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Pancakes and crooked graphs

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Abstract
We give a very brief account of the work of Dima Fon-Der-Flaass on pancakes and distance-regular graphs.

Pancakes
Thinking back of Dima, I have only positive memories. Working with Dima was fun! Having him as a visitor was even more fun. I remember very well that on one of his visits to Tilburg, he invited me over for lunch at the hotel. No ordinary lunch, no: he baked delicious pancakes in his hotel room. That is Dima as I will remember and miss him. Dima’s recipe for the pancakes: keep your feet on the ground, be creative and surprising.

Distance-regular graphs
A distance-regular graph with intersection array \( \{b_0, b_1, ..., b_{d-1}; c_1, ..., c_d\} \) is a connected graph with diameter \( d \) such that for every vertex \( x \) and vertex \( y \) at distance \( i \) from \( x \), the number of neighbours of \( y \) at distance \( i - 1 \) from \( x \) equals \( c_i \), and the number of neighbours of \( y \) at distance \( i + 1 \) from \( x \) equals \( b_i \), for all \( i \). Such a graph is regular with valency \( b_0 \) (for more basic information on distance-regular graphs, see [3]).

Crooked graphs
De Caen, Mathon, and Moorhouse [7] constructed distance-regular graphs with intersection array \( \{2^{2t-1}, 2^{2t-1}; 1, 2, 2^{2t-1}\} \). Such graphs are antipodal covers of the complete graph. The graphs are easily defined as follows. Let \( GF(2^{2t-1}) \times GF(2) \times GF(2^{2t-1}) \) be the vertex set. Two vertices \((a, i, \alpha)\) and \((b, j, \beta)\) are adjacent precisely if
\[
\alpha + \beta = a^2b + ab^2 + (i + j)(a^3 + b^3).
\]
Actually, the graphs are defined in somewhat greater generality; and they are related to the Preparata codes; see [7] for details. The construction also allows for taking quotients. In this way distance-regular graphs with intersection arrays \( \{2^{2t-1}, 2^{2t-1}; 1, 2, 2^{2t-1}\} \)

for \( i = 1, \ldots, 2t \) arise. Prior to this construction, no distance-regular graphs with these intersection arrays were known for \( i < t \).

Together with Bending [1], Dima very creatively introduced the concept of crooked functions to generalize this construction. Let \( V \) be an \( n \)-dimensional vector space over \( GF(2) \). A function \( Q : V \to V \) is called crooked if it satisfies the following three properties:

\[
Q(0) = 0; \\
Q(x) + Q(y) + Q(z) + Q(x + y + z) \neq 0 \text{ for any three distinct } x, y, z; \\
Q(x) + Q(y) + Q(z) + Q(x + a) + Q(y + a) + Q(z + a) \neq 0 \text{ if } a \neq 0 \text{ (} x, y, z \text{ arbitrary)}.
\]

A crooked function \( Q \) is a bijection such that \( H_a(Q) := \{ Q(x) + Q(x + a) : x \in V \} \) is the complement of a hyperplane for every \( a \neq 0 \). These sets \( H_a(Q) \) are all distinct, so every complement of a hyperplane appears among them exactly once.

Given a crooked function \( Q \), a distance-regular ‘crooked’ graph with the same intersection array as the above one can be defined. Now the vertex set is \( V \times GF(2) \times V \), and two vertices \((a, i, \alpha)\) and \((b, j, \beta)\) are adjacent precisely if

\[
\alpha + \beta = Q(a + b) + (i + j + 1)(Q(a) + Q(b)).
\]

The crooked function that gives the graphs of De Caen, Mathon, and Moorhouse is given by \( Q(x) = x^3 \) on \( V = GF(2^n) \) for odd \( n \) (and more generally \( Q(x) = x^{e+1} \) with \( \gcd(e, n) = 1 \)).

Crooked functions form a special class of almost bent functions, which in turn form a special class of almost perfect nonlinear functions. Recently, a lot of new quadratic almost perfect nonlinear functions have been discovered. For odd \( n \), each such function is almost bent. If in addition the function is bijective and maps 0 to 0, then it is also crooked (cf. [9, p. 92]). A new family of crooked functions was thus constructed by Budaghyan, Carlet, and Leander [4, Prop. 1]. See also [2], but beware that a less strict definition of crookedness (compared to Dima’s definition) is used there.

Dima and I [9] showed that almost bent (and hence crooked) functions can be used also to construct distance regular graphs with the same intersection array as a Kasami distance regular graph [3, Theorem 11.2.1, (13), \( q = 2 \)]. They are defined on vertex set \( V \times V \), with two vertices \((a, \alpha)\) and \((b, \beta)\) being adjacent precisely if

\[
\alpha + \beta = Q(a + b).
\]

Using crooked functions, we [8] also constructed symmetric five-class association schemes similar to those constructed in [5] from Kasami graphs, and uniformly packed codes with the same parameters as the double error-correcting BCH codes (Kasami codes) and Preparata codes.

**Nonlinear functions and accomplices**

Crooked functions, almost bent, and almost perfect nonlinear functions have, in some sense, an extremely high degree of nonlinearity. These type of functions play an important role in cryptography. In [9], Dima and I described several characterizations of the mentioned classes of nonlinear functions. We also gave an overview of constructions of all kinds of combinatorial objects from these functions, such as semi-biplanes, difference sets, distance
regular graphs, symmetric association schemes, and uniformly packed codes. We even came up with a further generalization of the crooked graphs using ‘accomplices’ of almost bent functions (not surprisingly, a crooked function is an accomplice of itself).

**Prolific constructions**

With De Caen [6], Dima obtained yet another generalization of the above mentioned distance-regular graphs, by using Latin squares. This initiated the prolific construction by Dima [13] of distance-regular \( n \)-covers of complete graphs \( K_{n^2} \) by using affine planes of order \( n \). Dima realized that, in general, his method produces many (potentially) non-isomorphic such graphs; at least \( 2^{\frac{1}{2}n^2\log n(1+o(1))} \) to be more precise. Computational results by Degraer and Coolsaet [10] confirm this. Muzychuk [14] later extended Dima’s prolific ideas further.

**Nonexistence and the Fon-Der-Flaass graph?**

There are quite some feasibility conditions known on possible intersection arrays of distance-regular graphs. Still, there are many intersection arrays for which it is undecided whether there can be distance-regular graphs with such an array. The monograph by Brouwer, Cohen, and Neumaier [3] contains a list of intersection arrays passing the known (in 1989 to the authors) conditions. Dima contributed by eliminating two possibilities from that list: \( \{5, 4, 3; 1, 1, 2\} \) [11] and \( \{5, 4, 3, 3; 1, 1, 1, 2\} \) [12].

One of the smaller — still ‘open’ — intersection arrays is \( \{7, 6, 6; 1, 1, 2\} \). Deciding whether a distance-regular graph with this intersection array exists is an intriguing open problem that would have been an ideal one to work on with Dima. In fact, I wouldn’t be surprised if he at least played a bit with this problem. So if someone wants to solve it: keep your feet on the ground, be creative and surprising. If such a graph exists, I would call it the Fon-Der-Flaass graph!

**References**


