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Abstract

This paper analyzes a single-machine scheduling problem with family setup times both from an optimization and a cost allocation perspective. In a so-called family sequencing situation jobs are processed on a single machine, there is an initial processing order on the jobs, and every job within a family has an identical cost function that depends linearly on its completion time. Moreover, a job does not require a setup when preceded by another job from the same family while a family specific setup time is required when a job follows a member of some other family.

Explicitly taking into account admissibility restrictions due to the presence of the initial order, we show that for any subgroup of jobs there is an optimal order, such that all jobs of the same family are processed consecutively. To analyze the allocation problem of the maximal cost savings of the whole group of jobs, we define and analyze a so-called corresponding cooperative family sequencing game which explicitly takes into account the maximal cost savings for any coalition of jobs. Using nonstandard techniques we prove that each family sequencing game has a non-empty core by showing that a particular marginal vector belongs to the core. Finally, we specifically analyze the case in which the initial order is family ordered.

Keywords: Single-machine scheduling, Family scheduling model, Setup times, Cooperative game, Core, Marginal vector

JEL classification code: C71

1 Introduction

Scheduling is about the optimal planning of processing a number of jobs through a number of machines. Economies of scale are fundamental to manufacturing operations. With respect to scheduling, this phenomenon manifests itself in

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efficiencies gained from grouping jobs together. In particular, so-called family scheduling problems have received considerable attention in the scheduling literature with setup considerations. These problems consider situations where the jobs can be classified into distinct families with respect to their production requirements such as the required tooling or container size. Ahn and Hyun (1990), Bruno and Sethi (1977), Mason and Anderson (1991), and Monma and Potts (1989) propose algorithms for minimizing total weighted flowtime on a single machine with family setup times. We refer to Webster and Baker (1995) and Liaee and Emmons (1997) for a review of scheduling literature on family scheduling problems. Further we note here that sequence-dependent setup times tend to make solutions difficult to find. We refer to Allahverdi et al. (1999) and Allahverdi et al. (2008) for a review of the scheduling literature with sequence-dependent and sequence-independent setup considerations.

In this paper we restrict attention to setup considerations of the following type. A job does not require a setup when following another job from the same family, but a "family setup time" is required when it follows a member of another family. An example of a specific application of this type of family scheduling problems is a production line of colored plastics (cf. Potts and Van Wassenhove (1992)). In this setting customer orders can be divided into color groups. A setup is required when switching from a job of one color to a job of another color. Furthermore our framework assumes that per family, a cost function is defined that depends linearly on the completion time of its members. Moreover we assume that there is an initial processing order σ_0 on the jobs which provides the initial right of each job to be completed at a certain time with a given set of preceding jobs.

Santos and Magazine (1985) show that for each family, an urgency index can be computed such that, if the jobs are processed in an order of non-increasing urgency indices, then the total costs are minimized. This result however, can not be applied when considering the optimization of subgroups of jobs since the initial order on jobs puts additional constraints on the order of jobs within a subgroup. An order is admissible if each job outside the subgroup is completed at least as early as in the initial order, and its set of preceding jobs remains unchanged. In this paper we show that for each subgroup there is an optimal order which (within components) processes all jobs of the same family consecutively, but which is not necessarily the urgency order of Santos and Magazine.

To analyze the allocation problem of the maximal cost savings of the whole group of jobs, we define a cooperative family sequencing game, corresponding to the family sequencing situation, which explicitly takes into account the maximal cost savings for any coalition of jobs. The game theoretic analysis of cost allocation problems arising from sequencing situations dates back to Curiel et al. (1989) in the setting of one machine sequencing situations with a finite number of jobs, linear cost functions and an initial order. It was shown that these games are convex and hence allow for core elements; efficient allocations that can not be improved upon by a subgroup of jobs. The following studies in this strand of literature have extended the basic model by considering ready times (Hamers et al. (1995)), due dates (Borm et al. (2002)), precedence relations (Hamers et al. (2005)) and controllable processing times (van Velzen (2006)). The current paper is one of the first to explicitly incorporate setup times. In Lohmann et al. (2010) sequencing situations are analyzed where for each job some setup is required which depends on its predecessor.

We show that for our class of family sequencing games the marginal vector which corresponds to the initial order, belongs to the core of the game. It is however also seen that these games in general are neither convex, σ_0-component additive, nor permutationally convex (Granot and Huberman (1982)) with respect to the initial order. Therefore, the proof of the result above does not rely on standard techniques, but requires a tailor made analysis.

Finally, we specifically analyze the case where the initial order of jobs is such that all members of the same family are processed consecutively. In this case it turns out that all subgames in which the last job with respect to the initial order is not participating, are convex. From this we are able to derive a core element for the corresponding family sequencing game, based on the Shapley value (Shapley (1953)).

The outline of the paper is as follows. Section 2 formally describes family sequencing situations and analyzes the optimization problem of all subgroups. With respect to the associated cost allocation problem, Section 3 shows that family sequencing games have a nonempty core. In section 4, the specific case of ordered family sequencing is analyzed.

2 Family Sequencing Situations

In this section, we consider a one machine sequencing situation in which a finite number of jobs are queued in front of a machine, waiting to be processed. The machine in the situation can handle at most one job at a time. The set of jobs is denoted by N. The jobs can be partitioned into f families with respect to their production requirements. Let F be the set of families with $|F| = f$. A family function $\mathcal{F} : N \to F$ associates to each job $i \in N$ the family $\mathcal{F}(i)$ that he belongs to. We denote by n_k the number of jobs in family k.

An *order on the set of jobs* is a bijection $\sigma : N \to \{1, ..., |N|\}$. We denote the set of all orders on N by $\Pi(N)$. Given an order $\sigma \in \Pi(N)$ the set of predecessors of a job $i \in N$ with respect to σ is defined as $P(\sigma, i) = \{j \in N | \sigma(j) < \sigma(i)\}$. Similarly, the set of successors of i with respect to σ is defined as $S(\sigma, i) = \{j \in N | \sigma(j) > \sigma(i)\}$. Moreover, let $\bar{P}(\sigma, i) = P(\sigma, i) \cup \{i\}$ and $\bar{S}(\sigma, i) = S(\sigma, i) \cup \{i\}$.

It is assumed that there is an initial order σ_0 on the jobs before the processing of the machine starts. If a job in family k follows a job of the same family, then it does not require a setup. However, the family setup time $s_k > 0$ is required if it is preceded by a job of a different family or if it is the first job. Observe that the setup times are independent of the family of the preceding job. We assume that each job of the same family requires the same processing time which is denoted by $p_k > 0$ for every family $k \in F$. For each job $i \in N$, the costs $c_i(t)$ of spending time t in the system is assumed to be linear in the completion time.

We assume that all jobs of family k have the same cost parameter $\alpha_k > 0$ such that $c_i(t) = \alpha_k t$ for all $i \in \mathcal{F}^{-1}(k)$.

A one machine sequencing situation as described above is called a *family sequencing situation* and is denoted by $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ with $s, p, \alpha \in \mathbb{R}^F_{++}$. In a family sequencing situation the completion time $C(\sigma, i)$ of job i when processed according to the order σ is given by

$$C(\sigma, i) = \sum_{j \in \bar{P}(\sigma,i)} \left(x_{\sigma,j} s_{\mathcal{F}(j)} + p_{\mathcal{F}(j)} \right),$$

where $x_{\sigma,j}$ equals 1 if job j requires a setup when the jobs are processed with respect to σ and 0 otherwise, i.e.

$$x_{\sigma,j} = \begin{cases} 1 & \text{if } \sigma^{-1}(\sigma(j)-1) \notin \mathcal{F}(j) \text{ or } \sigma(j) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The total costs if the jobs are processed according to the order σ equals

$$\sum_{j \in N} \alpha_{\mathcal{F}(j)} C(\sigma, j).$$

By reordering the jobs with respect to σ_0, the total costs can be reduced. We call an order *optimal* if it minimizes the total costs. It was proven by Santos and Magazine (1985) and, independently, by Dobson et al. (1987) that a *highest urgency comes first* (HUCF) order is optimal for family sequencing situations. An HUCF order processes the jobs of the same family together as a group (consecutively) and processes these family groups in nonincreasing order of the family-specific urgency index u_k defined by $u_k = \frac{n_k \alpha_k}{s_k + n_k p_k}$. Here the the numerator indicates that a family with a high cost parameter is likely to be processed in the beginning of the optimal order, from the denominator can be seen that families with a high total processing time are processed in the tail of the optimal order.

Theorem 2.1. ***(Santos and Magazine (1985))*** *For every family sequencing situation an HUCF order is optimal.*

Example 2.1. Consider the family sequencing situation $(N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ with $N = \{1, 2, 3, 4, 5\}$ and $F = \{1, 2\}$. Assume that $\mathcal{F}(1) = \mathcal{F}(4) = \mathcal{F}(5) = 1$ and $\mathcal{F}(2) = \mathcal{F}(3) = 2$. Further assume that $\sigma_0 = (1, 2, 3, 4, 5)$, $s = (1, 5)$, $p = (5, 2)$ and $\alpha = (5, 4)$. Then the urgencies for the families are $u_1 = \frac{15}{16}$ and $u_2 = \frac{8}{9}$, respectively. Hence, an HUCF order processes the jobs in family 1 first and then the jobs in family 2. In Figure 1, we depict the processing orders σ_0 and an HUCF order σ_N.

The cost savings obtained by N when using σ_N equals

$$\begin{aligned} & \sum_{j \in N} \alpha_{\mathcal{F}(j)} (C(\sigma_0, j) - C(\sigma_N, j)) \\ = \quad & 5(6-6) + 4(13-23) + 4(15-25) + 5(21-11) + 5(26-16) = 20. \end{aligned}$$

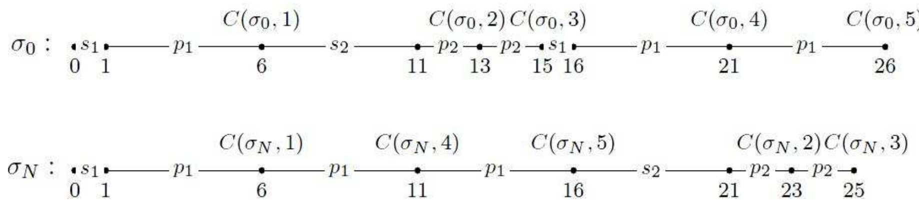

Figure 1: The orders σ_0 and σ_N.

$\diamondsuit$

For a family sequencing situation $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$, the costs of a subgroup T with respect to an order σ equals $\sum_{j \in T} \alpha_{\mathcal{F}(j)} C(\sigma, j)$. We want to determine the maximal cost savings of T when its members decide to cooperate. For this aim, we have to specify which orders are admissible for T with respect to the initial order. We assume that an order $\sigma \in \Pi(N)$ is *admissible* for a subgroup T with respect to σ_0 if it satisfies the following two conditions:

(i) $P(\sigma, i) = P(\sigma_0, i)$ for all $i \in N \backslash T$, and

(ii) $C(\sigma, i) \leq C(\sigma_0, i)$ for all $i \in N \backslash T$.

Condition (i) is the standard admissibility requirement in the sequencing literature and requires that T can achieve cost savings only by changing jobs within its σ_0-components, being the maximally connected subsets of T with respect to σ_0. However, in a family sequencing situation, a subgroup may negatively affect the jobs outside the subgroup by reordering its jobs within σ_0-components. Hence, we also adopt condition (ii) which guarantees that T cannot not harm the jobs outside T. The set of admissible orders of T is denoted by $\mathcal{A}(T)$. Then, the corresponding optimization problem for T in family sequencing situation $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ is given by

$$\min_{\sigma \in \mathcal{A}(T)} \sum_{j \in T} \alpha_{\mathcal{F}(j)} C(\sigma, j).$$

An admissible order for which the minimum is attained is called *optimal* for T.

Example 2.2. Reconsider the family sequencing situation from Example 2.1. Now consider the subgroup $S = \{1, 2, 3\}$. Observe that the order $\sigma = (2, 3, 1, 4, 5)$ is an admissible order for S with respect to σ_0: $P(\sigma, i) = P(\sigma_0, i)$ for all $i \in N \backslash S$ and both $C(\sigma, 4) \leq C(\sigma_0, 4)$ and $C(\sigma, 5) \leq C(\sigma_0, 5)$. This can be seen from Figure 2.
The cost savings obtained by S when using σ equals

$$\sum_{j \in S} \alpha_{\mathcal{F}(j)} (C(\sigma_0, j) - C(\sigma, j)) = 5(6 - 15) + 4(13 - 7) + 4(15 - 9) = 3.$$

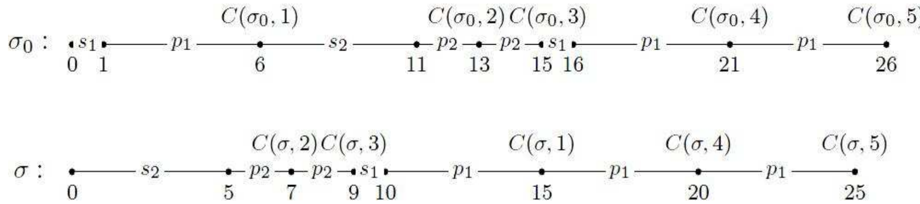

Figure 2: The orders σ_0 and σ.

Actually, σ is an optimal order for S. Notice that σ processes the jobs in family 2 first although with respect to the optimal order for all jobs, jobs in family 1 are processed first.

Now consider the subgroup $T = \{1, 2, 3, 5\}$. Clearly, σ is an admissible order for T. Actually, σ is also an optimal order for T. The cost savings obtained by T using σ equals

$$\begin{aligned}
\sum_{j \in T} \alpha_{\mathcal{F}(j)}(C(\sigma_0, j) - C(\sigma, j)) &= \sum_{j \in S} \alpha_{\mathcal{F}(j)}(C(\sigma_0, j) - C(\sigma, j)) \\
&\quad + \alpha_1(C(\sigma_0, 5) - C(\sigma, 5)) \\
&= 3 + 5(26 - 25) \\
&= 8.
\end{aligned}$$

That is, when the jobs 1,2 and 3 reorder themselves from σ_0 to σ, also job 5 profits from an earlier completion time and this profit is now taken into account. $\diamond$

Let $T \subset N$ and $\sigma \in \Pi(N)$ such that T is a connected subgroup with respect to σ. Define,

$$f(\sigma, T) = \arg\min_{j \in T} \ \sigma(j),$$

and

$$l(\sigma, T) = \arg\max_{j \in T} \ \sigma(j).$$

Clearly, $f(\sigma, T)$ is the first job within T, and $l(\sigma, T)$ the last job within T with respect to the order σ. We denote by $P(\sigma, T)$ the collection of jobs which stand in front of every member of T in the order σ, i.e.,

$$P(\sigma, T) = \{i \in N | \sigma(i) < \sigma(f(\sigma, T))\}$$

and $S(\sigma, T)$ the collection of jobs which stand behind every member of T in the order σ,

$$S(\sigma, T) = \{i \in N | \sigma(i) > \sigma(l(\sigma, T))\}.$$

Let $\sigma \in \Pi(N)$. We call a set of jobs R that are processed between two setups when the jobs are processed with respect to σ, a *run of* σ. Obviously, all jobs in the same run are of the same family. A run which consists of jobs of family k is called a *run of family k*.

An order σ is *family ordered* if it processes all jobs that belong to the same family consecutively, i.e. if for every pair of jobs $i, j \in N$ where $\mathcal{F}(i) = \mathcal{F}(j)$ it holds that $h \in \mathcal{F}(i)$ for every h with $\sigma(i) < \sigma(h) < \sigma(j)$.

Since a σ_0-component of a subgroup can affect the completion times of the members of another σ_0-component behind it (cf. Example 2.2), it is generally not easy to find an optimal admissible order for a subgroup. Nevertheless, there are useful properties regarding the structure of optimal admissible orders. In the following theorem we show that for every connected subgroup, there exist an optimal admissible order that processes the jobs of the same family consecutively.

Theorem 2.2. *Let $(N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ be a family sequencing situation and let $T \subset N$ be a subset of jobs. Then, there exists an optimal order for T which processes all jobs of the same family within a σ_0-component of T consecutively.*

Proof. Let $T = T_1 \cup T_2 \cup \cdots \cup T_l$ where for each $y \in \{1, \ldots, l\}$, T_y is a maximally connected subset of T with respect to σ_0. Let $\sigma \in \mathcal{A}(T)$ and suppose σ is an optimal order for T.

Fix $y \in \{1, 2, ..., l\}$ and suppose that with respect to σ, family k jobs in T_y are processed in different runs. Let K_1 and K_2 be the set of family k jobs in T_y that belong to the first and second run, respectively. Let M be the set of jobs (of other families) that are placed in between K_1 and K_2 with respect to σ. Let γ be the time to process and setup all jobs in M when they are processed with respect to σ, i.e., $\gamma = \sum_{j \in M}(x_{\sigma,j} s_{\mathcal{F}(j)} + p_{\mathcal{F}(j)})$. Let $i_1 = f(\sigma, K_1)$, $i_2 = f(\sigma, K_2)$, $m = f(\sigma, M)$, and $h = f(\sigma, S(\sigma, K_2))$.

Now consider the order $\sigma' \in \Pi(N)$ which is obtained from σ by moving all jobs in K_1 to the head of K_2 and the order $\sigma'' \in \Pi(N)$ which is obtained from σ by moving all jobs in K_2 to the tail of K_1. Figure 3 depicts the orders σ, σ' and σ''.

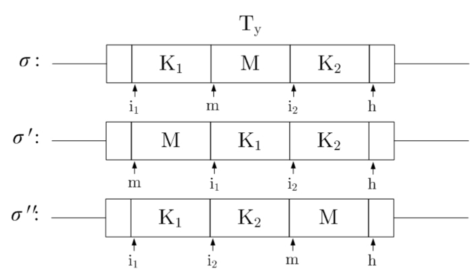

Figure 3: The orders σ, σ' and σ''.

We start by showing that σ' and σ'' are admissible. Next we show that either σ' or σ'' is also an optimal order, i.e.

$$\sum_{j\in T}\alpha_{\mathcal{F}(j)}(C(\sigma,j)-C(\sigma',j))\geq 0 \tag{1}$$

or

$$\sum_{j\in T}\alpha_{\mathcal{F}(j)}(C(\sigma,j)-C(\sigma'',j))\geq 0. \tag{2}$$

First we show that σ' is an admissible order for T. Clearly, $P(\sigma',i)=P(\sigma,i)=P(\sigma_0,i)$ for all $i\in N\backslash T$. It remains to show that $C(\sigma',i)\leq C(\sigma_0,i)$ for every $i\in N\backslash T$. Since σ is an admissible order, it is sufficient to show that $C(\sigma',i)\leq C(\sigma,i)$ for every $i\in N\backslash T$. Observe that $x_{\sigma',j}=x_{\sigma,j}$ for every $j\in N\backslash\{i_1,i_2,m\}$. Hence, $C(\sigma',i)=C(\sigma,i)$ for every $i\in P(\sigma,K_1)$ and

$$\begin{aligned}
C(\sigma,i)-C(\sigma',i) &= \sum_{j\in\bar{P}(\sigma,i)}\left(s_{\mathcal{F}(j)}x_{\sigma,j}+p_{\mathcal{F}(j)}\right)-\sum_{j\in\bar{P}(\sigma',i)}\left(s_{\mathcal{F}(i)}x_{\sigma',i}+p_{\mathcal{F}(i)}\right)\\
&= \sum_{j\in\{i_1,i_2,m\}}\left(x_{\sigma,j}-x_{\sigma',j}\right)s_{\mathcal{F}(j)}\\
&\geq 0,
\end{aligned}$$

for every $i\in S(\sigma,M)$. The inequality follows from $x_{\sigma,m}=1$ which implies $(x_{\sigma,m}-x_{\sigma',m})s_{\mathcal{F}(m)}\geq 0$. Further it holds that $x_{\sigma,i_2}=1$, $x_{\sigma',i_1}+x_{\sigma',i_2}=1$, and $s_{\mathcal{F}(i_1)}=s_{\mathcal{F}(i_2)}$ such that $\sum_{j\in\{i_1,i_2\}}\left(x_{\sigma,j}-x_{\sigma',j}\right)s_{\mathcal{F}(j)}=x_{\sigma,i_1}s_{\mathcal{F}(i_1)}\geq 0$. Hence, $\sigma'\in\mathcal{A}(T)$.

Next we show that σ'' is an admissible order for T. Obviously, $P(\sigma'',i)=P(\sigma_0,i)$ for every $i\in N\backslash T$. Since σ is an admissible order, it is sufficient to show that $C(\sigma'',i)\leq C(\sigma,i)$ for every $i\in N\backslash T$. It can be observed that $x_{\sigma'',i}=x_{\sigma,i}$ for every $i\in N\backslash\{i_2,h\}$. Hence, $C(\sigma'',i)=C(\sigma,i)$ for every $i\in P(\sigma,M)$ and

$$\begin{aligned}
C(\sigma,i)-C(\sigma'',i) &= \sum_{j\in\bar{P}(\sigma,i)}\left(s_{\mathcal{F}(j)}x_{\sigma,j}+p_{\mathcal{F}(j)}\right)\\
&\quad -\sum_{j\in\bar{P}(\sigma'',i)}\left(s_{\mathcal{F}(j)}x_{\sigma'',j}+p_{\mathcal{F}(j)}\right)\\
&= \sum_{j\in\{i_2,h\}}(x_{\sigma,j}-x_{\sigma'',j})s_{\mathcal{F}(j)},
\end{aligned}$$

for every $i\in S(\sigma,K_2)$. Clearly, $x_{\sigma,i_2}=1$ and $x_{\sigma'',i_2}=0$. However, $x_{\sigma,h}$ can either be 0 or 1. First assume that $x_{\sigma,h}=0$, i.e., h is member of family k. Hence, $x_{\sigma'',h}=1$ and

$$\sum_{j\in\{i_2,h\}}(x_{\sigma,j}-x_{\sigma'',j})s_{\mathcal{F}(j)}=(1-0)s_{\mathcal{F}(i_2)}+(0-1)s_{\mathcal{F}(h)}=(1-0)s_k+(0-1)s_k=0.$$

Next assume that $x_{\sigma,h}=1$. Then,

$$\begin{aligned}
\sum_{j\in\{i_2,h\}}(x_{\sigma,j}-x_{\sigma'',j})s_{\mathcal{F}(j)} &= (1-0)s_k+(1-x_{\sigma'',h})s_{\mathcal{F}(h)}\\
&\geq (1-0)s_k+(1-1)s_{\mathcal{F}(h)}=s_k.
\end{aligned}$$

Hence $C(\sigma,i)-C(\sigma'',i)\geq 0$ for every $i\in S(\sigma,K_2)$. This yields that $\sigma''\in\mathcal{A}(T)$.

To prove Theorem 2.2 it is sufficient to prove that either (1) or (2) is satisfied. First observe that

$$\sum_{j\in T}\alpha_{\mathcal{F}(j)}(C(\sigma,j)-C(\sigma',j))\geq\sum_{j\in K_1\cup M}\alpha_{\mathcal{F}(j)}\left(C(\sigma,j)-C(\sigma',j)\right), \tag{3}$$

where the inequality follows from the fact that $C(\sigma',i)=C(\sigma,i)$ for every $i\in P(\sigma,K_1)$ and $C(\sigma,i)-C(\sigma',i)\geq 0$ for every $i\in S(\sigma,M)$. Moreover, for every $i\in M$ it holds that

$$\begin{aligned}
C(\sigma,i)-C(\sigma',i) &= \sum_{j\in K_1}(x_{\sigma,j}s_{\mathcal{F}(j)}+p_{\mathcal{F}(j)})+(x_{\sigma,m}-x_{\sigma',m})s_{\mathcal{F}(m)}\\
&= x_{\sigma,i_1}s_k+|K_1|p_k+(x_{\sigma,m}-x_{\sigma',m})s_{\mathcal{F}(m)}\\
&= x_{\sigma,i_1}s_k+|K_1|p_k+(1-x_{\sigma',m})s_{\mathcal{F}(m)}\\
&\geq |K_1|p_k,
\end{aligned}\tag{4}$$

and for every $i\in K_1$

$$\begin{aligned}
C(\sigma,i)-C(\sigma',i) &= -\sum_{j\in M}(x_{\sigma',j}s_{\mathcal{F}(j)}+p_{\mathcal{F}(j)})-(x_{\sigma',i_1}-x_{\sigma,i_1})s_k\\
&\geq -(\gamma+s_k).
\end{aligned}\tag{5}$$

The inequality follows from $x_{\sigma',i_1}=1$ and

$$\sum_{j\in M}(x_{\sigma',j}s_{\mathcal{F}(j)}+p_{\mathcal{F}(j)})=\begin{cases}\gamma, & \text{if } x_{\sigma',m}=1,\\ \gamma-s_{\mathcal{F}(m)}, & \text{if } x_{\sigma',m}=0.\end{cases}$$

Consequently, by inequalities (3)-(5), we have that

$$\sum_{j\in T}\alpha_{\mathcal{F}(j)}\left(C(\sigma,j)-C(\sigma',j)\right)\geq|K_1|\left(p_k\sum_{j\in M}\alpha_{\mathcal{F}(j)}-(\gamma+s_k)\alpha_k\right). \tag{6}$$

Secondly, observe that

$$\sum_{j\in T}\alpha_{\mathcal{F}(j)}(C(\sigma,j)-C(\sigma'',j))\geq\sum_{j\in K_2\cup M}\alpha_{\mathcal{F}(j)}\left(C(\sigma,j)-C(\sigma'',j)\right), \tag{7}$$

which follows from $C(\sigma'',i)=C(\sigma,i)$ for every $i\in P(\sigma,M)$ and $C(\sigma,i)-C(\sigma'',i)\geq 0$ for every $i\in S(\sigma,K_2)$. Moreover,

$$C(\sigma,i)-C(\sigma'',i)=\begin{cases}\gamma+s_k, & \text{if } i\in K_2,\\ -|K_2|p_k, & \text{if } i\in M.\end{cases} \tag{8}$$

which implies that

$$\sum_{j\in T}\alpha_{\mathcal{F}(j)}(C(\sigma,j)-C(\sigma'',j))\geq|K_2|\left((\gamma+s_k)\alpha_k-p_k\sum_{j\in M}\alpha_{\mathcal{F}(j)}\right). \tag{9}$$

Either the righthandside in (6) or (9) is non-negative which shows that either (1) or (2) is satisfied. $\square$

From Theorem 2.2 it follows that for the optimization problem for a subgroup of jobs, the urgency indices of the jobs are not the only factor to take into consideration. Apparently, the structure of families within the subgroup is also of concern.

Let $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ be a family sequencing situation and let T be a connected subset with respect to σ_0. The *family urgency index* $u_{T,k}$ for T is defined as

$$u_{T,k} = \frac{n_{T,k}\alpha_k}{s_k + n_{T,k}p_k},$$

for all $k \in \mathcal{F}(T)$, where $n_{T,k}$ is the number jobs of family k in T and $\mathcal{F}(T) = \bigcup_{i\in T} \mathcal{F}(i)$ is the set of families associated to T.

Now assume that $\mathcal{F}(l(\sigma_0, T)) = \bar{k}(T)$. The *tail-adjusted family urgency index* $u'_{T,l}$ for T is defined as

$$u'_{T,l} = \begin{cases} u_{T,l} & \text{if } l \neq \bar{k}(T), \\ 0 & \text{if } l = \bar{k}(T). \end{cases}$$

for all $l \in \mathcal{F}(T)$.

An order $\sigma \in \Pi(N)$ is called *an HUCF order for* T, where T is connected with respect to σ_0, if

(i) $P(\sigma, i) = P(\sigma_0, i)$ for every $i \in N\backslash T$, and

(ii) σ is family ordered and processes the family groups in non-increasing order of the family urgency index for T.

An order $\sigma \in \Pi(N)$ is called *a tail-adjusted HUCF order for* T, where T is connected with respect to σ_0, if

(i) $P(\sigma, i) = P(\sigma_0, i)$ for every $i \in N\backslash T$, and

(ii) σ is family ordered and processes the family groups in non-increasing order of the tail-adjusted family urgency index for T.

Example 2.3. Reconsider the family sequencing situation from Example 2.2. Consider again $S = \{1, 2, 3\}$. Then, $\mathcal{F}(S) = \{1, 2\}$ and $u_{S,1} = \frac{5}{6}$ and $u_{S,2} = \frac{8}{9}$, respectively. Since $\bar{k}(S) = 2$, the tail-adjusted urgencies in S are $u'_{S,1} = \frac{5}{6}$ and $u'_{S,2} = 0$, respectively. Hence, $\sigma = (2, 3, 1, 4, 5)$ is an HUCF order and $\sigma' = \sigma_0 = (1, 2, 3, 4, 5)$ is a tail-adjusted HUCF order for S. $\diamondsuit$

Next we focus on the structure of the optimal orders for connected subgroups which include the job that is processed first with respect to σ_0. In the following lemma, we show that for connected subgroups of this type, either an HUCF order or a tail-adjusted HUCF order is optimal.

Lemma 2.1. *Let $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ be a family sequencing situation and T a connected subgroup with respect to σ_0 with $\sigma_0^{-1}(1) \in T$. Then,*

(i) If an HUCF order for T is admissible, then it is optimal for T.

(ii) If an HUCF order for T is not admissible, then a tail-adjusted HUCF order for T is optimal for T.

Proof. Let $\sigma \in \Pi(N)$ be an HUCF order for T.

(i) Let $\sigma \in \mathcal{A}(T)$. From Theorem 2.1 and Theorem 2.2 it immediately follows that σ is optimal.

(ii) Let $\sigma \notin \mathcal{A}(T)$. Then, clearly, $S(\sigma_0, T) \neq \emptyset$. Define $h = f(\sigma_0, S(\sigma_0, T))$. We first prove the following claim:

Claim: Let $\pi \in \Pi(N)$ be such that $P(\pi, i) = P(\sigma_0, i)$ for all $i \in N \setminus T$, π is family ordered for T, and $\pi \notin \mathcal{A}(T)$. Then $C(\pi, h) > C(\sigma_0, h)$, $x_{\pi,h} = 1$ and $x_{\sigma_0,h} = 0$.

Proof of the claim: Since $\pi \notin \mathcal{A}(T)$ and $P(\pi, i) = P(\sigma_0, i)$ for all $i \in N \setminus T$, it holds that

$$C(\pi, i) > C(\sigma_0, i), \tag{10}$$

for some $i \in S(\pi, T)$. Since $\sigma_0^{-1}(1) \in T$ and since π is family ordered, the number of setups in T are minimized, which implies the total setup time is minimized, and, consequently

$$C(\pi, l(\pi, T)) \leq C(\sigma_0, l(\sigma_0, T)).$$

Hence, since π and σ_0 coincide after the last job in T, for (10) to hold it must be the case that $x_{\pi,h} = 1$ and $x_{\sigma_0,h} = 0$, while in fact $C(\pi, h) > C(\sigma_0, h)$. This proves the claim. Clearly, the claim implies that

$$C(\sigma, h) > C(\sigma_0, h), \ x_{\sigma,h} = 1, \text{ and } x_{\sigma_0,h} = 0. \tag{11}$$

Let $\sigma' \in \Pi(N)$ denote a tail-adjusted HUCF order for T. Since σ' is family ordered, $P(\sigma', i) = P(\sigma_0, i)$ for all $i \in N \setminus T$, and $x_{\sigma',h} = 0$ we readily see from the claim that $\sigma' \in \mathcal{A}(T)$.

Now consider an arbitrary order $\tau \in \Pi(N)$ with $\tau \in \mathcal{A}(T)$. We prove that all jobs in T that are within $\mathcal{F}(h)$ are processed last in T. From (11) it follows that $T \cap \mathcal{F}(h) \neq \emptyset$. Now suppose $x_{\tau,h} = 1$ or that at least two members of $\mathcal{F}(h)$ within T are processed in different runs. Then,

$$\begin{aligned} C(\tau, h) &\geq \sum_{k \in \mathcal{F}(T)} (s_k + n_{T,k} p_k) + s_{\mathcal{F}(h)} + p_{\mathcal{F}(h)} \\ &= C(\sigma, h) \\ &> C(\sigma_0, h). \end{aligned}$$

The equality holds by the fact that σ is an HUCF order and $x_{\sigma,h} = 1$. The strict inequality follows from (11). This establishes a contradiction with the admissibility of τ. Hence, for each admissible order there is no setup required for job h, and the members of $\mathcal{F}(h)$ are not processed in different runs in T. This implies that for each admissible order all members of $\mathcal{F}(h)$ are sent to the back of T. The tail-adjusted HUCF order σ' satisfies this condition.

It remains to prove that σ' is optimal for the set $T' = \{i \in T | \mathcal{F}(i) \neq \mathcal{F}(h)\}$. This is obvious by applying *(i)* since σ' is defined as a HUCF order for T' which is admissible for T'.

$\square$

3 Family Sequencing Games

A *transferable utility* (TU) *game* is an ordered pair (N, v) where N is the finite set of players, and v the characteristic function v on 2^N, the collection of all subsets of N. The function v assigns to every *coalition* $T \in 2^N$ a real number $v(T)$ with $v(\emptyset) = 0$. Here, $v(T)$ is called the worth or value of the coalition T. The set of all TU-games with player set N is denoted by TU^N. Where no confusion arises, we write v rather than (N, v). A game v is called *monotonic* if $v(S) \leq v(T)$ for every $S \subset T$ and v is called *superadditive* if $v(S)+v(T) \leq v(S \cup T)$ for every $S, T \in 2^N$ with $T \cap S = \emptyset$. A game v is *convex* if a player's marginal contribution does not decrease if he joins a larger coalition, i.e., $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for every $i \in N$ and $S, T \subset N \backslash \{i\}$ with $S \subset T$.

The *core*, denoted by $Core(v)$, of a game v is defined as the set of efficient allocations for which no coalition has an incentive to split off from the grand coalition, i.e.,

$$Core(v) = \left\{ x \in \mathbb{R}^N \middle| \sum_{j \in N} x_j = v(N) \text{ and } \sum_{j \in S} x_j \geq v(S) \text{ for all } S \in 2^N \right\}.$$

A game with a nonempty core is called *balanced*.

A coalition $S \subset N$ is called *connected with respect to an order* $\sigma \in \Pi(N)$ if for all $i, j \in S$ and $h \in N$ such that $\sigma(i) < \sigma(h) < \sigma(j)$ it holds that $h \in S$. We denote with $con(\sigma)$ the set of coalitions that are connected with respect to σ. For a coalition S, $S \backslash \sigma$ denotes the set of σ-components of S.

Let $\sigma \in \Pi(N)$. A TU-game v is called σ*-component additive* if it satisfies the following three conditions:

(i) $v(\{i\}) = 0$ for all $i \in N$, and

(ii) v is superadditive, and

(iii) $v(S) = \sum_{T \in S \backslash \sigma} v(T)$ for all $S \in 2^N$.

Le Breton et al. (1992) showed that σ-component additive games are balanced.

In a *family sequencing game* corresponding to a family sequencing situation, players will correspond to jobs, and the value of a coalition T is defined as the maximum cost savings coalition T can achieve by means of an admissible orders in $\mathcal{A}(T)$. Formally, the family sequencing game v corresponding to a family sequencing situation $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ is defined by

$$v(T) = \max_{\sigma \in \mathcal{A}(T)} \left\{ \sum_{j \in T} \alpha_{\mathcal{F}(j)} \;\; (C(\sigma_0, j) - C(\sigma, j)) \right\}, \tag{12}$$

for every $T \subset N$. It readily follows that v is monotonic.

Example 3.1. Reconsider the family sequencing situation of Examples 2.1 - 2.3. Since $\sigma = (2,3,1,4,5)$ is an optimal order for $S = \{1,2,3\}$ and $T = \{1,2,3,5\}$ it holds for the corresponding family sequencing game v, that $v(S) = 3$ and $v(T) = 8$. From this can be seen that the game is not σ_0-component additive; $\{1,2,3\}$ and $\{5\}$ are the σ_0-components of $\{1,2,3,5\}$, but $v(\{1,2,3,5\}) = 8 \neq 3 = v(\{1,2,3\}) + v(\{5\})$.

The complete family sequencing game v is given by: $v(N) = v(\{2,3,4,5\}) = 20$, $v(\{1,2,3,4\}) = v(\{2,3,4\}) = 10$, $v(\{1,2,3,5\}) = 8$, $v(\{1,2,3\}) = 3$ and $v(S) = 0$ for every remaining coalition $S \in 2^N$. Also observe that this game is not convex

$$v(N) - v(\{2,3,4,5\}) = 0 < 3 = v(\{1,2,3\}) - v(\{2,3\}).$$

$\Diamond$

Let $v \in TU^N$. The *marginal vector* $m^\sigma(v) \in \mathbb{R}^N$ with respect to $\sigma \in \Pi(N)$ is defined to be the vector with for each $i \in N$

$$m_i^\sigma(v) = v(\bar{P}(\sigma, i)) - v(P(\sigma, i)),$$

where $P(\sigma, i)$ is the set of predecessors of job $i \in N$ and $\bar{P}(\sigma, i) = P(\sigma, i) \cup \{i\}$.

A concept that is closely related to convexity is permutationally convexity. The game v is said to be *permutationally convex* with respect to $\sigma \in \Pi(N)$ if

$$v(\bar{P}(\sigma, i) \cup T) - v(\bar{P}(\sigma, i)) \leq v(\bar{P}(\sigma, j) \cup T) - v(\bar{P}(\sigma, j)),$$

for every $i, j \in N$ with $\sigma(i) < \sigma(j)$ and $T \subset S(\sigma, j)$. Permutational convexity with respect to an order $\sigma \in \Pi(N)$ is a well-known sufficient condition for the corresponding marginal vector $m^\sigma(v)$ to be a core element (cf. Granot and Huberman (1982)). In the following example we show that family sequencing games need not be permutationally convex with respect to the initial order σ_0.

Example 3.2. Consider the family sequencing situation $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ with $N = \{1,2,3,4,5,6\}$ and $F = \{1,2,3,4\}$. Assume that $\mathcal{F}(1) = \mathcal{F}(3) = 1$, $\mathcal{F}(2) = 2$, $\mathcal{F}(4) = \mathcal{F}(5) = 3$ and $\mathcal{F}(6) = 4$. Furthermore, let $\sigma_0 = (1,2,3,4,5,6)$, $s = (2,2,1,5)$, $p = (1,2,2,5)$ and $\alpha = (10,10,10,1)$. Finally, let v be the family sequencing game corresponding to $\Sigma(N)$.

Consider the coalitions

$$S = \bar{P}(\sigma_0, 3) = \{1, 2, 3\}, \qquad S' = \bar{P}(\sigma_0, 3) \cup \{6\} = \{1, 2, 3, 6\},$$
$$W = \bar{P}(\sigma_0, 4) = \{1, 2, 3, 4\}, \qquad W' = \bar{P}(\sigma_0, 4) \cup \{6\} = \{1, 2, 3, 4, 6\}.$$

The urgency indices for S are $u_{S,1} = \frac{20}{4}$ and $u_{S,2} = \frac{10}{4}$. Hence, $\sigma_S = (1, 3, 2, 4, 5, 6)$ is an HUCF order for S. Clearly, σ_S is admissible for S. Then, by Lemma 2.1, σ_S is optimal for S.

The urgency indices for W are $u_{W,1} = \frac{20}{4}$, $u_{W,2} = \frac{10}{4}$ and $u_{W,3} = \frac{10}{3}$. Hence, $\sigma_W = (1, 3, 4, 2, 5, 6)$ is an HUCF order for W. It can easily be observed that σ_W is admissible for W. Then, by Lemma 2.1, σ_W is optimal for W.

Clearly, σ_S is also an optimal order for S' while σ_W is also an optimal order for W'. Then, v is not permutationally convex with respect to σ_0 since

$$2 = v(S') - v(S) > v(W') - v(W) = 1.$$

$\diamondsuit$

In the literature on sequencing games, balancedness of a game v is often proved by using the fact that $v \in TU^N$ is σ_0-component additive. However, in Example 3.1 is shown that family sequencing games need not be σ_0-component additive. For sequencing games with controllable processing times, van Velzen (2006) proved balancedness by using the property of permutationally convexity. From Example 3.2 it can be seen that family sequencing games are not permutationally convex with respect to σ_0. Hence, the balancedness of family sequencing games can not be proved using standard techniques. However, a direct but technically intricate proof shows the marginal vector that corresponds to initial order σ_0 does belongs to the core of a family sequencing game. For this proof we introduce some orders which we prove to be admissible and hence have a completion time smaller than or equal to the completion time of the initial order.

Theorem 3.1. *Let $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ be a family sequencing situation and let $v \in TU^N$ be the corresponding sequencing game. Then, $m^{\sigma_0}(v) \in Core(v)$.*

Proof. Let $T \backslash \sigma_0 = \{T_1, T_2, ..., T_l\}$ be such that $T_y \subset P(\sigma_0, T_{y+1})$ for every $y \in \{1, ..., l-1\}$. Let $\sigma \in \mathcal{A}(T)$ be an optimal order for T.

We have to show that $\sum_{j \in T} m_j^{\sigma_0}(v) \geq v(T)$. Since

$$\begin{aligned} v(T) &= \sum_{j \in T} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j) \right) \\ &= \sum_{y \in \{1,2,\ldots,l\}} \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j) \right), \end{aligned}$$

and

$$\sum_{j \in T} m_j^{\sigma_0}(v) = \sum_{y \in \{1,2,\ldots,l\}} \sum_{j \in T_y} m_j^{\sigma_0}(v),$$

it is sufficient to show that

$$\sum_{j \in T_y} m_j^{\sigma_0}(v) \geq \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j) \right), \tag{13}$$

for every $y \in \{1, 2, ..., l\}$.

Pick $y \in \{1, 2, ..., l\}$. Let $D = P(\sigma_0, T_y)$ and $E = D \cup T_y$. If $D = \emptyset$, then $y = 1$ and (13) follows from Lemma 2.1 and Theorem 2.2 which implies that σ is also an optimal order for T_1. Hence,

$$\sum_{j \in T_1} m_j^{\sigma_0}(v) = v(T_1) = \sum_{j \in T_1} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j) \right).$$

For the remainder, let $D \neq \emptyset$. Notice that $\sigma_0^{-1}(1) \in D$ and $\sum_{j \in T_y} m_j^{\sigma_0}(v) = v(E) - v(D)$. Hence inequality (13) boils down to

$$v(E) - v(D) - \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j) \right) \geq 0. \tag{14}$$

In the remainder of this proof we consider an order μ for E which is the combination of an optimal order for T_y and an optimal order for D. If μ is admissible for E, then inequality (14) can be shown directly. If μ is not admissible, then we construct an adjusted admissible order μ' to indirectly verify (14).

First, we denote by σ_y and by $\sigma_{\vec{y}}$ the orders defined by

$$\sigma_y(i) = \begin{cases} \sigma(i), & \text{if } i \in T_y, \\ \sigma_0(i), & \text{otherwise,} \end{cases}$$

$$\sigma_{\vec{y}}(i) = \begin{cases} \sigma(i), & \text{if } i \in \bigcup_{q=1}^{y} T_q, \\ \sigma_0(i), & \text{otherwise.} \end{cases}$$

Notice that $\sigma_{\vec{y}}$ is an admissible order for E. Moreover, let π be an optimal order for D. Then,

$$v(D) = \sum_{j \in D} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi, j) \right).$$

Since $\sigma_0^{-1}(1) \in D$, we can choose π to be either an HUCF order for D or a tail-adjusted HUCF order for D (cf. Lemma 2.1). Let $\mu \in \Pi(N)$ be the order defined by

$$\mu(i) = \begin{cases} \pi(i), & \text{if } i \in D, \\ \sigma_y(i), & \text{otherwise.} \end{cases}$$

Observe that $P(\sigma_0, i) = P(\mu, i)$ for every $i \in N \backslash E$.

Let $i_y = f(\sigma, T_y)$ and $k = \mathcal{F}(i_y)$. For all $i \in S(\sigma_0, D)$, observe that

$$\begin{aligned} C(\sigma_{\vec{y}}, i) - C(\mu, i) &= \sum_{j \in \bar{P}(\sigma_{\vec{y}}, i)} \left(s_{\mathcal{F}(j)} x_{\sigma_{\vec{y}}, j} + p_{\mathcal{F}(j)}\right) \qquad (15) \\ &\quad - \sum_{j \in \bar{P}(\mu, i)} \left(s_{\mathcal{F}(j)} x_{\mu, j} + p_{\mathcal{F}(j)}\right) \\ &= \sum_{j \in D} s_{\mathcal{F}(j)} (x_{\sigma_{\vec{y}}, j} - x_{\mu, j}) + s_k (x_{\sigma_{\vec{y}}, i_y} - x_{\mu, i_y}) \\ &\geq s_k (x_{\sigma_{\vec{y}}, i_y} - x_{\mu, i_y}). \qquad (16) \end{aligned}$$

Here the second equality follows from the fact that $\bar{P}(\sigma_{\vec{y}}, i) = \bar{P}(\mu, i)$ for every $i \in S(\sigma_0, D)$ and $x_{\sigma_{\vec{y}}, i} = x_{\mu, i}$ for every $i \in S(\sigma_0, D) \backslash \{i_y\}$. The inequality follows from the fact that, with respect to μ, the members of D are processed according to π which is an HUCF or a tail-adjusted HUCF order for D and these orders require the minimum total setup time to process the jobs in D. Hence, for all $h \in \mathcal{F}(D)$ it holds that

$$\sum_{j \in D, j \in \mathcal{F}^{-1}(h)} x_{\sigma_{\vec{y}}, j} \geq 1,$$

and

$$\sum_{j \in D, j \in \mathcal{F}^{-1}(h)} x_{\mu, j} = 1.$$

We now distinguish between two cases.

First assume $x_{\sigma_{\vec{y}}, i_y} - x_{\mu, i_y} \geq 0$. Then, μ is admissible for E. For this, first observe that $P(\sigma_0, i) = P(\mu, i)$ for every $i \in N \backslash E = S(\sigma_0, E)$. Next, by admissibility of $\sigma_{\vec{y}}$, it holds that $C(\sigma_0, i) \geq C(\sigma_{\vec{y}}, i)$ for every $i \in S(\sigma_0, E)$. Hence, by inequality (16) it holds that $C(\sigma_{\vec{y}}, i) \geq C(\mu, i)$ for every $i \in S(\sigma_0, E)$. We may conclude that μ is an admissible order for E and that

$$v(E) \geq \sum_{j \in E} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\mu, j)\right). \qquad (17)$$

Combining (16) and (17) one obtains

$$\begin{aligned} & v(E) - v(D) - \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) \\ \geq\ & \sum_{j \in E} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\mu, j)\right) - \sum_{j \in D} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi, j)\right) \\ & - \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, i) - C(\sigma, j)\right) \\ =\ & \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma, i) - C(\mu, j)\right) \\ =\ & \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_{\vec{y}}, i) - C(\mu, j)\right) \end{aligned}$$

$$\geq \quad s_k(x_{\sigma_{\vec{y}},i_y} - x_{\mu,i_y}) \sum_{j \in T_y} \alpha_{\mathcal{F}(j)}$$
$$\geq \quad 0.$$

The first equality follows from the fact that $C(\pi, i) = C(\mu, i)$ for all $i \in D$. The second equality holds by $C(\sigma, i) = C(\sigma_{\vec{y}}, i)$ for all $i \in T_y$. The second inequality follows from (16). Hence, inequality (14) is satisfied and the proof is finished.

Secondly, assume $x_{\sigma_{\vec{y}},i_y} - x_{\mu,i_y} < 0$, i.e., for the rest of the proof assume that $x_{\sigma_{\vec{y}},i_y} = 0$ and $x_{\mu,i_y} = 1$. Since $x_{\sigma_{\vec{y}},i_y} = 0$, $\mathcal{F}(i_y) = \mathcal{F}(l(\sigma_y, D)) = \mathcal{F}(l(\sigma_{\vec{y}}, D)) = k$. Observe that $l(\sigma_{\vec{y}}, D) = l(\sigma_0, D)$. Moreover, from $x_{\mu,i_y} = 1$ it follows that $\mathcal{F}(l(\mu, D)) \neq k$. Since $\mathcal{F}(l(\mu, D)) = \mathcal{F}(l(\pi, D))$ this implies that π can not be a tail-adjusted HUCF order for D. So, π can be chosen to be an HUCF order for D.

Define K_1, K_2, R_1, and M as follows

$$\begin{aligned}
K_1 &= \{i \in D \,|\, \mathcal{F}(i) = k \text{ and } \mathcal{F}(j) = k \text{ for all } j \in D \text{ with } \sigma_0(j) \geq \sigma_0(i)\}, \\
K_2 &= \{i \in T_y \,|\, \mathcal{F}(i) = k \text{ and } \mathcal{F}(j) = k \text{ for all } j \in T_y \text{ with } \sigma_{\vec{y}}(j) \leq \sigma_{\vec{y}}(i)\}, \\
R_1 &= \{i \in D \,|\, \mathcal{F}(i) = k\} \supset K_1, \\
M &= \{i \in D \,|\, \mu(i) \geq \mu(l(\mu, R_1))\}.
\end{aligned}$$

Note that $M \neq \emptyset$. Order μ' is obtained from μ by moving all jobs in K_2 to the tail of R_1. Figure 4 depicts the orders $\sigma_{\vec{y}}$, μ, and μ', and K_1, K_2, R_1, and M.

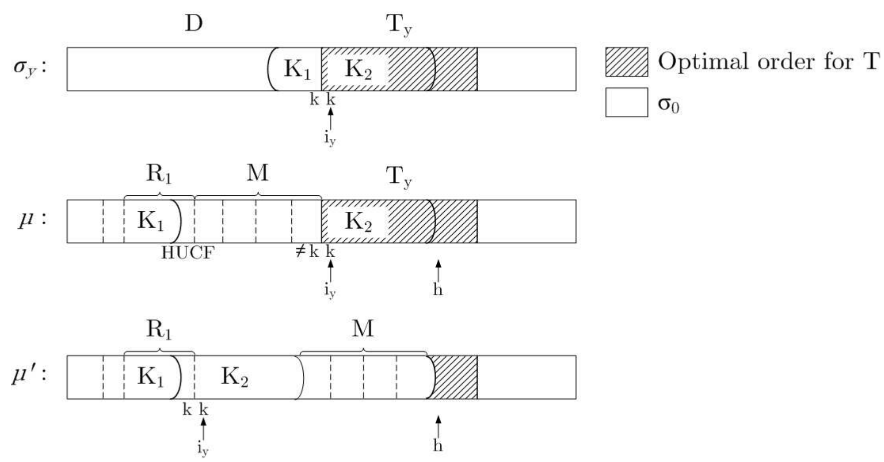

Figure 4: The orders $\sigma_{\vec{y}}$, μ and μ'.

To prove (14) it suffices to see that μ' is an admissible order for E and that

$$\sum_{j \in E} \alpha_{\mathcal{F}(j)} \left(C(\mu, j) - C(\mu', j) \right) \geq s_k \sum_{j \in T_y} \alpha_{\mathcal{F}(j)}. \tag{18}$$

Indeed, observe that (18) implies (13) since

$$
\begin{aligned}
& v(E) - v(D) - \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) \\
\geq \; & \sum_{j \in E} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\mu', j)\right) - v(D) - \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) \\
= \; & \sum_{j \in E} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\mu, j)\right) + \sum_{j \in E} \alpha_{\mathcal{F}(j)} \left(C(\mu, j) - C(\mu', j)\right) \\
& - \sum_{j \in D} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi, j)\right) - \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) \\
\geq \; & \sum_{j \in E} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\mu, j)\right) + s_k \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \\
& - \sum_{j \in D} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi, j)\right) - \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) \\
= \; & \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma, i) - C(\mu, j)\right) + s_k \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \\
= \; & \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_{\vec{y}}, i) - C(\mu, j)\right) + s_k \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \\
\geq \; & s_k (x_{\sigma_{\vec{y}}, i_y} - x_{\mu, i_y}) \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} + s_k \sum_{j \in T_y} \alpha_{\mathcal{F}(j)} \\
= \; & 0.
\end{aligned}
$$

Here the first inequality follows from the admissibility of μ' and the second from (18). The first equality follows from and the optimality of order π for D, the second from the fact that $C(\pi, i) = C(\mu, i)$ for all $i \in D$. The third equality holds by $C(\sigma, i) = C(\sigma_{\vec{y}}, i)$ for all $i \in T_y$. The last inequality holds due to (16). By assumption it holds that $x_{\sigma_{\vec{y}}, i_y} = 0$ and $x_{\mu, i_y} = 1$ such that the last equality holds.

First we prove that μ' is an admissible order for E. Clearly, $P(\mu', i) = P(\mu, i) = P(\sigma_0, i)$ for all $i \in N \setminus E$. So, it is sufficient to show that $C(\mu', i) \leq C(\sigma_0, i)$ for every $i \in N \setminus E$.

Let $h = f(\mu, S(\mu, K_2))$. Observe that h is not necessarily an element of T_y. For every $i \in S(\mu, K_2)$ it holds that

$$
\begin{aligned}
C(\mu, i) - C(\mu', i) &= \sum_{j \in \bar{P}(\mu, i)} (s_{\mathcal{F}(j)} x_{\mu, j} + p_{\mathcal{F}(j)}) - \sum_{j \in \bar{P}(\mu', i)} (s_{\mathcal{F}(j)} x_{\mu', j} + p_{\mathcal{F}(j)}) \\
&= \sum_{j \in \{i_y, h\}} (x_{\mu, j} - x_{\mu', j}) s_{\mathcal{F}(j)} \\
&= s_k + (x_{\mu, h} - x_{\mu', h}) s_{\mathcal{F}(h)}. \qquad (19)
\end{aligned}
$$

If $x_{\mu,h} = 1$, then for every $i \in S(\mu, K_2)$ it follows that

$$\begin{aligned} C(\mu', i) &= C(\mu, i) - s_k - (x_{\mu,h} - x_{\mu',h}) s_{\mathcal{F}(h)} \\ &\leq C(\mu, i) - s_k \\ &\leq C(\sigma_{\vec{y}}, i) \\ &\leq C(\sigma_0, i). \end{aligned}$$

Here the equality holds by (19). The first inequality holds by assumption, the second by (16), and the third by $\sigma_{\vec{y}} \in \mathcal{A}(E)$. Hence, μ' is an admissible order for E.

If $x_{\mu,h} = 0$ then by definition of K_2 it holds that $h \notin T_y$ and $T_y = K_2$. Hence, it follows that $C(\pi, i) = C(\mu, i)$ for all $i \in S(\sigma_0, E)$. Further it holds that $\mathcal{F}(h) = k$ and $x_{\mu',h} = 1$ such that for all $i \in N \setminus E$ it holds that

$$C(\mu', i) = C(\mu, i) = C(\pi, i) \leq C(\sigma_0, i).$$

Here the first equality follows from (19), where $x_{\mu,h} = 0$, $x_{\mu',h} = 1$, and $\mathcal{F}(h) = k$. The inequality holds since $\pi \in \mathcal{A}(D)$. Hence, μ' is an admissible order for E.

It remains to prove that inequality (18) holds. Let γ be the time to process and setup all jobs in M when they are processed with respect to μ, i.e., $\gamma = \sum_{j \in M} (x_{\mu,j} s_{\mathcal{F}(j)} + p_{\mathcal{F}(j)})$. For this, we first show that

$$\gamma \alpha_k - p_k \sum_{j \in M} \alpha_{\mathcal{F}(j)} \geq 0. \tag{20}$$

Let π' be the order obtained from π by taking the group R_1 behind M. Figure 5 depicts the two orders π and π'.

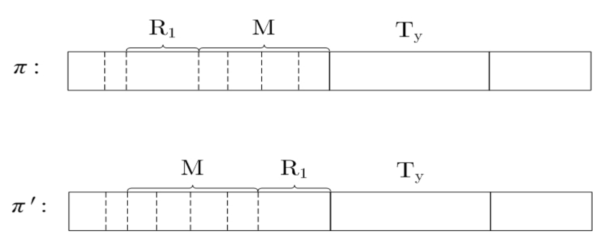

Figure 5: The orders π and π'.

Since π is optimal for D, π is not tail-adjusted HUCF, and since the number of setups in π is equal to the number of setups in π', order π' is admissible for D. Observe that for all $i \in D$

$$C(\pi', i) - C(\pi, i) = \begin{cases} 0 & \text{if } i \in D \backslash (M \cup R_1), \\ \gamma & \text{if } i \in R_1, \\ -(s_k + |R_1| p_k) & \text{if } i \in M. \end{cases} \tag{21}$$

Therefore,

$$
\begin{aligned}
& |R_1| \left(\gamma\alpha_k - p_k \sum_{j\in M} \alpha_{\mathcal{F}(j)} \right) - s_k \sum_{j\in M} \alpha_{\mathcal{F}(j)} \\
= \; & |R_1|\gamma\alpha_k - (s_k + |R_1| p_k) \sum_{j\in M} \alpha_{\mathcal{F}(j)} \\
= \; & \sum_{j\in R_1 \cup M} \alpha_{\mathcal{F}(j)} \left(C(\pi', j) - C(\pi, j) \right) \\
= \; & \sum_{j\in D} \alpha_{\mathcal{F}(j)} \left(C(\pi', j) - C(\pi, j) \right) \\
= \; & \sum_{j\in D} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi, j) \right) - \sum_{j\in D} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi', j) \right) \\
= \; & v(D) - \sum_{j\in D} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi', j) \right) \\
\geq \; & 0,
\end{aligned}
$$

where the second equality holds by (21) and the last equality follows from the fact that π is an optimal order for D. The inequality holds by the admissibility of π' for D. Hence,

$$
\gamma\alpha_k - p_k \sum_{j\in M} \alpha_{\mathcal{F}(j)} \geq \frac{s_k \sum_{j\in M} \alpha_{\mathcal{F}(j)}}{|R_1|} \geq 0, \tag{22}
$$

which proves (20).

With respect to (18), observe that for $i \in E$

$$
C(\mu, i) - C(\mu', i) = \begin{cases} 0, & \text{if } i \in P(\mu, M), \\ -|K_2| p_k, & \text{if } i \in M, \\ \gamma + s_k, & \text{if } i \in K_2, \\ s_k + (x_{\mu,h} - x_{\mu',h}) s_j & \text{if } i \in T_y \setminus K_2. \end{cases} \tag{23}
$$

Hence,

$$
\begin{aligned}
& \sum_{j\in E} \alpha_{\mathcal{F}(j)} \left(C(\mu, j) - C(\mu', j) \right) \\
= \; & \sum_{j\in M} \alpha_{\mathcal{F}(j)} \left(C(\mu, j) - C(\mu', j) \right) + \sum_{j\in K_2} \alpha_{\mathcal{F}(j)} \left(C(\mu, j) - C(\mu', j) \right) \\
& + \sum_{j\in T_y \setminus K_2} \alpha_{\mathcal{F}(j)} \left(C(\mu, j) - C(\mu', j) \right) \\
= \; & -|K_2| p_k \sum_{j\in M} \alpha_{\mathcal{F}(j)} + (\gamma + s_k) \sum_{j\in K_2} \alpha_{\mathcal{F}(j)} \\
& + (s_k + (x_{\mu,h} - x_{\mu',h}) s_h) \sum_{j\in T_y \setminus K_2} \alpha_{\mathcal{F}(j)}
\end{aligned}
$$

$$
\begin{aligned}
&\geq -|K_2| p_k \sum_{j\in M} \alpha_{\mathcal{F}(j)} + (\gamma + s_k) \sum_{j\in K_2} \alpha_{\mathcal{F}(j)} + s_k \sum_{j\in T_y \setminus K_2} \alpha_{\mathcal{F}(j)} \\
&= |K_2| \left(\gamma \alpha_k - p_k \sum_{j\in M} \alpha_{\mathcal{F}(j)} \right) + s_k \sum_{j\in K_2} \alpha_{\mathcal{F}(j)} + s_k \sum_{j\in T_y \setminus K_2} \alpha_{\mathcal{F}(j)} \\
&\geq s_k \sum_{j\in T_y} \alpha_{\mathcal{F}(j)}.
\end{aligned}
$$

The second equality follows from (23). The first inequality follows from the fact that if $T_y \setminus K_2 \neq \emptyset$, then $x_{\mu,h} = 1$. The last inequality follows from (22). Hence, (18) is verified. This concludes the proof. □

4 Ordered Family Sequencing Games

In this section we consider *ordered family sequencing situations*. A family sequencing situation is called *ordered* if the initial order σ_0 is family ordered. Note that since all family members are processed consecutively the number of setups is minimized in σ_0. In the proof of Theorem 4.2 we see that in this case several marginal vectors are in the core of ordered family sequencing games. This allows us to construct a Shapley-based core allocation for ordered family sequencing games.

Let $n = \sigma_0^{-1}(|N|)$. For a game $v \in TU^N$ the subgame $v^{-n} \in TU^{N\setminus\{n\}}$ is defined by

$$v^{-n}(S) = v(S),$$

for all $S \in 2^{N\setminus\{n\}}$. The next theorem shows for ordered family sequencing situations that v^{-n} is convex and that v has a component additive value structure.

Theorem 4.1. *Let $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ be an ordered family sequencing situation with corresponding family sequencing game $v \in TU^N$. Let $n = \sigma_0^{-1}(|N|)$. Then,*

(i) the subgame $v^{-n} \in TU^{N\setminus\{n\}}$ is convex, and

(ii) $v(S) = \sum_{T\in S\setminus\sigma_0} v(T)$ for all $S \in 2^N$.

Proof. Set $F = \{1, \ldots, f\}$. Let $F_k = \mathcal{F}^{-1}(k)$ be the set of family members of family $k \in F$ ordered in such a way that

$$\sigma_0(i) < \sigma_0(j),$$

whenever $i \in F_k, j \in F_l$ with $k, l \in \{1, \ldots, f\}$ and $k < l$. Note that $n \in F_f$. Clearly, a coalition S can be written as

$$S = \bigcup_{k\in K^S} F_k \cup \bigcup_{k\in P^S} G_k^S, \tag{24}$$

with $G_k^S \subsetneq F_k$ for all $k \in P^S$. Here $K^S \subset F$ denotes the set of families from which each member is in S, and $P^S \subset F$ denotes the set of families in S of whom at least one family member is not in S. Hence, G_k^S is the set of members of family $k \in P^S$ which are in S.

Since in σ_0 the number of setups is minimized, the number of setups equals f and

$$C(\sigma_0, n) = \min_{\sigma \in \Pi(N \setminus \{n\})} C(\sigma, n).$$

Consequently, each admissible order for an arbitrary coalition of $N \setminus \{n\}$ has to remain family ordered.

Consider the standard sequencing situation $(\hat{N}, \hat{\sigma}_0, \hat{p}, \hat{\alpha})$ with $\hat{N} = \{1, 2, \ldots, f-1\}$ and

$$\begin{aligned} \hat{\sigma}_0(k) &= k, \\ \hat{p}_k &= s_k + n_k p_k, \\ \hat{\alpha}_k &= n_k \alpha_k, \end{aligned}$$

for $k \in \hat{N}$. Denote the corresponding standard sequencing game by $\hat{v} \in TU^{\hat{N}}$. It follows from Curiel et al. (1989) that $\hat{v}$ is convex and $\hat{\sigma}_0$-component additive. Then explicitly using the fact that with respect to v each admissible order for $N \setminus \{n\}$ is family ordered, one readily checks that for every $\hat{S} \subset \hat{N}$ it holds that

$$\hat{v}(\hat{S}) = v\left(\bigcup_{k \in \hat{S}} F_k\right). \tag{25}$$

(i) Let $i \in N \setminus \{n\}$ and consider $S \subset T \subset N \setminus \{n, i\}$. For convexity of v^{-n}, it suffices to prove that

$$v^{-n}(S \cup \{i\}) - v^{-n}(S) \leq v^{-n}(T \cup \{i\}) - v^{-n}(T),$$

or equivalently, that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T). \tag{26}$$

From (24) and (25) and the fact that each admissible order for S has to remain family ordered it follows that

$$v(S) = v\left(\bigcup_{k \in K^S} F_k \cup \bigcup_{k \in P^S} G_k^S\right) = v\left(\bigcup_{k \in K^S} F_k\right) = \hat{v}(K^S).$$

The first equality follows from (24), the second from the fact that jobs in G^S are fixed, and the third from (25). Similarly, $v(T) = \hat{v}(K^T)$. Also note that $K^S \subset K^T$.

If $K^{S \cup \{i\}} = K^S$, then $v(S \cup \{i\}) = \hat{v}(K^S) = v(S)$ and (26) holds by monotonicity.

If $K^{S\cup\{i\}} \neq K^S$, then

$$S \cup \{i\} = \bigcup_{k\in K^S} F_k \cup F_{\mathcal{F}^{-1}(i)} \cup \bigcup_{k\in P^S\setminus\mathcal{F}^{-1}(i)} G_k^S,$$

and

$$T \cup \{i\} = \bigcup_{k\in K^T} F_k \cup F_{\mathcal{F}^{-1}(i)} \cup \bigcup_{k\in P^T\setminus\mathcal{F}^{-1}(i)} G_k^T.$$

Hence,

$$\begin{aligned} v(S\cup\{i\}) - v(S) &= \hat{v}(K^S \cup \mathcal{F}^{-1}(i)) - \hat{v}(K^S) \\ &\leq \hat{v}(K^T \cup \mathcal{F}^{-1}(i)) - \hat{v}(K^T) \\ &= v(T\cup\{i\}) - v(T). \end{aligned}$$

The inequality holds since $\hat{v}$ is convex.

(ii) Set $S\backslash\sigma_0 = \{S_1, \ldots, S_l\}$ such that $S_y \subset P(\sigma, S_{y+1})$ for every $y \in \{1, \ldots, l-1\}$. It follows that if $n \in S$, and $n \in S_l$. Let, as before,

$$S_y = \bigcup_{k\in K_y^S} F_k \cup \bigcup_{k\in P_y^S} G_k^{S_y},$$

for all $y \in \{1, \ldots, l\}$. Note that $K^S\backslash\hat{\sigma}_0 = \cup_{y=1}^l K_y^S$ where K_y^S may be empty for some $y \in \{1, \ldots, l\}$.

First let $S \subset N \setminus \{n\}$. Since each admissible order remains family ordered, it suffices to prove *(ii)* for a coalition $S = \bigcup_{k\in K^S} F_k$ (i.e. with $P^S = \emptyset$). Then,

$$\begin{aligned} v(S) &= v\left(\bigcup_{k\in K^S} F_k\right) \\ &= \hat{v}(K^S) \\ &= \sum_{y=1}^{l} \hat{v}(K_y^S) \\ &= \sum_{y=1}^{l} v\left(\bigcup_{k\in K_y^S} F_k\right) \\ &= \sum_{y=1}^{l} v(S_y) \\ &= \sum_{T\in S\backslash\sigma_0} v(T). \end{aligned}$$

The third equality holds since $\hat{v}$ is $\hat{\sigma}_0$-component additive. This proves *(ii)*.

Secondly, let $n \in S$. Let π be an optimal order for $\bigcup_{y=1}^{l-1} S_y$, μ an optimal order for S_l and define σ as follows

$$\sigma(i) = \begin{cases} \pi(i) & \text{if } i \in \bigcup_{y=1}^{l-1} S_y, \\ \mu(i) & \text{if } i \in S_l, \\ \sigma_0(i) & \text{otherwise.} \end{cases}$$

Using the fact that σ_0 is family ordered, one readily verifies that π is also family ordered and

$$C(\sigma, i) = C(\sigma_0, i), \tag{27}$$

for all $i \in N \setminus S$. Consequently, σ is also optimal for S. Thus,

$$\begin{aligned} v(S) &= \sum_{j \in S} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) \\ &= \sum_{j \in \cup_{y=1}^{l-1} S_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) + \sum_{j \in S_l} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\sigma, j)\right) \\ &= \sum_{j \in \cup_{y=1}^{l-1} S_y} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\pi, j)\right) + \sum_{j \in S_l} \alpha_{\mathcal{F}(j)} \left(C(\sigma_0, j) - C(\mu, j)\right) \\ &= v(\bigcup_{y=1}^{l-1} S_y) + v(S_l) \\ &= \sum_{y=1}^{l-1} v(S_y) + v(S_l) \\ &= \sum_{y=1}^{l} v(S_y) \\ &= \sum_{T \in S \setminus \sigma_0} v(T). \end{aligned}$$

where the fifth equality follows from (27). This finishes the proof of *(ii)*. □

Theorem 4.1 allows us to provide a suitable solution concept for family ordered sequencing situations. For this purpose we use the Shapley value (Shapley (1953)). The Shapley value is defined as the average of all marginal vectors, i.e.,

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v),$$

for all $v \in TU^N$.

Theorem 4.2. *Let $\Sigma(N) = (N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ be an ordered family sequencing situation with corresponding family sequencing game $v \in TU^N$. Let $n = \sigma_0^{-1}(|N|)$. Then,*

$$\left(\Phi(v^{-n}), v(N) - v(N \setminus \{n\})\right) \in Core(v).$$

Proof. To prove Theorem 4.2 it suffices to show that the marginal vector $m^{\sigma}(v)$ for an arbitrary $\sigma \in \Pi(N)$ such that $\sigma(n) = n$, belongs to the core of v. Let $\sigma \in \Pi(N)$. It suffices to prove that for every $S \subset N$ it holds that

$$\sum_{j \in S} m_j^{\sigma}(v) \geq v(S).$$

Define $\sigma' \in \Pi(N \setminus \{n\})$ by $\sigma'(i) = \sigma(i)$ for all $i \in N \setminus \{n\}$.

Let $S \subset N$ be such that $n \notin S$. Then,

$$\sum_{j \in S} m_j^{\sigma}(v) = \sum_{j \in S} m_j^{\sigma'}(v^{-n}) \geq v^{-n}(S) = v(S),$$

where the inequality follows from Theorem 4.1*(i)*.

Let $S \subset N$ such that $n \in S$. Set $S \backslash \sigma_0 = \{S_1, \ldots, S_l\}$ such that $S_y \subset P(\sigma, S_{y+1})$ for every $y \in \{1, \ldots, l-1\}$. It follows that $n \in S_l$. Choose $x \in Core(v)$. Then,

$$\begin{aligned}
\sum_{j \in S} m_j^{\sigma}(v) &= \sum_{y=1}^{l-1} \sum_{j \in S_y} m_j^{\sigma}(v) + \sum_{j \in S_l} m_j^{\sigma}(v) \\
&\geq \sum_{y=1}^{l-1} v(S_y) + \sum_{j \in S_l} m_j^{\sigma}(v) \\
&= \sum_{y=1}^{l-1} v(S_y) + v(N) - v(N \setminus S_l) \\
&\geq \sum_{y=1}^{l-1} v(S_y) + \sum_{j \in N} x_j - \sum_{j \in N \setminus S_l} x_j \\
&= \sum_{y=1}^{l-1} v(S_y) + \sum_{j \in S_l} x_j \\
&\geq \sum_{y=1}^{l-1} v(S_y) + v(S_l) \\
&= v(S).
\end{aligned}$$

The first inequality follows from Theorem 4.1*(i)*. The second and third inequalities are due to the fact that $x \in Core(v)$. The last equality follows from Theorem 4.1*(ii)*.

□

In the following example illustrates the solution concept from Theorem 4.2.

Example 4.1. Consider the family sequencing situation $(N, F, \mathcal{F}, \sigma_0, s, p, \alpha)$ with $N = \{1, \ldots, 10\}$ and $F = \{1, \ldots, 5\}$. Assume that $F_1 = \{1, 2\}$, $F_2 = \{3\}$, $F_3 = \{4, 5, 6\}$, $F_4 = \{7, 8\}$, $F_5 = \{9, 10\}$. Further assume that $\sigma_0 = (1, \ldots, 10)$,

$s = (8, 6, 1, 6, 4)$, $p = (2, 4, 3, 2, 3)$ and $\alpha = (1, 1, \frac{5}{3}, 5, 1)$. Let $S = \{2, \ldots, 7, 9\}$. Then $v(S) = v(\{3, 4, 5, 6\})$.

Consider the corresponding standard sequencing situation $(\hat{N}, \hat{\sigma}_0, \hat{p}, \hat{\alpha})$ with $\hat{N} = \{1, 2, 3, 4\}$, $\hat{\sigma}_0 = (1, 2, 3, 4)$, $\hat{p} = (10, 10, 10, 10)$, $\hat{\alpha} = (2, 1, 5, 10)$. Moreover, $v(S) = \hat{v}(\{2, 3\}) = 40$.

Then,

$$\big(\Phi(v^{-n}), v(N) - v(N \setminus \{n\})\big) = \big(15, 15, 80, 35, 35, 35, 37\tfrac{1}{2}, 37\tfrac{1}{2}, 0, 10\big) \in Core(v).$$

Note that $\Phi(v^{-n})(9) = 0$ due to the fact that if $9 \in S$, then $5 \in P^S$ (i.e. job 10 is never in S). $\Diamond$

Example 4.2. Reconsider the family sequencing game of Example 3.1 with set of jobs $N = \{1, 2, 3, 4, 5\}$ and $v(N) = 20$. Note that σ_0 is not family ordered. Then,

$$\big(\Phi(v^{-n}), v(N) - v(N \setminus \{n\})\big) = \big(\tfrac{1}{4}, 3\tfrac{7}{12}, 3\tfrac{7}{12}, 2\tfrac{7}{12}, 10\big) \notin Core(v).$$

This can be seen from the fact $v(N) = v(\{2345\})$ such that $x_1 = 0$ for all $x \in Core(v)$. $\Diamond$

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