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Dual Concepts of Almost Distance-Regularity and the Spectral Excess Theorem*

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Abstract

Generally speaking, ‘almost distance-regular’ graphs share some, but not necessarily all, of the regularity properties that characterize distance-regular graphs. In this paper we propose two new dual concepts of almost distance-regularity, thus giving a better understanding of the properties of distance-regular graphs. More precisely, we characterize $m$-partially distance-regular graphs and $j$-punctually eigenspace distance-regular graphs by using their spectra. Our results can also be seen as a generalization of the so-called spectral excess theorem for distance-regular graphs, and they lead to a dual version of it.

Keywords: Distance-regular graph, Distance matrices, Eigenvalues, Idempotents, Local spectrum, Predistance polynomials
2010 Mathematics Subject Classification: 05E30, 05C50

1 Preliminaries

Almost distance-regular graphs, recently studied in the literature, are graphs which share some, but not necessarily all, of the regularity properties that characterize distance-regular graphs. Two examples of the former are partially distance-regular graphs [14] and $m$-walk-regular graphs [6].

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In this paper we propose and characterize two dual concepts of almost distance-regularity, and study some cases where distance-regularity is attained. As in the theory of distance-regular graphs, the two proposed concepts lead to several duality results. Our results can also be seen as a generalization of the so-called spectral excess theorem for distance-regular graphs (see [9]; for short proofs, see [15, 10]). This theorem characterizes distance-regular graphs by their spectra and the average number of vertices at extremal distance. A dual version of this theorem is also derived.

We use standard concepts and results for distance-regular graphs [1, 2], spectral graph theory [4, 12], and spectral and algebraic characterizations of distance-regular graphs [8]. Moreover, for some more details and other concepts of almost distance-regularity (such as distance-polynomial and partially distance-regular graphs), we refer the reader to our recent paper [5]. In what follows, we recall the main concepts, terminology, and results involved.

Let $\Gamma$ be a simple, connected, $\delta$-regular graph, with vertex set $V$, order $n = |V|$, and adjacency matrix $A$. The distance between two vertices $u$ and $v$ is denoted by $\text{dist}(u, v)$, so the diameter of $\Gamma$ is $D = \max_{u,v \in V} \text{dist}(u, v)$. The set of vertices at distance $i$ from a given vertex $u \in V$ is denoted by $\Gamma_i(u)$, for $i = 0, 1, \ldots, D$. The distance-$i$ graph $\Gamma_i$ is the graph with vertex set $V$ and where two vertices $u$ and $v$ are adjacent if and only if $\text{dist}(u, v) = i$ in $\Gamma$. Its adjacency matrix $A_i$ is usually referred to as the distance-$i$ matrix of $\Gamma$. The spectrum of $\Gamma$ is denoted by $\text{sp}\Gamma = \{\lambda_0^m, \lambda_1^m, \ldots, \lambda_d^m\}$, where the different eigenvalues of $\Gamma$ are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$.

1.1 The predistance and preidempotent polynomials

From the spectrum of $\Gamma$, we consider the predistance polynomials $\{p_i\}_{0 \leq i \leq d}$ which are orthogonal with respect to the following scalar product in $\mathbb{R}_d[x]$:

$$\langle f, g \rangle_\Delta = \frac{1}{n} \text{tr} (f(A)g(A)) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\lambda_i)g(\lambda_i),$$

and which satisfy $\text{deg} p_i = i$ and $\langle p_i, p_j \rangle_\Delta = \delta_{ij} p_i(\lambda_0)$, for all $i, j = 0, 1, \ldots, d$. For more details, see [9]. Like every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_ip_i + \gamma_{i+1}p_{i+1}, \quad i = 0, 1, \ldots, d,$$

with $\beta_{-1} = \gamma_{d+1} = 0$. Some basic properties of these coefficients, such as $\alpha_i + \beta_i + \gamma_i = \lambda_0$ for $i = 0, 1, \ldots, d$, and $\beta_i n_i = \gamma_{i+1} n_{i+1} \neq 0$ for
\[ i = 0, 1, \ldots, d - 1, \text{ where } n_i = \|p_i\|_2^2 = p_i(\lambda_0), \] can be found in [3]. Let \( \omega_i \) be the leading coefficient of \( p_i \). Then, from the above recurrence and since \( p(0) = 1 \), it is immediate that \( \omega_i = (\gamma_1 \gamma_2 \cdots \gamma_i)^{-1} \) for \( i = 1, \ldots, d \).

For any graph, the sum of all the predistance polynomials gives the Hoffman polynomial \( H \) satisfying \( H(\lambda_i) = n \delta_{0i}, i = 0, 1, \ldots, d \), which characterizes regular graphs via the condition \( H(A) = J \), the all-1 matrix [13]. Note that the leading coefficient \( \omega_d \) of \( H \) (and also of \( p_d \)) is \( \omega_d = n/\pi \).

From the predistance polynomials, we define the so-called preidempotent polynomials \( q_j \), \( j = 0, 1, \ldots, d \), by
\[
q_j(\lambda_i) = \frac{m_j}{n_i} p_i(\lambda_j), \quad i = 0, 1, \ldots, d,
\]
which are orthogonal with respect to the scalar product
\[
\langle f, g \rangle = \frac{1}{n} \text{tr} (f\{A\} g\{A\}) = \frac{1}{n} \sum_{i=0}^{d} n_i f(\lambda_i) g(\lambda_i),
\]
where \( f\{A\} = \frac{1}{\sqrt{n}} \sum_{i=0}^{d} f(\lambda_i)p_i(A) \). Note that, since \( q_j(\lambda_0) = m_j \), the duality between the two scalar products (1) and (3) and their associated polynomials is made apparent by writing
\[
\langle p_i, p_j \rangle = \frac{1}{n} \sum_{l=0}^{d} n_l p_i(\lambda_l)p_j(\lambda_l) = \delta_{ij} n_i, \quad i, j = 0, 1, \ldots, d, \quad (4)
\]
\[
\langle q_i, q_j \rangle = \frac{1}{n} \sum_{l=0}^{d} n_l q_i(\lambda_l)q_j(\lambda_l) = \delta_{ij} m_i, \quad i, j = 0, 1, \ldots, d. \quad (5)
\]

### 1.2 Vector spaces, algebras and bases

Let \( \Gamma \) be a graph with diameter \( D \), adjacency matrix \( A \) and \( d + 1 \) distinct eigenvalues. We consider the vector spaces \( \mathbb{A} = \mathbb{R}_d[A] = \text{span}\{I, A, A^2, \ldots, A^d\} \) and \( \mathcal{D} = \text{span}\{I, A, A_2, \ldots, A_D\} \), with dimensions \( d + 1 \) and \( D + 1 \), respectively. Then, \( \mathbb{A} \) is an algebra with the ordinary product of matrices, known as the adjacency algebra, with orthogonal bases \( A_p = \{p_0(A), p_1(A), p_2(A), \ldots, p_d(A)\} \) and \( A_\lambda = \{E_0, E_1, \ldots, E_d\} \), where the matrices \( E_i \), \( i = 0, 1, \ldots, d \), corresponding to the orthogonal projections onto the eigenspaces, are the (principal) idempotents of \( A \). Besides, since \( I, A, A^2, \ldots, A^D \) are linearly independent, we have that \( \dim \mathbb{A} = d + 1 \geq D + 1 \) and, therefore, we always have \( D \leq d \) [1]. Moreover, \( \mathcal{D} \) forms an algebra with the entrywise or Hadamard product of matrices, defined by

\[ i = 0, 1, \ldots, d - 1, \text{ where } n_i = \|p_i\|_2^2 = p_i(\lambda_0), \] can be found in [3]. Let \( \omega_i \) be the leading coefficient of \( p_i \). Then, from the above recurrence and since \( p(0) = 1 \), it is immediate that \( \omega_i = (\gamma_1 \gamma_2 \cdots \gamma_i)^{-1} \) for \( i = 1, \ldots, d \).

For any graph, the sum of all the predistance polynomials gives the Hoffman polynomial \( H \) satisfying \( H(\lambda_i) = n \delta_{0i}, i = 0, 1, \ldots, d \), which characterizes regular graphs via the condition \( H(A) = J \), the all-1 matrix [13]. Note that the leading coefficient \( \omega_d \) of \( H \) (and also of \( p_d \)) is \( \omega_d = n/\pi \).

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\langle p_i, p_j \rangle = \frac{1}{n} \sum_{l=0}^{d} n_l p_i(\lambda_l)p_j(\lambda_l) = \delta_{ij} n_i, \quad i, j = 0, 1, \ldots, d, \quad (4)
\]
\[
\langle q_i, q_j \rangle = \frac{1}{n} \sum_{l=0}^{d} n_l q_i(\lambda_l)q_j(\lambda_l) = \delta_{ij} m_i, \quad i, j = 0, 1, \ldots, d. \quad (5)
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\((X \circ Y)_{uv} = X_{uv}Y_{uv}\). We call \(D\) the distance \(\circ\)-algebra, which has orthogonal basis \(D_\lambda = \{I, A, A_2, \ldots, A_d\}\).

From now on, we work with the vector space \(T = \mathcal{A} + D\), and relate the distance-\(i\) matrices \(A_i \in D\) to the matrices \(p_i(A) \in \mathcal{A}\). Note that \(I, A, J\) and \(J\) are matrices in \(\mathcal{A} \cap D\) since \(J = H(A) \in \mathcal{A}\). Recall that \(\mathcal{A} = D\) if and only if \(\Gamma\) is distance-regular (see [1, 2]). In this case, we have \(D = d\), and the predistance polynomials become the distance polynomials satisfying \(A_i = p_i(A)\). In \(T\), we consider the following scalar product:

\[
\langle R, S \rangle = \frac{1}{n} \text{tr}(RS) = \frac{1}{n} \text{sum}(R \circ S),
\]

where \(\text{sum}(M)\) denotes the sum of all entries of \(M\). Observe that the factor \(1/n\) assures that \(\|I\|^2 = 1\), whereas \(\|J\|^2 = n\). Note also that the average degree of \(\Gamma_i\) is \(\bar{\delta}_i = \|A_i\|^2\) and the average multiplicity of \(\lambda_j\) is \(\bar{m}_j = \frac{m}{n} = \|E_j\|^2\). According to (1), this scalar product of matrices satisfies \(\langle f(A), g(A) \rangle = \langle f, g \rangle_\Delta\).

2 Two dual approaches to almost distance-regularity

Here we limit ourselves to the case of graphs with spectrally maximum diameter (or the ‘non-degenerate’ case) \(D = d\). Consequently, we will use indiscriminately the two symbols, \(D\) and \(d\), depending on what we are referring to. In this context, let us consider the following two definitions of almost distance-regularity:

**Definition 2.1** For a given \(i\), \(0 \leq i \leq D\), a graph \(\Gamma\) is \(i\)-punctually distance-regular when there exist constants \(p_{ji}\) such that

\[
A_i E_j = p_{ji} E_j
\]

for every \(j = 0, 1, \ldots, d\); and \(\Gamma\) is \(m\)-partially distance-regular when it is \(i\)-punctually distance-regular for all \(i \leq m\).

**Definition 2.2** For a given \(j\), \(0 \leq j \leq d\), a graph \(\Gamma\) is \(j\)-punctually eigenspace distance-regular when there exist constants \(q_{ij}\) such that

\[
E_j \circ A_i = q_{ij} A_i
\]

for every \(i = 0, 1, \ldots, D\); and \(\Gamma\) is \(m\)-partially eigenspace distance-regular when it is \(j\)-punctually eigenspace distance-regular for all \(j \leq m\).
Notice that the concepts of $D$-partial distance-regularity and $d$-partial eigenspace distance-regularity coincide with the known dual definitions of distance-regularity (see [2]).

Some basic characterizations of punctual distance-regularity, in terms of the distance matrices and the idempotents, were given in [5].

**Proposition 2.3 ([5])** Let $D = d$. Then, $\Gamma$ is $i$-punctually distance-regular if and only if any of the following conditions holds:

(a1) $A_i \in \mathcal{A}$,

(a2) $p_i(A) \in \mathcal{D}$,

(a3) $A_i = p_i(A)$.

Following the duality between Definitions 2.1 and 2.2, it seems natural to conjecture the dual of this proposition: A graph $\Gamma$ is $j$-punctually eigenspace distance-regular if and only if any of the following conditions is satisfied:

(b1) $E_j \in \mathcal{D}$,

(b2) $q_j[A] \in \mathcal{A}$,

(b3) $E_j = q_j[A]$,

where $f[A] = \frac{1}{n} \sum_{i=0}^{d} f(\lambda_i)A_i$. However, although (b1) is clearly equivalent to Definition 2.2 and (b3) $\Rightarrow$ (b1), (b2), until now we have not been able to prove any of the other equivalences and we leave them as conjectures.

In order to derive some new characterizations of punctual distance-regularity, besides the already defined $\bar{\delta}_i$ and $\overline{m}_j$, we consider the following average numbers:

- **The average crossed local multiplicities** are
  
  $$\overline{m}_{ij} = \frac{1}{n\delta_i} \sum_{\text{dist}(u,v)=i} m_{uv}(\lambda_j) = \frac{\langle E_j, A_i \rangle}{\|A_i\|^2},$$

  where $m_{uv}(\lambda_j) = (E_j)_{uv}$ are the crossed local multiplicities.

- **The average number of shortest $i$-paths from a vertex** is
  
  $$\overline{P}_i = \frac{1}{n} \sum_{u \in V} P_i(u) = \frac{1}{n} \sum (A^i \circ A_i) = \langle A^i, A_i \rangle = \frac{1}{\omega_i} \langle p_i(A), A_i \rangle,$$

  (10)
where $P_i(u)$ denotes the number of shortest paths from a vertex $u$ to the vertices in $\Gamma_i(u)$ and $\omega_i = (\gamma_1\gamma_2 \cdots \gamma_i)^{-1}$ is the leading coefficient of $p_i$, $i = 1, \ldots, d$.

- The average number of shortest $i$-paths is

$$
\bar{d}_i^{(i)} = \frac{1}{n\delta_i} \sum (A^i \circ A_i) = \frac{P_i}{\delta_i}.
$$

Proposition 2.4 Let $\Gamma$ be a graph with predistance polynomials $p_i$ and recurrence coefficients $\gamma_i, \alpha_i, \beta_i$, $i = 0, 1, \ldots, d$. Then, $\Gamma$ is $i$-punctually distance-regular if and only if any of the following equalities holds:

(a1) $\frac{1}{\delta_i} = \sum_{j=0}^d \frac{m^2_{ij}}{m_j}$

(a2) $\overline{P}_i = \frac{1}{\omega} \sqrt{p_i(\lambda_0)\delta_i} = \sqrt{\beta_0\beta_1 \cdots \beta_{i-1}\delta_i\gamma_i\gamma_{i-1} \cdots \gamma_1}$

(a3) $\omega_i \bar{u}_i^{(i)} = 1$ and $\overline{\delta}_i = p_i(\lambda_0)$.

Moreover, $\Gamma$ is $j$-punctually eigenspace distance-regular if and only if

(b1) $m_j = \sum_{i=0}^D \overline{\delta}_i m^2_{ij}$.

Proof. (a1) This is a result from [5].

(a2) From (10) and the Cauchy-Schwarz inequality, we get

$$
\omega_i \overline{P}_i = \langle p_i(A), A_i \rangle \leq \|p_i(A)\| \|A_i\| = \sqrt{p_i(\lambda_0)\delta_i} = \sqrt{\beta_0\beta_1 \cdots \beta_{i-1}\delta_i\gamma_i\gamma_{i-1} \cdots \gamma_1}.
$$

Moreover, equality occurs if and only if the matrices $p_i(A)$ and $A_i$ are proportional, which is equivalent to $\Gamma$ being $i$-punctually distance-regular by Proposition 2.3.

(a3) From (11) and (12) we have that $\omega_i \bar{u}_i^{(i)} \leq \sqrt{p_i(\lambda_0)/\delta_i}$, with equality if and only if $\Gamma$ is $i$-punctually distance-regular. Thus, if the conditions in (a3) hold, $\Gamma$ satisfies the claimed property. Conversely, if $\Gamma$ is $i$-punctually distance-regular, both equalities in (a3) are simple consequences of $p_i(A) = A_i$. Indeed, the first one comes from considering the $uv$-entries, with $\text{dist}(u, v) = i$, in the above matrix equation, whereas the second one is obtained by taking square norms.
(b1) From (9), we find that the orthogonal projection of $E_j$ on $D$ is $\hat{E}_j = \sum_{i=0}^{D} m_{ij} A_i$. Now, from $\|\hat{E}_j\|^2 \leq \|E_j\|^2$ we get

$$
\sum_{i=0}^{D} m_{ij}^2 \|A_i\|^2 = \sum_{i=0}^{D} \delta_i m_{ij}^2 \leq m_j
$$

and, in the case of the equality, Definition 2.2 applies with $q_{ij} = \overline{m}_{ij}$. □

Notice the duality between (a1) and (b1) with $\frac{1}{\delta_i}$ and $m_j$.

Now, let us consider the more global concept of partial distance-regularity.

In this case, we also have the following new result where, for a given $0 \leq i \leq d$, $s_i = \sum_{j=0}^{i} p_j$, $t_i = H - s_{i-1} = \sum_{j=i}^{d} p_j$, $S_i = \sum_{j=0}^{i} A_j$, and $T_i = J - S_{i-1} = \sum_{j=i}^{d} A_j$.

**Proposition 2.5** A graph $\Gamma$ is $m$-partially distance-regular if and only if any of the following conditions holds:

(a1) $\Gamma$ is $i$-punctually distance-regular for $i = m, m-1, \ldots, \max\{2, 2m-d\}$.

(a2) $\Gamma$ is $m$-punctually distance-regular and $t_{m+1}(A) \circ S_m = O$.

(a3) $s_i(A) = S_i$ for $i = m, m-1$.

**Proof.** In all cases, the necessity is clear since $p_i(A) = A_i$ for every $0 \leq i \leq m$ (for (a2), note that $t_{m+1}(A) = J - s_m(A)$). Then, let us prove sufficiency. The result in (a1) is basically Proposition 3.7 in [5]. In order to prove (a2), we show by (backward) induction that $p_i(A) = A_i$ and $t_{i+1}(A) \circ S_i = O$ for $i = m, m-1, \ldots, 0$. By assumption, these equations are valid for $i = m$. Suppose now that $p_i(A) = A_i$ and $t_{i+1}(A) \circ S_i = O$ for some $i > 0$. Then, $t_i(A) \circ S_i = A_i$ and, multiplying both terms by $S_{i-1}$ (with the Hadamard product), we get $t_i(A) \circ S_{i-1} = O$. So, what remains is to show that $p_{i-1}(A) = A_{i-1}$. To this end, let us consider the following three cases:

(i) For $\text{dist}(u, v) > i - 1$, we have $(p_{i-1}(A))_{uv} = 0$.

(ii) For $\text{dist}(u, v) = i - 1$, we have $(t_{i+1}(A))_{uv} = 0$, so $(p_{i-1}(A))_{uv} = (s_{i-1}(A))_{uv} = (s_{i-1}(A))_{uv} + (A_i)_{uv} = (s_i(A))_{uv} = 1 - (t_{i+1}(A))_{uv} = 1$.

(iii) For $\text{dist}(u, v) < i - 1$, we use the recurrence (2) to write

$$
x t_i = \sum_{j=i}^{d} x p_j = \sum_{j=i}^{d} (\beta_{j-1} p_{j-1} + \alpha_j p_j + \gamma_{j+1} p_{j+1})
$$
\[
\begin{align*}
\beta_i - 1 p_i - 1 & + \sum_{j=1}^{d} (\alpha_j + \beta_j + \gamma_j) p_j \\
\beta_i - 1 p_i - 1 & - \gamma_i p_i + \delta t_i,
\end{align*}
\]

which gives
\[
A t_i(A) = \beta_i - 1 p_i - 1 (A) - \gamma_i A_i + \delta t_i(A).
\]

Then, since \((t_i(A))_{uv} = (A_i)_{uv} = 0\) and \(\beta_i - 1 \neq 0\), we get
\[
(p_i - 1(A))_{uv} = \frac{1}{\beta_i - 1} (A t_i(A))_{uv} = \frac{1}{\beta_i - 1} \sum_{w \in \Gamma(u)} (t_i(A))_{uv} = 0,
\]

because \(\text{dist}(v, w) \leq \text{dist}(v, u) + \text{dist}(u, w) \leq i - 1\) for the relevant \(w\).

From \((i), (ii), \) and \((iii), \) we have that \(p_i - 1(A) = A_i - 1, \) so by induction \(\Gamma\) is \(m\)-partially distance-regular, and the sufficiency of \((a2)\) is proven. Finally, the sufficiency of \((a3)\) follows from that of \((a2)\) because \(s_i(A) = S_i\) for every \(i \in \{m - 1, m\}\) implies that \(p_m(A) = (s_m - s_{m-1})(A) = S_m - S_{m-1} = A_m\) and \(t_{m+1}(A) \circ S_m = (J - s_m(A)) \circ S_m = (J - S_m) \circ S_m = O. \)

\[\square\]

Given some vertex \(u\) and an integer \(i \leq \text{ecc}(u)\), we denote by \(N_i(u)\) the \(i\)-neighborhood of \(u\), which is the set of vertices that are at distance at most \(i\) from \(u\). In [8] it was proved that \(s_i(\lambda_0)\) is upper bounded by the harmonic mean of the numbers \(|N_i(u)|\) and equality is attained if and only if \(s_i(A) = S_i\). A direct consequence of this property and Proposition 2.5(a3) is the following characterization.

**Theorem 2.6** A graph \(\Gamma\) is \(m\)-partially distance-regular if and only if, for every \(i \in \{m - 1, m\}\),
\[
s_i(\lambda_0) = \frac{n}{\sum_{u \in V} |N_i(u)|^{-1}}.
\]

### 3 Distance-regular graphs

Let us particularize our results to the case of distance-regular graphs. With this aim, we use the following theorem giving some known characterizations.

**Theorem 3.1** ([7, 11]) A graph \(\Gamma\) with \(d+1\) distinct eigenvalues and diameter \(D = d\) is distance-regular if and only if any of the following statements is satisfied:

\[\square\]
(a) $\Gamma$ is $D$-punctually distance-regular.

(b) $\Gamma$ is $j$-punctually eigenspace distance-regular for $j = 1, d$.

In fact, notice that (a) corresponds to any of the conditions in Proposition 2.5 with $m = d$. Moreover, the duality between (a) and (b) is made apparent when they are stated as follows:

(a) $A_0(= I), A_1(= A), A_D \in A$;

(b) $E_0(= \frac{1}{n} J), E_1, E_d \in D$.

Then, by using Theorem 3.1 and Proposition 2.4(a1) and (b1), and Theorem 2.6 (with $m = d$), we have the spectral excess theorem [9] in the next condition (a), its dual form in (b), and its harmonic mean version [8, 15] in (c).

**Theorem 3.2** A regular graph $\Gamma$ with $D = d$ is distance-regular if and only if any of the following equalities holds:

(a) $\frac{1}{\delta_d} = \sum_{j=0}^{d} \frac{m^2_{dj}}{m_j}$.

(b) $m_j = \sum_{i=0}^{d} \delta_i m^2_{ij}$ for $j = 1, d$.

(c) $s_{d-1}(\lambda_0) = \frac{n}{\sum_{u \in V} |N_{d-1}(u)|^{-1}}$.

In fact, condition (a) is usually written in its equivalent form $\overline{\delta_d} = p_d(\lambda_0)$ as, when $i = d$, the first condition in Proposition 2.4(a.3) always holds since

$$
\overline{\delta_d} = \frac{1}{\delta_d} (A_d, A_d) = \frac{1}{\delta_d \omega_d} (H(A), A_d) = \frac{1}{\delta_d \omega_d} (J, A_d) = \frac{1}{\delta_d \omega_d} \|A_d\|^2 = \frac{1}{\omega_d}.
$$

Notice also that, in (c), we do not need to impose the condition of Theorem 2.6 for $i = d$ since $s_d(\lambda_0) = H(\lambda_0) = N_d(u) = n$ for every $u \in V$.

**References**


