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Abstract An omnibus test for spherical symmetry in $\mathbb{R}^2$ is proposed, employing localized empirical likelihood. The thus obtained test statistic is distribution free under the null hypothesis. The asymptotic null distribution is established and critical values for typical sample sizes, as well as the asymptotic ones, are presented. In a simulation study, the good performance of the test is demonstrated. Furthermore, a real data example is presented.

Keywords Asymptotic distribution · Distribution free · Empirical likelihood · Hypothesis test · Spherical symmetry

Mathematics Subject Classification (2000) 62G10 · 62G20 · 62G30 · 62H15 · 60F05

1 Introduction

Spherically symmetric distributions are an important class of distributions: They are a generalization of the multivariate standard normal distribution and include, amongst others, also multivariate Laplace and $t$ distributions. Furthermore, spherical symmetry is a distributional assumption which is associated with many statistical models, see Fang et al. (1990). For instance, only recently a relationship between $L_1$ spheri- cal symmetry and Archimedean copulas was discovered in McNeil and Nešlehová (2009). Another example is Laurent (1974), where univariate general linear models
are considered with an error term that is spherically symmetric distributed. More applications of spherically symmetric distributions in statistics, such as in minimax estimation or stochastic processes, are discussed in Chmielewski (1981). For a general introduction to symmetry see Serfling (2006). Our focus is on spherical symmetry in $\mathbb{R}^2$.

There exist several approaches to test for spherical symmetry, cf. the survey papers Fang and Liang (1999) or Liang et al. (2008) for a good overview. An often used basis, which is also underlying this paper, is the stochastic representation: Let $X = (X_1, X_2)$ be a bivariate random vector. Define the radius $S := \sqrt{X_1^2 + X_2^2}$ and the direction $Z := X/S$. Then $X$ is bivariate spherically symmetric (in the $L_2$-norm) around the origin if and only if $S$ is independent of $Z$, and $Z$ is uniformly distributed on the unit circle. Other nonparametric tests based on this stochastic representation include Smith (1977) and Baringhaus (1991), whereas the test proposed in Koltchinskii and Li (1998) uses multivariate distribution functions (df’s) and a multivariate extension of quantile functions.

We will moreover use that uniform random variables on a circle which are projected on a tangent to that circle are Cauchy distributed on that tangent (see, amongst others, Szabłowski 1998). More precisely, set $Y := X_2/X_1; (1, Y)$ is the projection of $Z$ on the tangent at $(1, 0)$. If $Z$ is uniformly distributed on the unit circle it follows that $Y$ is standard Cauchy distributed. Since such a projection cannot distinguish between $(X_1, X_2)$ and $(-X_1, -X_2)$, we project those $(X_1, X_2)$ with $X_1 > 0$ on the line $x_1 = 1$, whereas the $(X_1, X_2)$ with $X_1 < 0$ are projected on $x_1 = -1$. Denoting $\delta := \text{sign}(X_1)$, both $Y|\delta = -1$ and $Y|\delta = 1$ are then also standard Cauchy distributed. It also holds the other way around: if both $Y|\delta = -1$ and $Y|\delta = 1$ are standard Cauchy distributed, $P(\delta = -1) = \frac{1}{2}$, and $S, Y$ and $\delta$ are independent, then $Z$ is uniformly distributed on the unit circle and $S$ and $Z$ are independent.

We wish to test

$$H_0 : X \text{ is spherically symmetric around the origin}$$

on the basis of $(S, Y, \delta)$, but, for the first time, the test is developed in an empirical likelihood framework. The empirical likelihood method has the nice features which are known from parametric likelihood theory, but the data are used directly, i.e. in a nonparametric manner (see the monograph Owen 2001). By localizing a functional equation, see Einmahl and McKeague (2003), we create an omnibus test for spherical symmetry. More precisely, a functional equation is ‘split up’ in infinitely many pointwise equations and then standard empirical likelihood theory is used to deal with these pointwise constraints. Finally the infinitely many likelihood ratios are considered simultaneously as a stochastic process and an integral of this stochastic process is taken.

In Sect. 2, we derive the test statistic and present its limiting behavior under $H_0$. The test is consistent against all alternatives. In Sect. 3, critical values are computed, and in a simulation study we examine the performance of the test by power calculations for normal distributions and by a comparison to the test proposed in Koltchinskii and Li (1998). Furthermore, an application to a financial data set is presented. An extension of the test statistic to $\mathbb{R}^3$ is given in Sect. 4. The proof of the main result is deferred to Sect. 5.
2 Main result

Let \((S, Y, \delta)\), as introduced in Sect. 1, have df \(F\) with marginals \(F_S\), \(F_Y\), and \(F_\delta\). Define the subdistribution functions by \(F^-(s, y) := F(s, y, -1)\) and \(F^+(s, y) := F(s, y, 1) - F^-(s, y)\) and denote their marginals with \(F_S^\pm\) and \(F_Y^\pm\). Then the null hypothesis of spherical symmetry can be written as

\[
H_0: F^-(s, y) = F^+(s, y) = \frac{1}{2} F_S(s) G(y), \quad \text{for all } s \in \mathbb{R}^+, \ y \in \mathbb{R},
\]

with \(G\) denoting the standard Cauchy df.

Consider \(n\) independent random variables \((X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})\) distributed as \((X_1, X_2)\). Write \((S_i, Y_i, \delta_i), \ i = 1, \ldots, n\), for the transformed random vectors and denote with \(F_n\) their empirical df. Define the nonparametric likelihood \(L(\tilde{F}) = \prod_{i=1}^n \tilde{P}((S_i, Y_i, \delta_i))\), where \(\tilde{P}\) is the probability measure corresponding to \(\tilde{F}\). Furthermore, define for fixed \((s, y) \in \mathbb{R}^+ \times \mathbb{R}\) the localized empirical likelihood ratio

\[
R(s, y) = \frac{\sup^* \{L(\tilde{F})\}}{\sup \{L(F)\}},
\]

where \(\sup^*\) is the supremum taken under the constraints given by \(H_0\) and the corresponding marginal constraints:

\[
\tilde{F}^-(s, y) = \tilde{F}_S^-(s) G(y), \quad \tilde{F}^+(s, y) = \tilde{F}_S^+(s) G(y),
\]

\[
\tilde{F}_Y^-(y) = \frac{1}{2} G(y), \quad \tilde{F}_Y^+(y) = \frac{1}{2} G(y),
\]

\[
\tilde{F}_S^-(s) = \tilde{F}_S^+(s) = \frac{1}{2} \tilde{F}_S(s), \quad \tilde{F}^-(\infty, \infty) = \tilde{F}^+(\infty, \infty) = \frac{1}{2},
\]

and \(\sup\) is the maximum over the unrestricted likelihood obtained at \(\tilde{F} = F_n\), i.e., giving each observation mass \(\frac{1}{n}\).

Define, for either choice of sign, the bivariate empirical subdistribution functions

\[
F_n^\pm(s, y) = \frac{1}{n} \sum_{i=1}^n 1_{\left[0, s\right] \times \left(-\infty, y\right] \times \left\{ \pm 1 \right\}}(S_i, Y_i, \delta_i),
\]

and write \(N := n F_n^- (\infty, \infty)\). Observe that \(N\) is the number of data points with \(X_{1i} \leq 0\).

Consider for \((S_i, Y_i, \delta_i), i = 1, \ldots, n\), and either choice of sign, the regions

\[
A_3^\pm = [0, s] \times (y, \infty) \times \{\pm 1\}, \quad A_4^\pm = (s, \infty) \times (y, \infty) \times \{\pm 1\},
\]

\[
A_1^\pm = [0, s] \times (-\infty, y] \times \{\pm 1\}, \quad A_2^\pm = (s, \infty) \times (-\infty, y] \times \{\pm 1\}.
\]
Denote with $P_n$ the empirical measure corresponding to $F_n$. Let $F_{Sn}^\pm$ and $F_{Yn}^\pm$ denote the respective marginal df’s of $F_n^\pm$. Observe that

$$P_n(A_j^\pm) = F_{Sn}^\pm(s) - F_n^\pm(s, y), \quad P_n(A_k^\pm) = F_n^\pm(\infty, \infty) - F_{Sn}^\pm(s) - F_{Yn}^\pm(y) + F_n^\pm(s, y)$$

To maximize the numerator of (2), $\tilde{F}$ should put equal mass $p_j^-$, say, on each observation in $A_j^-$ and mass $p_j^+$ on each observation in $A_j^+$, $j = 1, \ldots, 4$. Hence we need to maximize

$$\prod_{j=1}^{4} (p_j^-)^n P_n(A_j^-) (p_j^+)^n P_n(A_j^+)$$

under the constraints

$$nP_n(A_1^-) p_1^- = (nP_n(A_1^-) p_1^- + nP_n(A_2^-) p_2^-) G(y),$$

$$nP_n(A_1^+) p_1^+ = (nP_n(A_1^+) p_1^+ + nP_n(A_3^+) p_3^+) G(y),$$

$$nP_n(A_1^-) p_1^+ + nP_n(A_2^+) p_2^- = \frac{1}{2} G(y),$$

$$nP_n(A_1^+) p_1^+ + nP_n(A_2^-) p_2^+ = \frac{1}{2} G(y),$$

$$nP_n(A_3^-) p_3^- = nP_n(A_3^+)^2 + nP_n(A_3^+) p_3^+,$$

$$\sum_{j=1}^{4} p_j^- nP_n(A_j^-) = \frac{1}{2},$$

$$\sum_{j=1}^{4} p_j^+ nP_n(A_j^+) = \frac{1}{2}.$$

This yields, for either choice of sign, the maximum empirical likelihood estimators

$$\hat{p}_3^\pm = \frac{F_{Sn}(s)(1 - G(y))}{2n P_n(A_3^\pm)}, \quad \hat{p}_4^\pm = \frac{(1 - F_{Sn}(s))(1 - G(y))}{2n P_n(A_4^\pm)},$$

$$\hat{p}_1^\pm = \frac{F_{Sn}(s) G(y)}{2n P_n(A_1^\pm)}, \quad \hat{p}_2^\pm = \frac{(1 - F_{Sn}(s)) G(y)}{2n P_n(A_2^\pm)}.$$

Define, for either choice of sign,

$$\log R^\pm(s, y) = n P_n(A_1^\pm) \log \frac{F_{Sn}(s) G(y)}{2 P_n(A_1^\pm)} + n P_n(A_3^\pm) \log \frac{(1 - F_{Sn}(s)) G(y)}{2 P_n(A_2^\pm)} + n P_n(A_3^\pm) \log \frac{F_{Sn}(s)(1 - G(y))}{2 P_n(A_3^\pm)}$$

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\[ + n P_n \left( A^\pm_4 \right) \log \frac{(1 - F_{Sn}(s))(1 - G(y))}{2 P_n \left( A^\pm_4 \right)}, \tag{3} \]

where \( 0 \log(a/0) = 0 \), then we have

\[ \log R(s, y) = \log R^-(s, y) + \log R^+(s, y). \]

Consider the test statistic

\[ T_n = -2 \int_{-\infty}^{\infty} \int_{0}^{\infty} \log R(s, y) \, dF_{Sn}(s) \, dG(y). \]

Clearly, \( T_n \) is distribution free under \( H_0 \); selected critical values are provided in Table 1.

We now consider the limiting distribution of \( T_n \). In order to define the limiting random variable, we denote with \( W \) a standard Wiener process on \([0, 1]^3\), i.e. a centered Gaussian process with \( \text{Cov}(W(u, v, w), W(\tilde{u}, \tilde{v}, \tilde{w})) = (u \wedge \tilde{u})(v \wedge \tilde{v})(w \wedge \tilde{w}) \), and with \( B(u, v, w) = W(u, v, w) - uvwW(1, 1, 1) \) the standard trivariate Brownian bridge. We also define \( B^-(u, v) := B(u, v, \frac{1}{2}) \) and \( B^+(u, v) := B(u, v, 1) - B^-(u, v) \). Observe \( B^-(1, 1) = -B^+(1, 1) \). Furthermore, let, for either choice of sign, \( W_0^\pm \) be a four-sided tied-down “half” Wiener process on \([0, 1]^2\) defined by \( W_0^\pm(u, v) := B^\pm(u, v) - vB^\pm(u, 1) - uB^\pm(1, v) + uvB^\pm(1, 1) \). Finally write

\[ K(u, v) = \frac{W_0^-(u, v)^2 + W_0^+(u, v)^2}{2u(1 - u)v(1 - v)} + 4B^-(1, 1)^2 \]
\[ + \frac{[B^-(u, 1) - uB^-(1, 1) - B^+(u, 1) + uB^+(1, 1)]^2}{u(1 - u)} \]
\[ + \frac{[B^-(1, v) - vB^-(1, 1)]^2 + [B^+(1, v) - vB^+(1, 1)]^2}{v(1 - v)}. \]

**Theorem 1** Let \( F_S \) be continuous. Then, under \( H_0 \),

\[ T_n \xrightarrow{d} \int_0^1 \int_0^1 K(u, v) \, du \, dv. \]

The proof of the theorem is given in Sect. 5.

Note that for fixed \( s \) and \( y \), under \( H_0 \),

\[ -2 \log R(s, y) \xrightarrow{d} K \left( F_S(s), G(y) \right) \xrightarrow{d} \chi^2_6. \]

This is a special case of Owen’s (Owen 2001) nonparametric version of the classical Wilks theorem.

Also note that within the localized empirical likelihood framework a test based directly on \((S, Z)\) can be constructed as well, but such a test has typically less power.
**Table 1** Critical values for the test for bivariate spherical symmetry

<table>
<thead>
<tr>
<th>$n$</th>
<th>Percentage points</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90%</td>
<td>95%</td>
<td>97.5%</td>
<td>99%</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>8.83</td>
<td>10.01</td>
<td>11.23</td>
<td>12.81</td>
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</tr>
<tr>
<td>100</td>
<td>8.83</td>
<td>9.99</td>
<td>11.20</td>
<td>12.80</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>8.77</td>
<td>9.96</td>
<td>11.17</td>
<td>12.74</td>
<td></td>
</tr>
<tr>
<td>$\infty$</td>
<td>8.61</td>
<td>9.83</td>
<td>11.02</td>
<td>12.66</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2** Size and power of the test for bivariate normal distributions with different correlations for sample sizes $n = 100$ and $n = 200$

<table>
<thead>
<tr>
<th></th>
<th>$n = 100$</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>$n = 200$</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10%</td>
<td>5%</td>
<td>2.5%</td>
<td>1%</td>
<td>10%</td>
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<td>2.5%</td>
<td>1%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.00</td>
<td>0.10</td>
<td>0.05</td>
<td>0.03</td>
<td>0.01</td>
<td>0.10</td>
<td>0.06</td>
<td>0.03</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.10</td>
<td>0.05</td>
<td>0.03</td>
<td>0.01</td>
<td>0.17</td>
<td>0.08</td>
<td>0.04</td>
<td>0.02</td>
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<tr>
<td></td>
<td>0.20</td>
<td>0.20</td>
<td>0.11</td>
<td>0.07</td>
<td>0.03</td>
<td>0.37</td>
<td>0.22</td>
<td>0.12</td>
<td>0.06</td>
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<tr>
<td></td>
<td>0.38</td>
<td>0.38</td>
<td>0.24</td>
<td>0.14</td>
<td>0.06</td>
<td>0.68</td>
<td>0.54</td>
<td>0.41</td>
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<td>0.61</td>
<td>0.61</td>
<td>0.46</td>
<td>0.31</td>
<td>0.17</td>
<td>0.94</td>
<td>0.86</td>
<td>0.78</td>
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<td>0.86</td>
<td>0.86</td>
<td>0.75</td>
<td>0.62</td>
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<td>0.99</td>
<td>0.98</td>
<td>0.93</td>
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<tr>
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<td>0.98</td>
<td>0.98</td>
<td>0.95</td>
<td>0.90</td>
<td>0.75</td>
<td>1.00</td>
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<td></td>
<td>1.00</td>
<td>1.00</td>
<td>0.99</td>
<td>0.97</td>
<td>0.75</td>
<td>1.00</td>
<td>1.00</td>
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<td>1.00</td>
<td>1.00</td>
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</tbody>
</table>

### 3 Simulation study and real data example

Table 1 provides selected critical values for the proposed test statistic $T_n$. The values for $n = 50, 100$ and $200$ are based on 100,000 samples in each case. For $n = \infty$, the quantiles of the limiting distribution are given, also based on 100,000 repetitions. The critical values for the finite sample sizes are well approximated by the asymptotic critical values.

To evaluate the power of the test (based on the critical values from Table 1), we regard data from a bivariate normal distribution with means 0, variances 1, and correlation $\rho$. The calculations, which are presented in Table 2, are based on 1000 replications. Note that for $\rho = 0$, we obtain the empirical size of the test. (Since the test statistic is distribution free under $H_0$ any other spherically symmetric distribution yields the same type I error probability.) At the 5% significance level we see a high power for $\rho = 0.6$ ($n = 100$), and for $n = 200$, $\rho = 0.4$ is already well detected.

Next, we compare the performance of our localized empirical likelihood (LEL-) test with the test proposed in Koltchinskii and Li (1998) (KL-test), see Table 3. It needs to be pointed out that the null hypothesis in Koltchinskii and Li (1998) is broader: There the center is unknown. Therefore the powers cannot be likened directly: A positive comparison for the LEL-test can be seen as an advise to use that test in case a center is given. We consider all the alternatives introduced in Koltchinskii and Li (1998):
be performed assuming spherical symmetry. Further statistical analysis of these data leads to more accurate inference, since it can

significance level. Therefore the null hypothesis is not rejected. As a consequence, a

(Note that the KL-test in Koltchinskii and Li (1998) also applies to higher dimensions, but the simulation results in there are only for dimension 2.)

$H_1^{(1)}$: $X_1 \sim \text{Exp}(1)$ and $X_2 \sim \text{Exp}(2)$, $X_1$ and $X_2$ independent, with $\text{Exp}(\lambda)$ the exponential distribution with mean $1/\lambda$;

$H_1^{(2)}$: $X_1 \sim N(0, 1)$ and $X_2 \sim \text{Exp}(1)$, $X_1$ and $X_2$ independent;

$H_1^{(3)}$: Mixture (with parameter 1/2) of two normal distributions with identity covariance matrices and with means $(-1.5, 0)$ and $(1.5, 0)$;

$H_1^{(4)}$: Uniform distribution on an equilateral triangle, centered at the origin.

(Not that the KL-test in Koltchinskii and Li (1998) also applies to higher dimensions, but the simulation results in there are only for dimension 2.)

Especially $H_1^{(1)}$ and $H_1^{(2)}$ are clearly visible as non-symmetric by the naked eye and should therefore lead to a high power. To center the data around the origin, we transform the data of $H_1^{(1)}$ and $H_1^{(2)}$ by subtracting the medians, hence we consider $(X_1 - \text{med}(X_1), X_2 - \text{med}(X_2))$. This is in line with Koltchinskii and Li (1998), where the empirical spatial median is chosen to estimate the center. The results are again based on 1000 repetitions of the LEL-test, whereas the results for the KL-test are taken from Koltchinskii and Li (1998) (100 repetitions). The LEL-test outperforms the KL-test in nearly every setting and typically performs even considerably better. For the alternative hypotheses $H_1^{(1)}$, $H_1^{(3)}$, and $H_1^{(4)}$, the LEL-test has for $n = 100$ already about the same power as the KL-test for $n = 200$. Only for $H_1^{(2)}$, $n = 100$, both tests have comparable power.

Finally we present a real data application. The bivariate data are the daily exchange rate log-returns of the Yen to the Dollar and the Pound to the Euro from January 2nd, 2009, to December 31st, 2009. The data set has size $n = 251$ and is available from http://wrds-web.wharton.upenn.edu, see Fig. 1. The returns are known to be centered at the origin; this is affirmed by an estimated spatial median of $(-3.3 \times 10^{-5}, -3.0 \times 10^{-4})$. We want to test whether these data are spherically symmetric and find $T_n = 6.84$, which is clearly below the asymptotic critical value at the 10% significance level. Therefore the null hypothesis is not rejected. As a consequence, a further statistical analysis of these data leads to more accurate inference, since it can be performed assuming spherical symmetry.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$n$</th>
<th>Significance level</th>
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<tbody>
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<td></td>
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</tr>
<tr>
<td></td>
<td>LEL</td>
<td>KL</td>
</tr>
<tr>
<td>$H_1^{(1)}$</td>
<td>100</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>1.00</td>
</tr>
<tr>
<td>$H_1^{(2)}$</td>
<td>100</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>1.00</td>
</tr>
<tr>
<td>$H_1^{(3)}$</td>
<td>100</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>1.00</td>
</tr>
<tr>
<td>$H_1^{(4)}$</td>
<td>100</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.99</td>
</tr>
</tbody>
</table>
As an example, consider the estimation of the probability $p$ that $X_1$ and $X_2$ are both positive (gains for Dollar and Euro) and that the radius $S$ (the size of the gains) is above a certain threshold $s_0$. In general, we can estimate this with the empirical probability $\hat{p}$, but under the null hypothesis we can estimate it with $\frac{1}{4} \hat{p}_s s_0$, with $\hat{p}_s s_0$ the empirical probability of $\{(x_1, x_2) : x_1^2 + x_2^2 > s_0^2\}$. For $s_0 = 0.015$ this leads to an asymptotic 95% confidence interval of $(0.0303, 0.0534)$, whereas the confidence interval based on $\hat{p}$ is more than double as wide: $(0.0214, 0.0742)$.

4 Extension of the test statistic to dimension 3

Let $X = (X_1, X_2, X_3)$ and define analogue to Sect. 1 $S = \sqrt{X_1^2 + X_2^2 + X_3^2}$ as the radius, $Z = X/S$ as the direction, and set $Y_1 = X_2/X_1$, $Y_2 = X_3/X_1$, and $\delta = \text{sign}(X_1)$. If $Z$ is uniformly distributed on the unit ball, it follows that $(Y_1, Y_2)$, as well as $(Y_1, Y_2) | \delta = -1$ and $(Y_1, Y_2) | \delta = 1$ are standard bivariate Cauchy distributed; recall that the corresponding density is given by

$$g(y_1, y_2) = \frac{1}{2\pi(1 + y_1^2 + y_2^2)^{3/2}}, \quad (y_1, y_2) \in \mathbb{R}^2.$$ 

Similarly to Sects. 1 and 2, we develop a test for (1) on the basis of $(S, Y_1, Y_2, \delta)$ in an empirical likelihood framework.

Consider the notation of Sect. 2, with now $Y = (Y_1, Y_2)$. Then, the null hypothesis of spherical symmetry can be written as

$$H_0: \quad F^{-}(s, y_1, y_2) = F^{+}(s, y_1, y_2) = \frac{1}{2} F_S(s) G(y_1, y_2),$$

for all $s \in \mathbb{R}^+, (y_1, y_2) \in \mathbb{R}^2$, with $G$ denoting the bivariate standard Cauchy df, with marginal df $G_1$.

Consider $n$ independent random vectors $(X_{11}, X_{21}, X_{31}), \ldots, (X_{1n}, X_{2n}, X_{3n})$ distributed as $(X_1, X_2, X_3)$ and write $(S_i, Y_{1i}, Y_{2i}, \delta_i), i = 1, \ldots, n$, for the transformed random vectors; $F_n$ denotes their empirical df. Define the localized empirical
likelihood ratio as in (2), where sup* is the supremum taken under the constraints given by $H_0$ and the corresponding marginal constraints:

$$
\tilde{F}^- (s, y_1, y_2) = \tilde{F}^- (s) G(y_1, y_2),
\tilde{F}^+ (s, y_1, y_2) = \tilde{F}^+ (s) G(y_1, y_2),
\tilde{F}^- (s, Y_1) = \tilde{F}^- (s) G_1(y_1),
\tilde{F}^+ (s, Y_1) = \tilde{F}^+ (s) G_1(y_1),
\tilde{F}^- (s, Y_2) = \tilde{F}^- (s) G_1(y_2),
\tilde{F}^+ (s, Y_2) = \tilde{F}^+ (s) G_1(y_2),
\tilde{F}^- (y_1) = \frac{1}{2} G_1(y_1),
\tilde{F}^+ (y_1) = \frac{1}{2} G_1(y_1),
\tilde{F}^- (y_2) = \frac{1}{2} G_1(y_2),
\tilde{F}^+ (y_2) = \frac{1}{2} G_1(y_2),
\tilde{F}^- (y_1, y_2) = \frac{1}{2} G(y_1, y_2),
\tilde{F}^+ (y_1, y_2) = \frac{1}{2} G(y_1, y_2),
\tilde{F}^- (s) = \tilde{F}^+ (s) = \frac{1}{2} \tilde{F}_S(s),
\tilde{F}^-(\infty, \infty, \infty) = \tilde{F}^+(\infty, \infty, \infty) = \frac{1}{2}.
$$

Define, analogue to Sect. 2, the bivariate empirical subdistribution functions $F_n^\pm (s, y_1, y_2)$ and consider for $(S_i, Y_{1i}, Y_{2i}, \delta_i)$, $i = 1, \ldots, n$, and either choice of sign, the regions

$$
\begin{align*}
A_1^\pm &= [0, s] \times (-\infty, y_1] \times (-\infty, y_2] \times \{ \pm 1 \}, \\
A_2^\pm &= (s, \infty) \times (-\infty, y_1] \times (-\infty, y_2] \times \{ \pm 1 \}, \\
A_3^\pm &= [0, s] \times (y_1, \infty) \times (-\infty, y_2] \times \{ \pm 1 \}, \\
A_4^\pm &= (s, \infty) \times (y_1, \infty) \times (-\infty, y_2] \times \{ \pm 1 \}, \\
A_5^\pm &= [0, s] \times (-\infty, y_1] \times (y_2, \infty) \times \{ \pm 1 \}, \\
A_6^\pm &= (s, \infty) \times (-\infty, y_1] \times (y_2, \infty) \times \{ \pm 1 \}, \\
A_7^\pm &= [0, s] \times (y_1, \infty) \times (y_2, \infty) \times \{ \pm 1 \}, \\
A_8^\pm &= (s, \infty) \times (y_1, \infty) \times (y_2, \infty) \times \{ \pm 1 \}.
\end{align*}
$$

Denote with $P_n$ the empirical measure corresponding to $F_n$. Let $F_{S_n}^\pm$, $F_{Y_{1n}}^\pm$, $F_{Y_{2n}}^\pm$, $F_{S,Y_{1n}}^\pm$, $F_{S,Y_{2n}}^\pm$, and $F_{Y_{1n},Y_{2n}}^\pm$ denote the respective marginal df’s of $F_n^\pm$. Observe that

$$
\begin{align*}
P_n(A_1^\pm) &= F_n^\pm (s, y_1, y_2), \\
P_n(A_2^\pm) &= F_{Y_{1n},Y_{2n}}^\pm (y_1, y_2) - F_n^\pm (s, y_1, y_2), \\
P_n(A_3^\pm) &= F_{S,Y_{2n}}^\pm (s, y_2) - F_n^\pm (s, y_1, y_2), \\
P_n(A_4^\pm) &= F_{Y_{2n}}^\pm (y_2) - F_{S,Y_{2n}}^\pm (s, y_2) - F_{Y_{1n},Y_{2n}}^\pm (y_1, y_2) + F_n^\pm (s, y_1, y_2), \\
P_n(A_5^\pm) &= F_{S,Y_{1n}}^\pm (s, y_1) - F_n^\pm (s, y_1, y_2), \\
P_n(A_6^\pm) &= F_{Y_{1n}}^\pm (y_1) - F_{S,Y_{1n}}^\pm (s, y_1) - F_{Y_{1n},Y_{2n}}^\pm (y_1, y_2) + F_n^\pm (s, y_1, y_2), \\
\end{align*}
$$
\[ P_n(A^+_{\pm}) = F^\pm_{Sn}(s) - F^\pm_{S,Y_{1n}}(s,y_1) - F^\pm_{S,Y_{2n}}(s,y_2) + F^\pm_n(s,y_1,y_2). \]

\[ P_n(A^-_{8}) = F^\pm_n(\infty, \infty, \infty) - F^\pm_{Sn}(s) - F^\pm_{Y_{1n}}(y_1) - F^\pm_{Y_{2n}}(y_2) + F^\pm_{S,Y_{1n}}(s,y_1) \]
\[ + F^\pm_{S,Y_{2n}}(s,y_2) + F^\pm_{Y_{1,Y_{2n}}}(y_1,y_2) - F^\pm_n(s,y_1,y_2). \]

To maximize the numerator of (2), we need to maximize
\[
\prod_{j=1}^{8} (p^-_jn P_n(A^-_j))(p^+_jn P_n(A^+_j))
\]
under the constraints
\[
n P_n(A^+_1)p^+_1 = G(y_1, y_2) \sum_{j \in \{1,3,5,7\}} p^+_jn P_n(A^+_j),
\]
\[
n P_n(A^+_1)p^+_1 + n P_n(A^+_3)p^+_3 = G_1(y_1) \sum_{j \in \{1,3,5,7\}} p^+_jn P_n(A^+_j),
\]
\[
n P_n(A^+_1)p^+_1 + n P_n(A^+_3)p^+_3 = G_1(y_2) \sum_{j \in \{1,3,5,7\}} p^+_jn P_n(A^+_j),
\]
\[
\sum_{j \in \{1,2,5,6\}} p^-_jn P_n(A^-_j) = \frac{1}{2} G_1(y_1),
\]
\[
\sum_{j \in \{1,2,3,4\}} p^-_jn P_n(A^-_j) = \frac{1}{2} G_1(y_2),
\]
\[
n P_n(A^+_1)p^+_1 + n P_n(A^+_2)p^+_2 = \frac{1}{2} G(y_1, y_2),
\]
\[
\sum_{j \in \{1,3,5,7\}} p^-_jn P_n(A^-_j) = \sum_{j \in \{1,3,5,7\}} p^+_jn P_n(A^+_j),
\]
\[
\sum_{j=1}^{8} p^+_jn P_n(A^+_j) = \frac{1}{2}.
\]

Similarly as in Sect. 2, we derive the maximum likelihood estimators \( \hat{p}^\pm_j, j = 1, \ldots, 8. \) This leads to
\[
\log R(s,y_1,y_2) = \log R^-(s,y_1,y_2) + \log R^+(s,y_1,y_2),
\]
with, for either choice of sign,
\[
\log R^\pm(s,y_1,y_2) = n P_n(A^\pm_1) \log \frac{F_{Sn}(s)G(y_1,y_2)}{2 P_n(A^\pm_1)}
\]
\[ + n P_n(A_2^\pm) \log \frac{(1 - F_{S_n}(s))G(y_1, y_2)}{2P_n(A_2^\pm)} \]
\[ + n P_n(A_3^\pm) \log \frac{F_{S_n}(s)(G(1) - G(y_1, y_2))}{2P_n(A_3^\pm)} \]
\[ + n P_n(A_4^\pm) \log \frac{(1 - F_{S_n}(s))(G(1) - G(y_1, y_2))}{2P_n(A_4^\pm)} \]
\[ + n P_n(A_5^\pm) \log \frac{F_{S_n}(s)(1 - G(1)) - G(1) + G(y_1, y_2))}{2P_n(A_5^\pm)} \]
\[ + n P_n(A_6^\pm) \log \frac{(1 - F_{S_n}(s))(1 - G(1) - G(1) + G(y_1, y_2))}{2P_n(A_6^\pm)} \]

The test statistic for testing spherical symmetry around the origin in \(\mathbb{R}^3\) is now given by

\[ T_n = -2 \int_{\mathbb{R}^3} \int_0^\infty \log R(s, y_1, y_2) dF_{S_n}(s) dG(y_1, y_2); \]

\(T_n\) is distribution-free under \(H_0\).

The derivation of the asymptotic behavior of \(T_n\) or the computation of the critical values of the test is much more difficult than for dimension 2 and beyond the scope of this paper.

5 Proof of Theorem 1

Write \(Q_S, Q\) for the quantile functions corresponding to \(F_S, G\), set \(U_i = F_S(S_i)\) and \(V_i = G(Y_i)\), and let \(\Gamma_n\) be the empirical df of the \((U_i, V_i, F_\delta(\delta_i))\) and \(\Gamma_{S_n}, \Gamma_{Y_n}, \Gamma_{\delta_n}\) the corresponding marginals. Furthermore, write \(\Gamma_n^-(u, v) := \Gamma_n(u, v, \frac{1}{2})\), hence \(\Gamma_n^-\) is the empirical subdistribution function of the \((U_i, V_i)\), for which \(\delta_i = -1\), with marginals \(\Gamma_{S_n}^-\) and \(\Gamma_{Y_n}^-\), and note that \(\Gamma_n^-(1, 1) = \frac{N}{n}\). Define \(\Gamma_n^+\) similarly.

Let \(0 < \varepsilon \leq \frac{1}{2}\). It suffices to show that, as \(n \to \infty\),

\[ T_{1n} = -2 \int_{Q(1-\varepsilon)}^{Q(\varepsilon)} \int_{Q(1-\varepsilon)}^{Q(\varepsilon)} \log R(s, y) dF_{S_n}(u) dG(y) \]
\[ \to \int_\varepsilon^{1-\varepsilon} \int_\varepsilon^{1-\varepsilon} K(u, v) du dv, \quad (4) \]

and

\[ T_{2n} = T_n - T_{1n} = O_P(\sqrt{\varepsilon}) \quad (5) \]
uniformly in $\varepsilon$; see Billingsley (1968, Theorem 4.2).

First, consider $T_{1n}$ and decompose it further to

$$T_{1n} = -2 \int_{\varepsilon}^{1-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} \log R^-(Q_S(u), Q(v)) \, d\Gamma_{Sn}(u) \, dv$$

$$- 2 \int_{\varepsilon}^{1-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} \log R^+(Q_S(u), Q(v)) \, d\Gamma_{Sn}(u) \, dv =: T_{1n}^- + T_{1n}^+.$$

Because of symmetry, we will first only consider $T_{1n}^-$. From (3), applying a Taylor expansion of $\log(1 + x)$, it follows that, uniformly in $s \in [Q_S(\varepsilon), Q_S(1-\varepsilon)]$ and $y \in [Q(\varepsilon), Q(1-\varepsilon)],$

$$\log R^-(s, y) = \frac{n}{2} - N - \frac{n}{8} \left[ \frac{(F_{Sn}(s)G(y) - 2P_n(A^-_1))^2}{P_n(A^-_1)} \right.$$

$$+ \frac{((1 - F_{Sn}(s))G(y) - 2P_n(A^-_2))^2}{P_n(A^-_2)}$$

$$+ \frac{(F_{Sn}(s)(1 - G(y)) - 2P_n(A^-_3))^2}{P_n(A^-_3)}$$

$$+ \frac{((1 - F_{Sn}(s))(1 - G(y)) - 2P_n(A^-_4))^2}{P_n(A^-_4)} \left] + o_P(1) \right.$$

$$= \frac{n}{2} - N - \frac{F_{Sn}(s)}{8P_n(A^-_1)P_n(A^-_3)} \left[ \sqrt{n}(F_{Sn}(s)G(y) - 2P_n(A^-_1)) \right]^2$$

$$- \frac{N}{n} \frac{F_{Sn}(s)}{8P_n(A^-_2)P_n(A^-_4)} \left[ \sqrt{n}((1 - F_{Sn}(s))G(y) - 2P_n(A^-_2)) \right]^2$$

$$- \frac{N}{n} \frac{F_{Sn}(s)}{8P_n(A^-_3)P_n(A^-_4)} \left[ \sqrt{n}(F_{Sn}(s) - F_{Sn}(s)) \right]^2 - \frac{[\sqrt{n}(1 - 2 \frac{N}{n})]^2}{8P_n(A^-_4)}$$

$$+ 2 \sqrt{n} \frac{(1 - 2 \frac{N}{n})}{8P_n(A^-_4)} \left[ \sqrt{n}(F_{Sn}(s) - F_{Sn}(s)) + \sqrt{n}((1 - F_{Sn}(s))G(y) - 2P_n(A^-_2)) \right]$$

$$- 2P_n(A^-_2) \right] + 2 \sqrt{n} \frac{(F_{Sn}(s) - F_{Sn}(s))}{8P_n(A^-_3)}$$

$$\times \sqrt{n}(F_{Sn}(s)G(y) - 2P_n(A^-_1))$$

$$- 2 \sqrt{n}(F_{Sn}(s) - F_{Sn}(s))$$

$$\times \sqrt{n}((1 - F_{Sn}(s))G(y) - 2P_n(A^-_2)) + o_P(1).$$

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Observe that
\[
\sqrt{n}(F_{S_n}(s)G(y) - 2P_n(A_{1}^-)) = \sqrt{n}(F_{S_n}(s) - F_S(s))G(y) - 2\sqrt{n}\left(P_n(A_{1}^-) - \frac{1}{2}F_S(s)G(y)\right),
\]
\[
\sqrt{n}\left((1 - F_{S_n}(s))G(y) - 2P_n(A_{2}^-)\right) = -2\sqrt{n}\left(F_{T_n}^{-}(y) - \frac{1}{2}G(y)\right) - \sqrt{n}(F_{S_n}(s)G(y) - 2P_n(A_{1}^-)),
\]
and
\[
\sqrt{n}(F_{S_n}^{+}(s) - F_{S_n}^{-}(s)) = \sqrt{n}(F_{S_n}^{+}(s) - F_{S}^{+}(s)) - \sqrt{n}(F_{S_n}(s) - F_{S}^{-}(s)).
\]
It follows from the Glivenko–Cantelli theorem that
\[
\frac{N}{n} \rightarrow \frac{1}{2},
\]
\[
\sup_{Q_{S}(\varepsilon) \leq s \leq Q_{S}(1-\varepsilon)} \frac{F_{S_n}^{-}(s)}{8P_n(A_{1}^-)P_n(A_3)} - \frac{1}{4F_S(s)G(y)(1 - G(y))} = o_P(1),
\]
\[
\sup_{Q_{S}(\varepsilon) \leq s \leq Q_{S}(1-\varepsilon)} \frac{N}{n} - F_{S_n}^{-}(s) \frac{8P_n(A_{1}^-)P_n(A_3)}{4(1 - F_S(s))G(y)(1 - G(y))} = o_P(1),
\]
\[
\sup_{Q_{S}(\varepsilon) \leq s \leq Q_{S}(1-\varepsilon)} \frac{N}{n} - F_{T_n}^{-}(y) \frac{8P_n(A_{1}^-)P_n(A_3)}{4F_S(s)(1 - F_S(s))(1 - G(y))} = o_P(1),
\]
\[
\sup_{Q_{S}(\varepsilon) \leq s \leq Q_{S}(1-\varepsilon)} \frac{1}{8P_n(A_{3}^-)} - \frac{1}{4F_S(s)(1 - G(y))} = o_P(1),
\]
\[
\sup_{Q_{S}(\varepsilon) \leq s \leq Q_{S}(1-\varepsilon)} \frac{1}{8P_n(A_{4}^-)} - \frac{1}{4(1 - F_S(s))(1 - G(y))} = o_P(1).
\]

Writing \(\alpha_n^-(u, v) := \sqrt{n}(\Gamma_{n}^{-}(u, v) - \frac{1}{2}uv), \alpha_n^+(u, v) := \sqrt{n}(\Gamma_{n}^{+}(u, v) - \frac{1}{2}uv)\), and \(\alpha_n(u, v) := \alpha_n^-(u, v) + \alpha_n^+(u, v)\), we have, using (6) uniformly for \(\varepsilon \leq u, v \leq 1 - \varepsilon\),

\[
\log R^{-}(Q_{S}(u), Q(v)) = \frac{n}{2} - N - \frac{[\sqrt{n}(\Gamma_{S_n}(u) - u)v - 2\sqrt{n}(\Gamma_{n}^{-}(u, v) - \frac{1}{2}uv)]^2}{4uv(1 - v)}
\]
\[
- \frac{[2\sqrt{n}(\Gamma_{Y_n}^{-}(v) - \frac{1}{2}v) + \sqrt{n}(\Gamma_{S_n}(u) - u)v - 2\sqrt{n}(\Gamma_{n}^{-}(u, v) - \frac{1}{2}uv)]^2}{4(1 - u)v(1 - v)}
\]
\[
\frac{1}{4u(1-u)} \left( \sqrt{n}(\Gamma_{S_n}^+(u) - \frac{1}{2}u) - \sqrt{n}(\Gamma_{S_n}^-(u) - \frac{1}{2}u) \right)^2 - \frac{1}{4u(1-u)} \left( \sqrt{n}(\Gamma_{S_n}^+(1,1) - \frac{1}{2}) \right)^2 \\
- \frac{\sqrt{n}(\Gamma_{S_n}^-(1,1) - \frac{1}{2})}{(1-u)(1-v)} \left[ \sqrt{n}(\Gamma_{S_n}^+(u) - \frac{1}{2}u) - \sqrt{n}(\Gamma_{S_n}^-(u) - \frac{1}{2}u) \right] \\
- 2\sqrt{n} \left( \Gamma_{S_n}^-(v) - \frac{1}{2}v \right) - \sqrt{n}(\Gamma_{S_n}^-(u) - u) v + 2\sqrt{n} \left( \Gamma_{S_n}^- (u, v) - \frac{1}{2}uv \right) \\
+ \frac{\sqrt{n}(\Gamma_{S_n}^+(u) - \frac{1}{2}u) - \sqrt{n}(\Gamma_{S_n}^-(u) - \frac{1}{2}u)}{2u(1-u)(1-v)} \left[ \sqrt{n}(\Gamma_{S_n}^-(u) - u) v \\
- 2\sqrt{n} \left( \Gamma_{S_n}^- (u, v) - \frac{1}{2}uv \right) + 2u\sqrt{n} \left( \Gamma_{S_n}^- (v) - \frac{1}{2}v \right) \right] + o_P(1)
\]

\[
= \frac{n}{2} - N - \frac{[v\alpha_n(u, 1) - 2\alpha_n^-(u, v)]^2}{4uv(1-v)} - \frac{[2\alpha_n^-(1, v) + v\alpha_n(u, 1) - 2\alpha_n^-(u, v)]^2}{4(1-u)v(1-v)}
\]

\[
- \frac{[\alpha_n^+(u, 1) - \alpha_n^-(u, 1)]^2}{4u(1-u)(1-v)} - \frac{\alpha_n^-(1, 1)^2}{(1-u)(1-v)} - \frac{\alpha_n^-(1, 1)}{(1-u)(1-v)} \\
\times \left[ \alpha_n^+(u, 1) - \alpha_n^-(u, 1) - 2\alpha_n^-(1, v) - v\alpha_n(u, 1) + 2\alpha_n^-(u, v) \right] \\
+ \frac{\alpha_n^+(u, 1) - \alpha_n^-(u, 1)}{2u(1-u)(1-v)} \left[ v\alpha_n(u, 1) + 2u\alpha_n^-(1, v) - 2\alpha_n^-(u, v) \right] + o_P(1). \quad (7)
\]

Applying

\[
- \frac{[v\alpha_n(u, 1) - 2\alpha_n^-(u, v)]^2}{4uv(1-v)} - \frac{[2\alpha_n^-(1, v) + v\alpha_n(u, 1) - 2\alpha_n^-(u, v)]^2}{4(1-u)v(1-v)}
\]

\[
= - \frac{[v\alpha_n(u, 1) - 2\alpha_n^-(u, v)]^2}{4u(1-u)v(1-v)} - \frac{\alpha_n^-(1, 1)}{(1-u)v(1-v)} \left[ v\alpha_n(u, 1) - 2\alpha_n^-(u, v) \right]
\]

\[
- \frac{\alpha_n^- (1, v)^2}{(1-u)v(1-v)},
\]

\[
\frac{[v\alpha_n(u, 1) - 2\alpha_n^-(u, v)]^2}{4uv(1-v)(1-u)} - \frac{[\alpha_n^+(u, 1) - \alpha_n^-(u, 1)]^2}{4u(1-u)(1-v)}
\]

\[
= - \frac{[v\alpha_n^-(u, 1) - \alpha_n^-(u, 1)]^2}{4u(1-u)} \\
\frac{[v\alpha_n^-(u, 1) - \alpha_n^-(u, 1)]^2}{u(1-u)v(1-v)} - \frac{\alpha_n^+(u, 1) - \alpha_n^-(u, 1)}{2u(1-u)(1-v)} \left[ v\alpha_n(u, 1) - 2\alpha_n^-(u, v) \right],
\]

\[
- \frac{[v\alpha_n^-(u, 1) - \alpha_n^-(u, 1)]^2}{uv(1-v)(1-u)} - \frac{\alpha_n^-(1, v)^2}{(1-u)v(1-v)}
\]

\[
= 2 \frac{\alpha_n^-(1, v)}{(1-u)v(1-v)} \left[ v\alpha_n^-(u, 1) - \alpha_n^-(u, v) \right]
\]
Hence we find to the right-hand side of (7) yields

\[
\begin{align*}
- \frac{[v\alpha_n^- (u, 1) - \alpha_n^- (u, v) + u\alpha_n^- (1, v)]^2}{u(1-u)v(1-v)} &= - \frac{\alpha_n^- (1, v)^2}{v(1-v)}, \\
- \frac{[v\alpha_n^- (u, 1) - \alpha_n^- (u, v) + u\alpha_n^- (1, v)]^2}{uv(1-v)(1-u)} &= - \frac{\alpha_n^- (1, 1)^2}{(1-u)(1-v)} \\
= - \frac{[v\alpha_n^- (u, 1) + u\alpha_n^- (1, v) - \alpha_n^- (u, v) - uv\alpha_n^- (1, 1)]^2}{uv(1-v)(1-u)} \\
&\times \frac{\alpha_n^- (1, 1)^2}{(1-u)(1-v)} - 2 \frac{\alpha_n^- (1, 1)}{(1-u)(1-v)}[v\alpha_n^- (u, 1) + u\alpha_n^- (1, v) - \alpha_n^- (u, v)], \\
&- \frac{\alpha_n^- (1, v)^2}{v(1-v)} + \frac{uv\alpha_n^- (1, 1)^2}{(1-u)(1-v)} \\
&= - \frac{[\alpha_n^- (1, v) - v\alpha_n^- (1, 1)]^2}{v(1-v)} + 2 \frac{\alpha_n^- (1, 1)}{1-v} \alpha_n^- (1, 1) + \frac{v\alpha_n^- (1, 1)^2}{(1-u)(1-v)}, \\
&- \frac{[\alpha_n^+ (u, 1) - \alpha_n^- (u, 1)]^2}{4u(1-u)} = - \frac{\alpha_n^- (1, 1)}{1-u}[\alpha_n^+ (u, 1) - \alpha_n^- (u, 1)] \\
&= - \frac{[\alpha_n^+ (u, 1) - \alpha_n^- (u, 1) + 2u\alpha_n^- (1, 1)]^2}{4u(1-u)} + \frac{u\alpha_n^- (1, 1)^2}{1-u},
\end{align*}
\]

to the right-hand side of (7) yields

\[
\log R^- (Q_S(u), Q(v)) = \frac{n}{2} - N - \frac{[v\alpha_n^- (u, 1) + u\alpha_n^- (1, v) - \alpha_n^- (u, v) - uv\alpha_n^- (1, 1)]^2}{uv(1-v)(1-u)} - \alpha_n^- (1, 1)^2 \\
- \frac{[\alpha_n^- (1, v) - v\alpha_n^- (1, 1)]^2}{v(1-v)} - \frac{[\alpha_n^+ (u, 1) - \alpha_n^- (u, 1) + 2u\alpha_n^- (1, 1)]^2}{4u(1-u)} + o_P(1).
\]

Because of symmetry we obtain a similar expression for \(\log R^+(Q_S(u), Q(v))\). Hence we find

\[
-2 \log R(Q_S(u), Q(v)) = \frac{[\alpha_n^- (u, v) - v\alpha_n^- (u, 1) - u\alpha_n^- (1, v) + uv\alpha_n^- (1, 1)]^2}{\frac{1}{2}u(1-u)v(1-v)} \\
+ \frac{[\alpha_n^+ (u, v) - v\alpha_n^+ (u, 1) - u\alpha_n^+ (1, v) + uv\alpha_n^+ (1, 1)]^2}{\frac{1}{2}u(1-u)v(1-v)} \\
+ \frac{[\alpha_n^- (1, v) - v\alpha_n^- (1, 1)]^2}{\frac{1}{2}v(1-v)} + \frac{[\alpha_n^+ (1, v) - v\alpha_n^+ (1, 1)]^2}{\frac{1}{2}v(1-v)} + \frac{\alpha_n^- (1, 1)^2}{\frac{1}{4}} \\
+ \frac{[\alpha_n^+ (u, 1) - u\alpha_n^+ (1, 1) - \alpha_n^- (u, 1) + u\alpha_n^- (1, 1)]^2}{u(1-u)} + o_P(1).
\]
Standard empirical process theory and the Skorohod construction (but keeping the same notation), yield, for either choice of sign,

$$\sup_{0 \leq u, v \leq 1} \left| \alpha_n^\pm(u, v) - B^\pm(u, v) \right| \to 0 \quad \text{a.s.}$$

Hence $T_{1n}$ can be replaced by

$$\int_{\varepsilon}^{1-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} K(u, v) \, dv \, d\Gamma_{Sn}(u).$$

Because the integrand is uniformly continuous, this implies (4) by the Helly–Bray theorem.

To show (5), we only consider integration over the L-shaped region

$$C_\varepsilon = \left\{ (u, v) \in (0, 1)^2 : 0 < u \leq \varepsilon, 0 < v \leq \frac{1}{2} \text{ or } 0 < u \leq \frac{1}{2}, 0 < v \leq \varepsilon \right\},$$

because of symmetry arguments. Consider the following five regions:

$$C_{\varepsilon,1,1} = \left\{ (u, v) \in (0, 1)^2 : 0 < u \leq n^{-3/5}, n^{-3/8} \leq v \leq \frac{1}{2} \right\},$$

$$C_{\varepsilon,1,2} = \left\{ (u, v) \in (0, 1)^2 : n^{-3/8} \leq u \leq \frac{1}{2}, 0 < v \leq n^{-3/5} \right\},$$

$$C_{\varepsilon,2} = \left\{ (u, v) \in (0, n^{-3/8}]^2 \right\},$$

$$C_{\varepsilon,3,1} = \left\{ (u, v) \in (0, 1)^2 : n^{-3/5} < u \leq \varepsilon, n^{-3/8} \leq v \leq \frac{1}{2} \right\},$$

$$C_{\varepsilon,3,2} = \left\{ (u, v) \in (0, 1)^2 : n^{-3/8} \leq u \leq \frac{1}{2}, n^{-3/5} < v \leq \varepsilon \right\},$$

which cover $C_\varepsilon$. We will use the following bound: For any $\eta > 0$ there exists a positive constant $M_\eta$, such that

$$\mathbb{P}\left( \Gamma_{Sn}(u) \leq u M_\eta, \Gamma_{Yn}(u) \leq u M_\eta, \text{ for all } 0 \leq u \leq 1 \right) > 1 - \eta,$$ (8)

see Shorack and Wellner (1986, p. 419).

For $C_{\varepsilon,1,1}, C_{\varepsilon,1,2},$ and $C_{\varepsilon,2}$ we only consider $\log R^-$, $\log R^+$ is treated similarly. We regard the four terms of (3) separately. For $C_{\varepsilon,1,1}$ and $C_{\varepsilon,2}$ we get, with (8) and if $P_n(A_j^\pm) \geq \frac{1}{n}, j = 1, \ldots, 4,$ with probability $1 - \eta,$

$$\left| n \Gamma^-(u, v) \log \frac{\Gamma_{Sn}(u)v}{2 \Gamma^-(u, v)} \right| \leq n \Gamma_{Sn}(u) \log \left( \frac{v}{n} \lor \frac{2 \Gamma_{Yn}(v)}{\Gamma_{Sn}(u)v} \right) \leq M_\eta un \log \left( n \lor \frac{2 M_\eta}{n} \right) \leq M_\eta un \log(2 M_\eta n),$$
and
\[ n \left( \Gamma_n^- (u) - \Gamma_n^- (u, v) \right) \log \frac{\Gamma_n(u)(1 - v)}{2 \Gamma_n^- (u) - \Gamma_n^- (u, v)} \]
\[ \leq n \Gamma_n(u) \log \left( n \sqrt{\frac{2(1 - \Gamma_n(v))}{\Gamma_n(u)(1 - v)}} \right) \leq M \eta u n \log (4n), \]
and, with \(|\log(1 + x)| \leq 2|x|\) for \(x \geq -0.5\), with probability \(1 - \eta\),
\[ n \left( \frac{N}{n} - \Gamma_n^- (u) - \Gamma_n^- (v) + \Gamma_n^- (u, v) \right) \log \left( \frac{1 - \Gamma_n(u)(1 - v)}{2 \frac{N}{n} - \Gamma_n^- (u) - \Gamma_n^- (v) + \Gamma_n^- (u, v)} \right) \]
\[ \leq n \left| \Gamma_n(u)(1 - v) - 2 \frac{N}{n} + 2 \Gamma_n^- (u) + 2 \Gamma_n^- (v) - 2 \Gamma_n^- (u, v) \right| \]
\[ \leq n \left| \Gamma_n(u)v - 2 \Gamma_n^- (u, v) \right| + n \left| 2 \Gamma_n^- (v) - v \right| + n \left| \Gamma_n(u) - 2 \Gamma_n^- (u) \right| \]
\[ + n \left| 1 - 2 \frac{N}{n} \right| \]
\[ \leq 6n \eta u + 2n \left| \Gamma_n^- (v) - \frac{1}{2} \right| + 2n \left| \Gamma_n^- (1, 1) - \frac{1}{2} \right| \]
\[ = 6n \eta u + 2n^{1/2} \left| \alpha_n^- (1, v) \right| + 2n^{1/2} \left| \alpha_n^- (1, 1) \right|. \]  
(9)
Furthermore, for \(C_{\varepsilon,2}\), we have, with probability \(1 - \eta\),
\[ n \left( \Gamma_n^- (v) - \Gamma_n^- (u, v) \right) \log \left( \frac{1 - \Gamma_n(u)(1 - v)}{2 \Gamma_n^- (v) - \Gamma_n^- (u, v)} \right) \leq M \eta v n \log (2M \eta n), \]
and for \(C_{\varepsilon,1,1}\), employing the Taylor expansion as in (9), with probability \(1 - \eta\),
\[ n \left( \Gamma_n^- (v) - \Gamma_n^- (u, v) \right) \log \left( \frac{1 - \Gamma_n(u)(1 - v)}{2 \Gamma_n^- (v) - \Gamma_n^- (u, v)} \right) \]
\[ \leq n \left| (1 - \Gamma_n(u))v - 2 \Gamma_n^- (v) + 2 \Gamma_n^- (u, v) \right| \leq 3n \eta u + 2n \left| \Gamma_n^- (v) - \frac{1}{2} \right| \]
\[ = 3n \eta u + 2n^{1/2} \left| \alpha_n^- (1, v) \right|. \]
Combining the above, we have with probability \(1 - 2\eta\)
\[ \int \int_{C_{\varepsilon,1,1}} \left| \log R^- (Q_S(u), Q(v)) \right| d \Gamma_n(u) dv \]
\[ \leq \int \int_{C_{\varepsilon,1,1}} M \eta u n \log (2M \eta n) \]
$$\begin{align*}
&+ \log(4n) + 9nM_\eta u + 4n^{1/2}\ |\alpha_n^-(1, v) | + 2n^{1/2}\ |\alpha_n^-(1, 1) | \ d\Gamma_{S_n}(u) \ dv \\
&\leq \left( 2M_\eta n^{2/5} \log(4M_\eta n) + 9M_\eta n^{2/5} + 4n^{1/2} \ \sup_{0 \leq v \leq 1} |\alpha_n^-(1, v) | + 2n^{1/2} |\alpha_n^-(1, 1) | \right) \\
&\times \int_{n^{-3/5}}^{1/2} \int_0^n d\Gamma_{S_n}(u) \ dv \to 0,
\end{align*}$$

and

$$\begin{align*}
&\int_{C_{\varepsilon, 2}} \left| \log R^-(Q_S(u), Q(v)) \right| d\Gamma_{S_n}(u) \ dv \\
&\leq \int_{C_{\varepsilon, 2}} (M_\eta u + M_\eta v)n \log(2M_\eta n) \\
&+ M_\eta un \log(4n) + 6nM_\eta u + 2n^{1/2} |\alpha_n^-(1, v) | + 2n^{1/2} |\alpha_n^-(1, 1) | \ d\Gamma_{S_n}(u) \ dv \\
&\leq \left( 3M_\eta n^{5/8} \log(4M_\eta n) + 6M_\eta n^{5/8} + 2n^{1/2} \ \sup_{0 \leq v \leq 1} |\alpha_n^-(1, v) | + 2n^{1/2} |\alpha_n^-(1, 1) | \right) \\
&\times \int_0^n d\Gamma_{S_n}(u) \ dv \to 0.
\end{align*}$$

The region $C_{\varepsilon, 1.2}$ can be treated in a similar way as $C_{\varepsilon, 1.1}$.

For $C_{\varepsilon, 3.1}$ and $C_{\varepsilon, 3.2}$ we use $|\log(1 + x) - x| \leq x^2$, for $x \geq -0.5$, and the convergence in probability of $P_n / P$ uniform over certain rectangles (the $A_{t+}$) to 1. This follows from, e.g., Einmahl (1987, Inequality 2.9 or Theorem 3.3). Then, with probability tending to 1,

$$\begin{align*}
&\left| \log R(Q_S(u), Q(v)) \right| \\
&\leq \frac{[v\alpha_n^-(1, 1) - u\alpha_n^-(1, 1) - \alpha_n^-(u, v) + u\alpha_n^-(1, v)]^2}{\frac{1}{2} u(1 - u)v(1 - v)} \\
&+ \frac{[v\alpha_n^+(u, 1) - u\alpha_n^+(1, u) - \alpha_n^+(u, v) + u\alpha_n^+(1, v)]^2}{\frac{1}{2} u(1 - u)v(1 - v)} \\
&+ \frac{[\alpha_n^-(1, v) - v\alpha_n^-(1, 1)]^2}{\frac{1}{2} v(1 - v)} + \frac{[\alpha_n^+(1, v) - v\alpha_n^+(1, 1)]^2}{\frac{1}{2} v(1 - v)} \\
&+ \frac{[\alpha_n^+(u, 1) - u\alpha_n^+(1, 1) - \alpha_n^-(u, 1) + u\alpha_n^-(1, 1)]^2}{u(1 - u)} + 4\alpha_n^-(1, 1)^2 \\
&\leq 4\alpha_n^-(u, v)^2 + v^2\alpha_n^-(u, 1)^2 + u^2\alpha_n^-(1, v)^2 + u^2v^2\alpha_n^-(1, 1)^2 \\
&+ \frac{4\alpha_n^+(u, v)^2 + v^2\alpha_n^+(u, 1)^2 + u^2\alpha_n^+(1, v)^2 + u^2v^2\alpha_n^+(1, 1)^2}{\frac{1}{2} u(1 - u)v(1 - v)}
\end{align*}$$
\[+ 2 \frac{\alpha_n^-(1, v)^2 + v^2 \alpha_n^-(1, 1)^2}{2v(1-v)} + 2 \frac{\alpha_n^+(1, v)^2 + v^2 \alpha_n^+(1, 1)^2}{2v(1-v)} + 2 \alpha_n^+(1, 1)^2\]
\[+ 4 \frac{\alpha_n^+(u, 1)^2 + u^2 \alpha_n^+(1, 1)^2 + \alpha_n^-(u, 1)^2 + u^2 \alpha_n^-(1, 1)^2}{u(1-u)} + 2 \alpha_n^-(1, 1)^2\]
\[\leq 32 \left[ \frac{\alpha_n^-(u, v)^2 + \alpha_n^+(u, v)^2}{uv} + \frac{\alpha_n^-(u, 1)^2 + \alpha_n^+(u, 1)^2}{u} + \frac{\alpha_n^-(1, v)^2 + \alpha_n^+(1, v)^2}{v} + 2 \alpha_n^-(1, 1)^2 \right].\]

Theorem 3.1 in Einmahl (1987) yields, for either choice of sign,
\[
\sup_{0 < u, v \leq 1} \frac{\alpha_n^\pm(u, v)}{(uv)^{1/4}} = O_P(1).
\]

Hence we find
\[
\int\int_{C_{\varepsilon, 3,1} \cup C_{\varepsilon, 3,2}} \left| \log R(Q_S(u), Q(v)) \right| d\Gamma_{Sn}(u) dv
\]
\[= O_P(1) \cdot \int\int_{C_{\varepsilon, 3,1} \cup C_{\varepsilon, 3,2}} \left( \frac{1}{\sqrt{uv}} + \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}} + 1 \right) d\Gamma_{Sn}(u) dv
\]
\[= O_P(1) \cdot \int\int_{C_{\varepsilon, 3,1} \cup C_{\varepsilon, 3,2}} \frac{1}{\sqrt{uv}} d\Gamma_{Sn}(u) dv = O_P(\sqrt{\varepsilon}),
\]
uniformly in \(\varepsilon\), because of (8). This completes the proof of (5). \(\square\)

References


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