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Open-Loop Nash Equilibria in the Non-cooperative Infinite-planning Horizon LQ Game*

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Abstract

In this note we reconsider Nash equilibria for the linear quadratic differential game for an infinite planning horizon. We consider an open-loop information structure. In the standard literature this problem is solved under the assumption that every player can stabilize the system on his own. In this note we relax this assumption and provide both necessary and sufficient conditions for existence of Nash equilibria for this game under the assumption that the system as a whole is stabilizable.

Keywords: linear-quadratic differential games, open-loop Nash equilibrium, solvability conditions, Riccati equations.

JEL-codes: C61, C72, C73.

1 Introduction

In this note we reconsider the linear quadratic differential game to minimize

$$\lim_{T\to\infty} J_i(x_0,u_1,u_2,T) \tag{1}$$

where

$$J_i=\int_0^T \{x^T(t)Q_ix(t)+u_i^T(t)R_iu_i(t)\}dt,$$

subject to the dynamic state equation

$$\dot{x}(t)=Ax(t)+B_1u_1(t)+B_2u_2(t),\ x(0)=x_0,\ \text{where } (u_1,u_2)\in\mathcal{U}_s. \tag{2}$$

Here $x(t)\in I\!R^n$, $u_i\in I\!R^{m_i}$, matrix R_i is positive definite and Q_i are symmetric, $i=1,2$.
We assume that the players have an open-loop information structure about the game. That is, at time $t=t_0$ they have all information about the game, determine their actions, which are then enforced as binding agreements for the whole planning horizon. Moreover, we assume that players

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will play a Nash strategy, i.e., they look for actions that have the property that a unilateral deviation from these actions makes them worse off.

This problem has been considered by many authors in the past (see e.g. [2] for references). This problem has been solved under the assumption that every player is capable to stabilize the system by his own, i.e. the pairs (A, B_i), $i = 1, 2$, are stabilizable. In this note we will relax this assumption. We will provide both necessary and sufficient conditions for existence of Nash equilibria for this game under the assumption that the matrix pair $(A, [B_1\ B_2])$ is stabilizable.

Assuming that the set of control functions considered by the players consists of the following set

$$\mathcal{U}_s(x_0) = \left\{ u \in L_{2,loc} \mid J_i(x_0, u) \text{ exists in } I\!R \cup \{-\infty, \infty\}, \lim_{t\to\infty} x(t) = 0 \right\},$$

where $L_{2,loc}$ is the set of *locally square-integrable* functions, i.e.,

$$L_{2,loc} = \{u[0,\infty) \mid \forall T > 0, \int_0^T u^T(s)u(s)ds < \infty\},$$

we show in the Appendix the next Theorem 1.1.

To that purpose we first introduce some notation. Throughout we will use 0 to denote a zero matrix that has dimensions that follow from the context where it is used (in particular, in some cases, it may happen that in fact it does not appear (i.e. it reduces to the empty matrix)). Let T_i be nonsingular transformation matrices which transform (A, B_i) into its controllable canonical form, $i = 1, 2$, (see Lemma 3.1, with n_i the dimension of the controllability subspace). Let

$$A_i := [I_{n_i}\ 0]T_iAT_i^{-1}\begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix};\ \tilde{S}_i := B_iR_i^{-1}B_i^TT_i^T\begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix};\ \tilde{Q}_i := [I_{n_i}\ 0]T_i^{-1^T}Q_i.$$

Theorem 1.1 *Consider the linear quadratic differential game (1,2). Assume that $(A, [B_1\ B_2])$ is stabilizable. Consider matrix*

$$M = \begin{bmatrix} A & -\tilde{S}_1 & -\tilde{S}_2 \\ -\tilde{Q}_1 & -A_1^T & 0 \\ -\tilde{Q}_2 & 0 & -A_2^T \end{bmatrix}. \tag{3}$$

If the linear quadratic differential game (1,2) has an open-loop Nash equilibrium for every initial state, then

1. *M has at least n stable eigenvalues (counted with algebraic multiplicities). More in particular, there exists a p-dimensional stable M-invariant subspace S, with $p \geq n$, such that*

$$\text{Im}\begin{bmatrix} I_n \\ V_1 \\ V_2 \end{bmatrix} \subset S,$$

   for some $V_i \in I\!R^{n_i\times n}$.

2. *With $Q_{11i} := [I_{n_i}\ 0]T_i^{-1^T}Q_iT_i^{-1}\begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix}$* and *$S_i := [I_{n_i}\ 0]T_iB_iR_i^{-1}B_i^TT_i^T\begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix}$, the two algebraic Riccati equations,*

$$A_i^TK_i + K_iA_i - K_iS_iK_i + Q_{11i} = 0, \tag{4}$$

   have a symmetric solution $K_i(.)$ such that $A_i - S_iK_i$ is stable, $i = 1, 2$.

Conversely, if the two algebraic Riccati equations (4) have a stabilizing solution and $v^T(t) =: [x^T(t), \psi_1^T(t), \psi_2^T(t)]$ is an asymptotically stable solution of

$$\dot{v}(t) = Mv(t),\ x(0) = x_0, \tag{5}$$

then,

$$u_i^* := -R_i^{-1}B_i^T T_i^T \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix} \psi_i(t),\ i = 1, 2,$$

provides an open-loop Nash equilibrium for the linear quadratic differential game (1,2). □

Notice that the matrices $\tilde{S}_i$ and $\tilde{Q}_i$ that appear in M are, in general, not symmetric and the dimensions of matrices A_i may differ. This is a clear distinction with the standard case considered in literature (see e.g. [2]).

Now, let

$$\tilde{A}_2 := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix};\ B := [B_1\ B_2];\ \tilde{S} := [\tilde{S}_1\ \tilde{S}_2];\ \text{ and } \tilde{Q} := \begin{bmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \end{bmatrix}.$$

Consider the set of (coupled) algebraic Riccati equations

$$0 = \tilde{A}_2^T P + PA - P\tilde{S}P + \tilde{Q}. \tag{6}$$

Definition 1.2 A solution $P^T =: [P_1^T, P_2^T]$, with $P_i \in \mathbb{R}^{n_i \times n}$, of the set of algebraic Riccati equations (6) is called

a. *stabilizing*, if $\sigma(A - \tilde{S}P) \subset \mathbb{C}^-$; [1]

b. *left-right stabilizing*[2] (LRS) if

i. it is a stabilizing solution, and

ii. $\sigma(-\tilde{A}_2^T + P\tilde{S}) \subset \mathbb{C}_0^+$; □

Similar to, e.g., [2] it can be shown that the next relationship between certain invariant subspaces of matrix M and solutions of the Riccati equation (6) applies. This property can be used to calculate the (left-right) stabilizing solutions of (6).

Lemma 1.3 *Let $V \subset \mathbb{R}^{n+n_1+n_2}$ be an n-dimensional invariant subspace of M, and let $X_i \in \mathbb{R}^{n_i \times n}$, $i =$ $0, 1, 2$, (with $n_0 = n$) be three real matrices such that*

$$V = \text{im}\left[X_0^T,\ X_1^T,\ X_2^T\right]^T.$$

If X_0 is invertible, then $P_i := X_i X_0^{-1}$, $i = 1, 2$, solves (6) and $\sigma(A - \tilde{S}P)) = \sigma(M|_V)$. Furthermore, (P_1, P_2) is independent of the specific choice of basis of V. □

[1] $\sigma(H)$ denotes the spectrum of matrix H; $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) < 0\}$; $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) \geq 0\}$.

[2] In [2] such a solution is called strongly stabilizing.

Lemma 1.4

1. *The set of algebraic Riccati equations (6) has a LRS solution $P^T = [P_1^T\ P_2^T]$ if and only if matrix M has an n-dimensional stable graph subspace and M has n_1+n_2 eigenvalues (counting algebraic multiplicities) in $\mathbb{C}_0^+$.*

2. *If the set of algebraic Riccati equations (6) has a LRS solution, then it is unique.*

Proof.

1. Assume that (6) has a LRS solution P. Then with $T := \begin{bmatrix} I_n & 0 \\ -P & I_{n_1+n_2} \end{bmatrix}$,

$$TMT^{-1} = \begin{bmatrix} A - \tilde{S}P & -\tilde{S} \\ 0 & -\tilde{A}_2^T + P\tilde{S} \end{bmatrix}.$$

Since P is a LRS solution, by Definition 1.2, matrix M has exact n stable eigenvalues and $n_1 + n_2$ eigenvalues (counted with algebraic multiplicities) in $\mathbb{C}_0^+$. Furthermore, obviously, the stable subspace is a graph subspace.

The converse statement is obtained similarly using the result of Lemma 1.3.

2. See, e.g., Kremer [4, Section 3.2]. □

Similar to [2] it can be shown that the next two important corollaries hold.

Corollary 1.5 *The infinite-planning horizon two-player linear quadratic differential game (1,2) has for every initial state an open-loop Nash set of equilibrium actions (u_1^*, u_2^*) which permit a feedback synthesis if and only if*

1. *there exist P_1 and P_2 that are solutions of the set of coupled algebraic Riccati equations (6) satisfying the additional constraint that the eigenvalues of $A_{cl} := A - S_1P_1 - S_2P_2$ are all situated in the left half complex plane, and*

2. *the two algebraic Riccati equations (4) have a symmetric solution $K_i(.)$ such that $A_i - S_iK_i$ is stable, $i = 1, 2$.*

If (P_1, P_2) is a set of stabilizing solutions of the coupled algebraic Riccati equations (6), the actions

$$u_i^*(t) = -R_i^{-1}B_i^TT_i^T \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix} P_i\Phi(t,0)x_0,\ i = 1, 2, \tag{7}$$

where $\Phi(t, 0)$ satisfies the transition equation $\dot{\Phi}(t, 0) = A_{cl}\Phi(t, 0)$; $\Phi(0, 0) = I$, yield an open-loop Nash equilibrium.

The costs, by using these actions, for the players are

$$x_0^TM_ix_0,\ i = 1, 2, \tag{8}$$

where M_i is the unique solution of the Lyapunov equation

$$A_{cl}^TM_i + M_iA_{cl} + Q_i + P_i^T[I_{n_i}\ 0]T_iS_iT_i^T[I_{n_i}\ 0]^TP_i = 0. \tag{9}$$

□

Corollary 1.6 *The linear quadratic differential game (1,2) has a unique open-loop Nash equilibrium for every initial state if and only if*

1. *The set of coupled algebraic Riccati equations (6) has a strongly stabilizing solution, and*

2. *the two algebraic Riccati equations (4) have a stabilizing solution.*

Moreover, the unique equilibrium actions are given by (7). □

Remark 1.7 **1.** In case (A, B_i) is controllable, matrix $T_i = I_n$ and $n_i = n$. In that case the above results coincide with the results presented, e.g., in [2][Chapter 7.4].
2. In case (A, B_i) is stabilizable but not controllable, the above results might be used to find the open-loop Nash equilibria more efficiently from a numerical point of view. For instance, matrix M in (3) has size $(n + n_1 + n_2) \times (n + N - 1 + n_2)$, whereas according the current literature, see e.g. [2][Theorem 7.11], one has to analyze a $3n \times 3n$ matrix to obtain the equilibria.
3. It can be straightforwardly verified that $\psi_i(t)$ in Theorem 1.1 coincides with the costate variable associated with the minimization of J_i (see, e.g., (14)) subject to the state equation (see, e.g., (15)). In case (6) has a set of stabilizing solutions it follows that $\psi_i(t) = P_i x(t)$, where $x(t)$ solves $\dot{x}(t) = (A - \tilde{S}_1 P_1 - \tilde{S}_2 P_2)x(t),\ x(0) = x_0$, solves (5). From (8) it follows that $\frac{\partial J_i}{\partial x_0} = 2M_i x_0$ which usually (see (9) differs from $\psi_i(0) = P_i x_0$. This implies that the correct shadow price interpretation of $x(t)$ is provided by $2M_i x(t)$ instead of $\psi_i(t)$. See, e.g., [1] for more details on this issue.
4. The results can be straightforwardly generalized for the N-player case. All results concerning matrix M should then be substituted by

$$M = \begin{bmatrix} A & -\tilde{S} \\ -\tilde{Q} & -\tilde{A}_2^T \end{bmatrix},$$

where $\tilde{S} = [\tilde{S}_1 \ \cdots \tilde{S}_N]$, $\tilde{Q} = [\tilde{Q}_1^T \ \cdots \ \tilde{Q}_N^T]$ and $\tilde{A}_2 = diag(A_i)$. In the next section we will illustrate this in a 3-player game. □

2 Example

Consider the problem to find the open-loop Nash equilibria for the next three player game. The game might be interpreted as a debt stabilization problem within a two country setting which engaged in a monetary union. Within that setting the variables $x_i(t)$ can be interpreted as the government debt, scaled to the level of national output, of country i. u_i as the primary fiscal deficit, also scaled to output, whereas the monetary financing undertaken by the central bank, measured as a fraction of aggregate output, is denoted by u_E. All parameters are assumed to be positive. The welfare loss-function of country $i,\ i = 1, 2$, and central bank, respectively, is given by

$$\begin{aligned} J_1(u_1(.)) &= \frac{1}{2}\int_0^\infty u_1^2(t) + \beta_1 x_1^2(t)dt && (10) \\ J_2(u_2(.)) &= \frac{1}{2}\int_0^\infty u_2^2(t) + \beta_2 x_2^2(t)dt && (11) \\ J_E(u_3(.)) &= \frac{1}{2}\int_0^\infty u_E^2(t) + \beta_E(\omega x_1(t) + (1-\omega)x_2(t))^2 dt. && (12) \end{aligned}$$

The evolution of debt in both countries over time is assumed to be described by the differential equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2(t) + \begin{bmatrix} -\gamma_1 \\ -\gamma_2 \end{bmatrix} u_E(t). \tag{13}$$

Let

$$\begin{aligned} A &:= \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}; \; B_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \; B_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \; B_E := -\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}; \; B := [B_1 \; B_2 \; B_E]; \\ Q_1 &:= \begin{bmatrix} \beta_1 & 0 \\ 0 & 0 \end{bmatrix}; \; Q_2 := \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 \end{bmatrix}; \; Q_E := \beta_E \begin{bmatrix} \omega^2 & \omega(1-\omega) \\ \omega(1-\omega) & (1-\omega)^2 \end{bmatrix}; \text{ and } S_i := B_i B_i^T, \; i = 1, 2, E. \end{aligned}$$

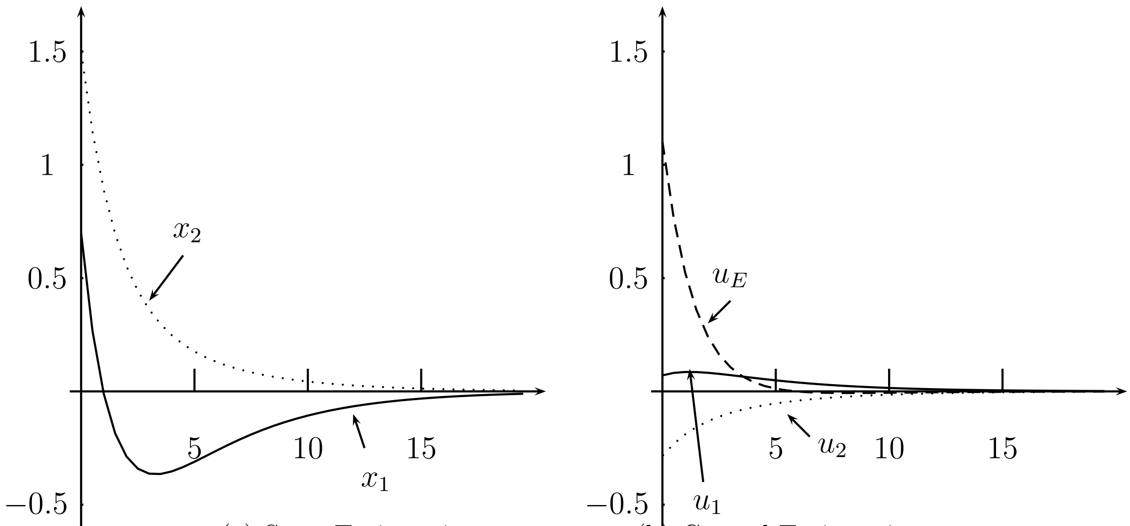
Obviously, the pairs (A, B_i), $i = 1, 2$ are not stabilizable. Consequently we cannot directly use the standard theory on linear quadratic differential games to find the open-loop Nash equilibria (see e.g. [2]). However, notice that the pair (A, B) is stabilizable. With $T_1 = I$, $T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T_E = I$, $n_1 = n_2 = 1$ and $n_E = 2$ in Theorem 1.1 we have that matrix M equals

$$M = \begin{bmatrix} \alpha_1 & 0 & -1 & 0 & -\gamma_1^2 & -\gamma_1\gamma_2 \\ 0 & \alpha_2 & 0 & -1 & -\gamma_1\gamma_2 & -\gamma_2^2 \\ -\beta_1 & 0 & -\alpha_1 & 0 & 0 & 0 \\ 0 & -\beta_2 & 0 & -\alpha_2 & 0 & 0 \\ -\beta_E\omega^2 & -\beta_E\omega(1-\omega) & 0 & 0 & -\alpha_1 & 0 \\ -\beta_E\omega(1-\omega) & -\beta_E(1-\omega)^2 & 0 & 0 & 0 & -\alpha_2 \end{bmatrix}.$$

By straightforward calculating one can verify that the determinant of $M - \lambda I$ equals

$$det := \lambda^6 + \eta_5 s^5 - \eta_4 s^4 - \eta_3 s^3 + \rho_2 s^2 + \eta_1 s + \eta_0, \text{ where } \eta_i > 0.$$

So, independent of the sign of ρ_2, this sixth order polynomial has 2 sign changes (if one orders it by descending variable exponent). So according Descartes' rule of signs matrix M has either two or no positive real roots. Furthermore, with λ replaced by $-\lambda$ the corresponding polynomial has 4 sign changes. By Descartes' rule M therefore has either 4, 2 or no negative real roots. So, in case *det* has six real roots, 2 of them will be positive and 4 of them negative. So, generically, in that case the game will have an infinite number of open-loop Nash equilibria and $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ equilibria for which the corresponding equilibrium strategies permit a feedback synthesis.
This is confirmed in case we chose, e.g., $x_1(0) = 0.7$, $x_2(0) = 1.5$, $\alpha_1 = 0.03$, $\alpha_2 = 0.08$, $\gamma_1 = 1$, $\gamma_2 = 0.5$, $\beta_1 = 0.04$, $\beta_2 = 0.08$, $\beta_E = 0.04$ and $\omega = 0.3$. In that case matrix M has the eigenvalues $\{0.7002, 0.2426, -0.0421, -0.0594, -0.2522, -0.6992\}$, and it is easily verified that there are 6 different strategies that permit a feedback synthesis.
By including a discount factor into the cost the number of equilibria reduces. In this example it turns out that if the discount factor is larger than 12% there will be a unique equilibrium. We calculated the resulting equilibrium strategies if a discount factor of 14% is used. Then, $P_1 = [0.1028 \; -0.0951]$, $P_2 = [-0.0494 \; 0.2133]$ and $P_E = \begin{bmatrix} 0.1316 & 0.2666 \\ 0.3343 & 0.6574 \end{bmatrix}$. So, $u_1(t) = -P_1 x(t)$, $u_2(t) = -P_2 x(t)$ and



(a) State Trajectories (b) Control Trajectories

Figure 1: Fiscal debt stabilization using a discount factor of 14%.

$u_E(t) = [1\ \frac{1}{2}]P_E x(t) = [0.2987\ 0.5953]x(t)$, where $x(t) = (A - B_1P_1 - B_2P_2 + B_E[1\ \frac{1}{2}]P_E)x(t),\ x(0) = x_0$.

In this specific case we observe that the central bank responds approximately with a three times higher monetary policy as the corresponding fiscal authorities to stabilize debt in their country. Furthermore we see that fiscal authorities negatively respond on a debt that has occurred in the other country. We plotted both the evolution of debt and control instruments in Figure 1.
From this figure we see that the initially strong monetary policy pursued by the central bank implies that the country with the initially smallest debt runs into a surplus. Furthermore, after five years the expansionary monetary financing policy of the central bank is replaced by a small contracting monetary financing.

3 Concluding remarks

In this paper we reconsidered the infinite planning horizon open-loop linear quadratic differential game. We derived under less stringent conditions than in current literature both necessary and sufficient conditions under which this game has an open-loop Nash equilibrium. Based on the controllable canonical form of the system we showed along the lines of the proof of [2] that one can derive Nash equilibria for stabilizable systems similar to the standard case considered in literature. By determining for each individual player its controllable canonical form, one can derive matrix M that has a similar structure as in the standard case. Since the size of this square matrix is usually smaller than $3n$ it may be advantageous from a computational point of view, also in the standard case, to use this reduced form to calculate the Nash equilibria. We illustrated the theory in a small example.

Appendix

First we recall the next well-known controllable state-space decomposition lemma (see e.g. [6, Theorem3.6]).

Lemma 3.1 *Consider the linear system $\dot{x}(t) = Ax(t) + Bu(t)$, with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Assume the controllability matrix $C := [B|AB|\cdots|A^{n-1}B]$ has rank n_1 (or stated differently the con-*

trollability subspace has dimension n_1). Let $v_i,\ i = 1,\cdots,n_1$ be n_1 linearly independent columns of C and $v_{n_1+1},\cdots,v_n$ be such that $V := [v_1 \cdots v_n]$ is invertible. Then, with $T := V^{-1}$, $\bar{x}(t) = Tx(t)$ satisfies $\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t)$, and $(\bar{A},\bar{B})$ have the next properties:

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix};\ \bar{B} = TB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix},$$

where $\bar{A}_{11} \in I\!R^{n_1 \times n_1}$, $\bar{B}_1 \in I\!R^{n_1 \times m}$ and $(\bar{A}_{11},\bar{B}_1)$ is controllable. □

Proof of Theorem 1.1 "⇒ part" Suppose that u_1^*, u_2^* are a Nash solution. That is,

$$J_1(u_1,u_2^*) \geq J_1(u_1^*,u_2^*) \text{ and } J_2(u_1^*,u_2) \geq J_2(u_1^*,u_2^*).$$

From the first inequality we see that for every $x_0 \in I\!R^n$ the (nonhomogeneous) linear quadratic control problem to minimize

$$J_1 = \int_0^\infty \{x^T(t)Q_1x(t) + u_1^T(t)R_1u_1(t)\}dt,$$

subject to the (nonhomogeneous) state equation

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2^*(t),\ x(0) = x_0,$$

has a solution. Notice that by assumption $x(t) \to 0$. So, since this problem has a minimum and $R_1 > 0$ it follows that $u_1(t) \to 0$ too.
From Lemma 3.1 we have that there exists then a state-space transformation $\tilde{x}(t) = T_1x(t)$ such that the above minimization problem can be rewritten as the (nonhomogeneous) linear quadratic control problem to minimize

$$J_1 = \int_0^\infty \{\tilde{x}_1^T(t)Q_{111}\tilde{x}_1(t) + 2\tilde{x}_2^T(t)Q_{121}^T\tilde{x}_1(t) + \tilde{x}_2^T(t)Q_{221}^T\tilde{x}_2(t) + u_1^T(t)R_1u_1(t)\}dt, \tag{14}$$

subject to the (nonhomogeneous) state equation

$$\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} \tilde{B}_{21} \\ \tilde{B}_{22} \end{bmatrix} u_2^*(t),\ \tilde{x}(0) = T_1x_0, \tag{15}$$

where $(A_1,\tilde{B}_1)$ is controllable and $\begin{bmatrix} Q_{111} & Q_{121} \\ Q_{121}^T & Q_{221} \end{bmatrix} := T_1^{-1^T}Q_1T_1^{-1}$. By assumption this minimization problem has a solution for every initial state. This implies, see [3, Theorem A.1], that the algebraic Riccati equation

$$A_1^TK_1 + K_1A_1 - K_1S_1K_1 + Q_{111} = 0, \text{ with } S_1 := \tilde{B}_1R_1^{-1}\tilde{B}_1^T = [I_{n_1}\ 0]T_1B_1R_1^{-1}B_1^TT_1^T \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix},$$

has a stabilizing solution K_1. Moreover the unique minimum is attained by

$$u_1(t) = -R_1^{-1}\tilde{B}_1^T(K_1\tilde{x}_1(t) + \tilde{m}_1(t)). \tag{16}$$

Here $\tilde{m}_1(t)$ is given by

$$\tilde{m}_1(t) = \int_t^\infty e^{-(A_1-S_1K_1)^T(t-s)} \left[K_1(\tilde{B}_{21}u_2^*(s) + \tilde{A}_{12}\tilde{x}_2(s)) + Q_{121}\tilde{x}_2(s)\right] ds. \tag{17}$$

By straightforward differentiation of (17) we obtain

$$\dot{\tilde{m}}_1(t) = -(A_1 - S_1K_1)^T\tilde{m}_1(t) - K_1(\tilde{B}_{21}u_2^*(t) + \tilde{A}_{12}\tilde{x}_2(t)) - Q_{121}\tilde{x}_2(t). \tag{18}$$

Next, introduce

$$\psi_1(t) := K_1\tilde{x}_1(t) + \tilde{m}_1(t). \tag{19}$$

Using (18) we get

$$\begin{aligned}
\dot{\psi}_1(t) &= K_1\dot{\tilde{x}}_1(t) + \dot{\tilde{m}}_1(t) \\
&= K_1(A_1 - S_1K_1)\tilde{x}_1(t) - K_1S_1\tilde{m}_1(t) + K_1\tilde{A}_{12}\tilde{x}_2(t) + K_1\tilde{B}_{21}u_2^*(t) - (A_1 - S_1K_1)^T\tilde{m}_1(t) \\
&\quad -K_1(\tilde{B}_{21}u_2^*(t) + \tilde{A}_{12}\tilde{x}_2(t)) - Q_{121}\tilde{x}_2(t) \\
&= (-Q_{111} - A_1^TK_1)\tilde{x}_1(t) - K_1S_1\tilde{m}_1(t) - (A_1 - S_1K_1)^T\tilde{m}_1(t) - Q_{121}\tilde{x}_2(t) \\
&= -Q_{111}\tilde{x}_1(t) - A_1^T(K_1\tilde{x}_1(t) + \tilde{m}_1(t)) - Q_{121}\tilde{x}_2(t) \\
&= -[Q_{111}\ Q_{121}]\tilde{x}(t) - A_1^T\psi_1(t).
\end{aligned} \tag{20}$$

In a similar way it follows that there exists a state transformation $\hat{x}(t) = T_2x(t)$ such that the minimization of J_2 can be rewritten as

$$J_2 = \int_0^\infty \{\hat{x}_1^T(t)Q_{112}\hat{x}_1(t) + 2\hat{x}_2^T(t)Q_{122}^T\hat{x}_1(t) + \hat{x}_2^T(t)Q_{222}^T\hat{x}_2(t) + u_2^T(t)R_2u_2(t)\}dt,$$

subject to the (nonhomogeneous) state equation

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_2 & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{12} \end{bmatrix} u_1(t) + \begin{bmatrix} \hat{B}_2 \\ 0 \end{bmatrix} u_2^*(t),\ \hat{x}(0) = T_2x_0,$$

where $(A_2, \hat{B}_2)$ is controllable and $\begin{bmatrix} Q_{112} & Q_{122} \\ Q_{122}^T & Q_{222} \end{bmatrix} := T_2^{-1^T}Q_2T_2^{-1}$. The solution of this problem is

$$u_2(t) = -R_2^{-1}\hat{B}_2^T(K_2\hat{x}_1(t) + \hat{m}_1(t)). \tag{21}$$

Here $\hat{m}_1(t)$ is given by

$$\hat{m}_1(t) = \int_t^\infty e^{-(A_2-S_2K_2)^T(t-s)} \left[K_1(\hat{B}_{11}u_1^*(s) + \hat{A}_{12}\hat{x}_2(s)) + Q_{122}\hat{x}_2(s)\right] ds, \tag{22}$$

where K_2 is the stabilizing solution of the algebraic Riccati equation

$$A_2^TK_2 + K_2A_2 - K_2S_2K_2 + Q_{112} = 0,\ \text{where } S_2 := [I_{n_2}\ 0]T_2B_2R_2^{-1}B_2^TT_2^T \begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}.$$

Introducing

$$\psi_2(t) := K_2\hat{x}_1(t) + \hat{m}_1(t), \tag{23}$$

we obtain then similarly as above that $\dot{\psi}_2(t) = -[Q_{112}\ Q_{122}]\,\hat{x}(t) - A_2^T\psi_1(t)$. Consequently, with M as in (3) and $v^T(t) := [x^T(t),\ \psi_1^T(t),\ \psi_2^T(t)]$, $v(t)$ satisfies

$$\dot{v}(t) = Mv(t),\ \text{with } v_1(0) := [I_n\ 0\ 0]v(0) = x_0.$$

Since by assumption, for arbitrary x_0, $v_1(t)$ converges to zero it follows from e.g. [2, Lemma 7.36] that matrix M must have at least n stable eigenvalues (counting algebraic multiplicities). Moreover, the other statement follows from the second part of this lemma. Which completes this part of the proof.

"$\Leftarrow$ part" Let u_2^* be as claimed in the theorem, that is

$$u_2^*(t) = -R_2^{-1}B_2^TT_2^T\begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}\psi_2(t). \tag{24}$$

We next show that then necessarily u_1^* solves the optimization problem

$$\min_{u_1}\int_0^\infty \{\bar{x}^T(t)Q_1\bar{x}(t) + u_1^TR_1u_1(t)\}dt,$$

subject to

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B_1u_1(t) + B_2u^*(t),\ \bar{x}(0) = x_0.$$

Using the state transformation $\bar{x} = T_1\tilde{x}$ and notation of the proof of the first part of this theorem the above problem can then be rewritten as the minimization of (14) subject to the system (15).

Since, by assumption, the algebraic Riccati equation (4) has a stabilizing solution, according [2, Theorem 5.16], the above minimization problem has a solution. This solution has the same structure as the solution advertised in (16,17). The only difference is that $u_2^*(t)$ satisfies in this case (24). For that reason we will use the notation $m_1(t)$ instead of $\tilde{m}_1(t)$ in (16,17). Introducing

$$\tilde{\psi}_1(t) := K_1\tilde{x}(t) + m_1(t),$$

we have that $u_1(t) = -R_1^{-1}\tilde{B}_1^T\tilde{\psi}_1$ and similar to (20) we get that

$$\dot{\tilde{\psi}}_1 = -[Q_{111}\ Q_{121}]\tilde{x}(t) - A_1^T\tilde{\psi}_1.$$

Consequently, $x_d(t) := T_1x(t) - \tilde{x}(t)$ and $\psi_d(t) := \psi_1(t) - \tilde{\psi}_1(t)$ satisfy

$$\begin{aligned}\dot{x}_d(t) &= T_1(AT_1^{-1}T_1x(t) - B_1R_1^{-1}B_1^TT_1^T\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}\psi_1(t) - B_2R_2^{-1}B_2^TT_2^T\begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}\psi_2(t)) - \\ &\quad (T_1AT_1^{-1}\tilde{x}(t) - T_1B_1R_1^{-1}\tilde{B}_1^T\tilde{\psi}_1 - T_1B_2R_2^{-1}B_2^TT_2^T\begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}\psi_2(t))\end{aligned}$$

and

$$\dot{\psi}_d(t) = -\tilde{Q}_1T_1^{-1}T_1x(t) - A_1^T\psi_1(t) - (-[Q_{111}\ Q_{121}]\tilde{x}(t) - A_1^T\tilde{\psi}_1(t)),$$

respectively. Or, stated differently,

$$\begin{bmatrix} \dot{x}_d(t) \\ \dot{\psi}_d(t) \end{bmatrix} = \begin{bmatrix} A_1 & \tilde{A}_{12} & -S_1 \\ 0 & \tilde{A}_{22} & 0 \\ -Q_{111} & -Q_{121} & -A_1^T \end{bmatrix}\begin{bmatrix} x_d(t) \\ \psi_d(t) \end{bmatrix},\ \begin{bmatrix} x_d(0) \\ \psi_d(0) \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix},\ \text{for some } p \in I\!R^{n_1}.$$

Notice that matrix $\begin{bmatrix} A_1 & -S_1 \\ -Q_{111} & -A_1^T \end{bmatrix}$ is the Hamiltonian matrix associated with the algebraic Riccati equation (4). Recall that the spectrum of this matrix is symmetric w.r.t. the imaginary axis. Since by assumption the Riccati equation (4) has a stabilizing solution, we know that its stable invariant subspace is given by Span$[I\ K_1]^T$. Therefore, with E^u representing a basis for the unstable subspace, we can write

$$\begin{bmatrix} 0 \\ p \end{bmatrix} = \begin{bmatrix} I \\ K_1 \end{bmatrix} v_1 + E^u v_2,$$

for some vectors v_i, $i = 1, 2$. However, it is easily verified that due to our asymptotic stability assumption both $x_d(t)$ and $\psi_d(t)$ converge to zero if $t \to \infty$. So, v_2 must be zero. From this it follows now directly that $p = 0$. Since the solution of the differential equation is uniquely determined, and $[x_d(t)\ \psi_d(t)] = [0\ 0]$ solve it, we conclude that $\tilde{x}(t) = T_1 x(t)$ and $\tilde{\psi}_1(t) = \psi_1(t)$. Or stated differently, u_1^* solves the minimization problem.
In a similar way it is shown that for u_1 given by u_1^*, player two his optimal control is given by u_2^*. Which proves the claim. $\square$.

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