \text{for } q \in P$$

and for $k \in I_n$,

$$r(q) := \frac{\|q - p(q)\|_2}{\max_{h \in I_n} \{|q_h - p_h(q)|/w_h\}} q - p(q), \quad \text{for } q \in Q \setminus P.$$

This function $r(\cdot)$ has the following properties:

- $r(\cdot)$ is a continuous function;
- For all $q \in Q$ it holds that $-w_k \leq r_k(q) \leq w_k$ for all $k \in I_n$;
- If $q \in \operatorname{bd} Q$, then there exists $h \in I_n$ satisfying $|r_h(q)| = w_h$;
- $r(q) \in N(Q, q)$ if $q \in \operatorname{bd} Q$.

See [32] for the function $r(\cdot)$ in more detail. For $q \in Q$, the rationing scheme $(\ell(q), u(q)) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$ is defined by

$$\ell_k(q) = -w_k - r_k(q) \quad \text{and} \quad u_k(q) = w_k - r_k(q), \quad k \in I_n.$$

For every consumer $h \in I_m$ the reduced budget set $B^i(q)$ is defined by

$$B^i(q) := B^i\big(p(q), \ell(q), u(q)\big), \quad q \in Q$$

and his reduced demand $d^h\colon Q \to \mathbb{R}^n$ is given by

$$d^h(q) := \operatorname{arg\,max}\big\{u^h\big(x^h\big) \mid x^h \in B^h(q)\big\}, \quad q \in Q.$$

Finally, the reduced excess demand function $z : Q \to \mathbb{R}^n$ is given by

$$z(q) := \sum_{h=1}^{m} d^h(q) - w.$$

By the assumptions, the function z is continuous and $\langle p(q), z(q)\rangle = 0$ for all $q \in Q$ (Walras' law). In addition, for all $q \in Q$ it holds that $r_k(q) = -w_k$ implies $\ell_k(q) = 0$ and, therefore, $z_k(q) \geq 0$ and $r_k(q) = w_k$ implies $u_k(q) = 0$ and, therefore, $z_k(q) \leq 0$. Let $Q^- := \{q \in Q \mid \langle w, q\rangle \leq \langle w, q'\rangle$ for all $q' \in Q\}$ and $Q^+ := \{q \in Q \mid \langle w, q\rangle \geq \langle w, q'\rangle$ for all $q' \in Q\}$. Since Q is a nonempty convex and compact full-dimensional set and $w \neq 0^n$, obviously $Q^- \neq \emptyset$, $Q^+ \neq \emptyset$, and $Q^- \cap Q^+ = \emptyset$.

Theorem 6.2 *For the exchange economy under price rigidities, there exists a continuum of zero points of the reduced excess demand function z having a nonempty intersection with both Q^- and Q^+. Moreover, every zero point of the function z induces a constrained equilibrium.*

Proof Clearly, the function z satisfies the conditions of Theorem 6.1 with $X = Q$, $c = w$, and the $n \times n$ diagonal matrix $D(q)$ with kth diagonal element equal to $d_k(q) = p_k(q)/w_k$ for every $k \in I_n$ and for every $q \in Q$. Hence, there exists a continuum $C \subset Q$ of zero points of the function z such that $C \cap Q^- \neq \emptyset$ and $C \cap Q^+ \neq \emptyset$. Let $q \in Q$ be a zero point of the function z. We will show that $(p(q), \ell(q), u(q), (d^1(q), \ldots, d^m(q)))$ is a constrained equilibrium. Definition 6.1(1) and (2) are trivially satisfied. Since $0 \leq d^h(q) \leq w_k$ for all $k \in I_n$ and $h \in I_m$, it holds that

$$-w_k \leq d_k^h(q) - w_k^h \leq w_k.$$

Suppose $d_k^h(q) - w_k^h = \ell_k(q)$ for some $h \in I_m$ and $k \in I_n$. Then $d_k^h(q) - w_k^h = \ell_k(q) > -w_k$ and so $r_k(q) < 0$. Hence, for all $i \in I_m$ it holds that

$$d_k^h(q) - w_k^h < w_k - r_k(q) = u_k(q)$$

and, therefore, no consumer is rationed on his net demand of good k. Similarly, if $d_k^h(q) - w_k^h = u_k(q)$ for some $h \in I_m$ and $k \in I_n$, it holds that for all $i \in I_m$

$$d_k^h(q) - w_k^h > -w_k - r_k(q) = \ell_k(q)$$

and, therefore, no consumer is rationed on his net supply of good k. So Condition (3) is satisfied.

It remains to check condition (4). Notice that $r(q) \in N(P, p(q))$, since $r(q)$ is a nonnegative multiple of the vector $q - p(q)$ and $p(q)$ is the orthogonal projection of q on P. Moreover, $l(q) = -w - p(q)$ and $u(q) = w - r(q)$. □

7 Concluding Remarks

We conclude the paper with several remarks. This paper establishes a general theorem on the existence of a connected set of zero points of an upper semicontinuous correspondence, linking two distinct points on the boundary of its domain. It is demonstrated that the theorem extends and unifies several powerful existence theorems such as Browder's fixed point theorem and theorems of Herings, Talman, and Yang. We also establish the existence of a continuum of coincidences of two mappings, thereby generalizing Ky Fan's coincidence theorem to a connected set of such points. Furthermore, we introduce several general results on the existence of a continuum of fixed points, zero points, optima, and solutions to nonlinear variational inequalities. In contrast to nonconstructive (sometimes sophisticated) approaches used in the literature, a prominent feature of our approach is its constructive nature and simplicity in its arguments. More precisely, the main theorem is proved constructively by making use of an algorithm.

References

1. Browder, F.E.: On continuity of fixed points under deformation of continuous mapping. Summa Bras. Math. **4**, 183–191 (1960)
2. Herings, P.J.J., Talman, A.J.J., Yang, Z.: The computation of a continuum of constrained equilibria. Math. Oper. Res. **21**, 675–696 (1996)
3. Herings, P.J.J., Talman, A.J.J., Yang, Z.: Variational inequality problems with a continuum of solutions: existence and computation. SIAM J. Control Optim. **39**, 1852–1873 (2001)
4. Talman, A.J.J., Yamamoto, Y.: Continuum of zero points of a mapping on a compact, convex set. SIAM J. Optim. **14**, 1128–1139 (2004)
5. Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequalities III, pp. 103–113. Academic Press, New York (1972)
6. Yang, Z.: A simplicial algorithm for computing robust stationary points of a continuous function on the unit simplex. SIAM J. Control Optim. **34**, 491–506 (1996)
7. van der Laan, G., Talman, A.J.J., Yang, Z.: Existence and approximation of robust solutions of variational inequality problems over polytopes. SIAM J. Control Optim. **37**, 333–352 (1998)
8. van der Laan, G., Talman, A.J.J., Yang, Z.: Refinements of stationary points with applications to noncooperative games and economics. SIAM J. Optim. **16**, 854–870 (2006)
9. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I+II. Springer, Berlin (2003)
10. Scarf, H.: The approximation of fixed points of a continuous mapping. SIAM J. Appl. Math. **15**, 1328–1343 (1967)
11. Scarf, H.: The Computation of Economic Equilibria. Yale University Press, New Haven (1973)
12. Eaves, B.C.: Homotopies for computation of fixed points. Math. Program. **3**, 1–22 (1972)
13. Eaves, B.C., Saigal, R.: Homotopies for computation of fixed points on unbounded regions. Math. Program. **3**, 225–237 (1972)
14. Merrill, O.H.: Applications and extensions of an algorithm that computes fixed points of certain upper semi-continuous point-to-set mappings. Ph.D. Thesis, University of Michigan (1972)
15. van der Laan, G., Talman, A.J.J.: A restart algorithm for computing fixed points without an extra dimension. Math. Program. **17**, 74–84 (1979)
16. Allgower, E.L., Georg, K.: Numerical Continuation Methods: An Introduction. Springer, Berlin (1990)
17. Todd, M.J.: The Computation of Fixed Points and Applications. Springer, Berlin (1976)
18. Yang, Z.: Computing Equilibria and Fixed Points. Kluwer Academic, Boston (1999)
19. Herings, P.J.J., Koshevoy, G., Talman, A.J.J., Yang, Z.: A general existence theorem of zero points. J. Optim. Theory Appl. **120**, 375–394 (2004)

20. Allgower, E.L., Sommese, A.: Piecewise linear approximation of smooth compact fibers. J. Complex. Theory **18**, 573–588 (2002)
21. Dehez, P., Drèze, J.H.: On supply-constrained equilibria. J. Econ. Theory **33**, 172–182 (1984)
22. Drèze, J.H.: Existence of an exchange economy under price rigidities. Int. Econ. Rev. **16**, 310–320 (1975)
23. Drèze, J.H.: Walras-Keynes equilibria, coordination and macroeconomics. Eur. Econ. Rev. **41**, 1737–1762 (1997)
24. Cox, C.C.: The enforcement of public price controls. J. Polit. Econ. **88**, 887–916 (1980)
25. van der Laan, G.: Equilibrium under rigid prices with compensation for the consumers. Int. Econ. Rev. **21**, 53–73 (1980)
26. van der Laan, G.: Simplicial approximation of unemployment equilibria. J. Math. Econ. **9**, 83–97 (1982)
27. van der Laan, G.: Supply-constrained fixed price equilibria in monetary economies. J. Math. Econ. **13**, 171–187 (1984)
28. Kurz, M.: Unemployment equilibria in an economy with linked prices. J. Econ. Theory **26**, 110–123 (1982)
29. Weddepohl, C.: Supply-constrained equilibria in economies with indexed prices. J. Econ. Theory **43**, 203–222 (1987)
30. Herings, P.J.J.: On the existence of a continuum of constrained equilibria. J. Math. Econ. **30**, 257–273 (1998)
31. Schalk, S.: Equilibrium theory: a salient approach. Ph.D. Thesis, Tilburg University (1999)
32. Herings, P.J.J., van der Laan, G., Talman, A.J.J.: Equilibria with coordination failure. J. Math. Econ. **45**, 23–37 (2009)
33. Herings, P.J.J.: Static and Dynamic Aspects of General Disequilibrium Theory. Kluwer Academic, Dordrecht (1996)
34. Sun, N., Yang, Z.: On fair allocations and indivisibilities. Discussion Paper No. 1347, Cowles Foundation, Yale University (2001)
35. Yang, Z.: A combinatorial topological theorem and its application in welfare economics. Yokohama Bus. Rev. **25**, 119–132 (2005)
36. Dugundji, J.: Topology. Allyn and Bacon, Boston (1970)
37. Talman, A.J.J., Yamamoto, Y.: A simplicial algorithm for stationary point problems on polytopes. Math. Oper. Res. **14**, 383–399 (1989)
38. Fujishige, S., Yang, Z.: A lexicographic algebraic theorem and its applications. Linear Algebra Appl. **279**, 75–91 (1998)
39. Mas-Colell, A.: A note on a theorem of F. Browder. Math. Program. **6**, 229–233 (1974)