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**SOLUTION CONCEPTS FOR GAMES WITH GENERAL
COALITION STRUCTURE**

By Gleb Koshevoy, Dolf Talman

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Solution concepts for games with general coalitional structure¹

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Abstract

We introduce a theory on marginal values and their core stability for cooperative games with arbitrary coalition structure. The theory is based on the notion of nested sets and the complex of nested sets associated to an arbitrary set system and the M-extension of a game for this set. For a set system being a building set or partition system, the corresponding complex is a polyhedral complex, and the vertices of this complex correspond to maximal strictly nested sets. To each maximal strictly nested set is associated a rooted tree. Given characteristic function, to every maximal strictly nested set a marginal value is associated to a corresponding rooted tree as in [9]. We show that the same marginal value is obtained by using the M-extension for every permutation that is associated to the rooted tree. The GC-solution is defined as the average of the marginal values over all maximal strictly nested sets. The solution can be viewed as the gravity center of the image of the vertices of the polyhedral complex. The GC-solution differs from the Myerson-kind value defined in [2] for union stable structures. The HS-solution is defined as the average of marginal values over the subclass of so-called half-space nested sets. The NT-solution is another solution and is defined as the average of marginal values over the subclass of NT-nested sets. For graphical buildings the collection of NT-nested sets corresponds to the set of spanning normal trees on the underlying graph and the NT-solution coincides with the average tree solution. We also study core stability of the solutions and show that both the HS-solution and NT-solution belong to the core under half-space supermodularity, which is a weaker condition than convexity of the game.

For an arbitrary set system we show that there exists a unique minimal building set containing the set system. As solutions we take the solutions for this building covering by extending in a natural way the characteristic function to it by using its Möbius inversion.

Key words: Core, polytope, building set, nested set complex, Möbius inversion, permutations, normal fan, average tree solution, Myerson value

AMS subject classification: 47H10, 49J40, 52C40, 90C30, 91B50.

JEL code: C71.

1 Introduction

In the classical model of cooperative games it is assumed that any subset of players can form a coalition. In many situations of cooperations, however, there are restrictions for forming coalitions. One of the most well-known examples is the Myerson communication graph game, see [18], where the possibility of forming coalitions is modeled by means of a communication graph. The vertices of the graph are identified with the players, and players that are connected by an edge can communicate. A coalition is a subset of players satisfying that the induced subgraph on the subset is a connected set. Another example is a game on a partially ordered set, or poset, e.g. see [10, 14]. In such a game there is a partial order on the players set, for example a kind of hierarchy, and a coalition is a subset of players which form an ideal, or filter, that is all players which are dominated by a player, or dominate the player, have to be members of the coalition. An extension of games on posets was considered in Bilbao and Edelman [4]. In these games there is an anti-exchange closure operator on the set of players, and a coalition can be formed only if it is a closed set with respect to this closure operator. In [2] games are considered for which the coalitions that can be formed are elements of a union stable system. In all these models, marginal values are defined, although using different methods, and Shapley or Myerson-type of values were studied as solutions, that is the average of possibly multiple marginal values is taken as solution concept to determine how much payoff every player will get. For a game with general coalition structure one faces two problems. One is to find an appropriate notion of a "marginal value" and another one is to count properly such values.

We consider cooperative games with general coalitional structure. The structure is a set system \mathcal{F} , being a collection of subsets of a finite set of n elements, denoted by $[n] := \{1, \dots, n\}$. Elements of $[n]$ might be considered as economic agents, or players, and elements of \mathcal{F} , which will always include the grand coalition $[n]$, are coalitions which players are able to form. A game on the set system \mathcal{F} is a characteristic function $v : \mathcal{F} \rightarrow \mathbb{R}$ assigning to each coalition S in \mathcal{F} its worth $v(S)$. Given a set system, a solution is a mapping from the class of games on it to the n -dimensional vector space \mathbb{R}^n and assigns for any game a payoff to every player.

In this paper we first construct for arbitrary set systems on $[n]$ the *M-extension* of a game, by using its Möbius inversion, and then discuss several solution concepts. For set systems being stable under union the M-extension of a game coincides with the restricted game introduced in [1], see also [2]. By using the M-extension of a game we introduce its so-called M-value. This solves the problem of defining marginal values for a general coalition structure. In case the set system is given by the sets of vertices of connected subgraphs of a connected graph the M-value is the Myerson value introduced in [18]. For

set systems being stable under union the M-value coincides with the Myerson value as defined in [2]. However, the M-value does not solve the second problem, it does not count properly marginal values, since marginal values may coincide.

For solving the proper counting of marginal values, we first consider so-called partition systems or building sets, introduced by Algaba et al. [3] and Postnikov [19]. A set system on $[n]$ is a partition system or building set if it contains as coalitions that can be formed all singletons and together with any two non-disjoint sets also the union of these sets. The latter property is called union stability, see [1]. It means that any two coalitions in the building set having elements in common are able to merge to a new coalition. To a connected graph is associated the graphical building set consisting of the vertex sets of all connected subgraphs of the graph. The class of graph games, introduced in [18], is therefore a subclass of the class of games on buildings sets. In general, for any set system on $[n]$ there is a unique minimal building set, called its building covering, containing the set system.

An important property of a partition system or building set on $[n]$ is that any subset of $[n]$ has a unique partition that consists of maximal (with respect to set-inclusion) elements of the building set. Due to this property, to any permutation of $[n]$ a unique collection of n elements of the building set is associated. Namely, let $\sigma = (\sigma(1), \dots, \sigma(n))$ be a permutation of the elements of $[n]$. Then the subset $[n] \setminus \{\sigma^{-1}(n)\}$ is uniquely partitioned into maximal elements of the building set, say, S_1, \dots, S_k . For each subset S_j , $j = 1, \dots, k$, denote by r_j the maximal element of $[n]$ in S_j such that $\sigma^{-1}(r_j) \in S_j$, and next partition each set $S_j \setminus \{\sigma^{-1}(r_j)\}$ into maximal elements, and so on. In this way, we obtain a so-called strictly nested set, being a family of elements of the building set, including $[n]$ itself, such that any two of its elements are either disjoint or one is a subset of the other, and, moreover, the union of any collection of subsets of disjoint elements of the family is not an element of the building set. Nested sets for buildings were introduced in algebraic geometry, see, for example, [6, 15, 19]. When a strictly nested set consists of n different sets it is said to be maximal. For a building set system the set of maximal strictly nested sets is nonempty. Any such set describes a unique way in which the grand coalition $[n]$ can be built from elements of only the building set by, starting with the empty set, letting players join to, possibly more than one, coalitions of the building set to obtain larger coalitions of the building set until the grand coalition has been obtained when a last player joins.

In case the building set is the power set of $[n]$ there are $n!$ maximal strictly nested sets and each maximal strictly nested set is a chain, a line-tree, that corresponds to a unique permutation. In case for an arbitrary building set a maximal strictly nested set is not a chain, it corresponds to a rooted tree and therefore to collection of permutations, being the linear extensions of the same partially ordered set that is induced by the structure of the

strictly nested set. The maximal strictly nested sets are the vertices of the nested complex induced by the set system. Given a game on a building set, for any maximal strictly nested set we define its marginal value. The rooted tree corresponding to a maximal strictly nested set is endowed with a natural orientation of edges in the direction from the root to the leaves. Such an orientation of edges defines a partially ordered set on the vertices of the tree. The connected ideals of such a poset are all elements of the building set. Since a vertex of a nested complex corresponds to a rooted tree on $[n]$ and all connected ideals of this tree are elements of the building set, we may define the marginal value by the same rule as in [9]. We show that the marginal value with respect to any maximal strictly nested set defined in this way equals the marginal value of the M-extension of the game calculated with respect to any permutation being a linear extension of the poset corresponding to the induced rooted tree.

For a cooperative game on a building set we introduce several new solution concepts which take into account a proper counting of marginal values. The GC-solution (*Gravity Center* solution) is defined as the average of marginal values over all vertices of the nested complex (maximal strictly nested sets) of the building set. The GC-value differs from the value defined in [2].

The HS-solution (*Half-Space* solution) is defined as the average of marginal values over a specific class of maximal strictly nested sets, the class of so-called HS-nested sets. An element of a building set is a half-space if its complement is also an element of the building set. A maximal strictly nested set is then an HS-nested set if every element of it not being a singleton is a half-space. It means that in the nested set a player can only join coalitions to form a larger coalition if the set of players that have not joined yet also form a coalition.

The NT-solution (*Normal Tree* solution) is defined as the average of marginal values over another specific class of maximal strictly nested sets, the class of so-called NT-nested sets. A maximal strictly nested set is an NT-nested set if, for any $i \in [n]$ and successor j of i in the corresponding rooted tree, $\{i, j\}$ is an element of the building set. On graphical buildings the collection of NT-nested sets corresponds to the collection of spanning normal trees on the graph, introduced in [11]. It means that in the nested set a player can only join a coalition if he is connected to the player of the coalition which he dominates in the tree.

For arbitrary building sets the HS- or the NT-solution may not exist. For graphical buildings, however, the set of NT-nested sets is a nonempty subset of the set of HS-nested sets. In case the set system is the power set all solutions, including the M-value, coincide and are equal to the Shapley value. On the class of graphical buildings, the NT-solution coincides with the average tree solution, introduced in [17].

We study core stability of all these solutions. The core of a game consists of the payoff vectors that cannot be blocked by any coalition in the set system. We prove that, given a building set \mathcal{B} , all marginal values and therefore also both the GC-solution and the M-value belong to the core if the game is \mathcal{B} -supermodular and \mathcal{B} -superadditive. Both these concepts are defined with respect to the specific underlying building set. We introduce the notion of *half-space \mathcal{B} -supermodularity*, which weakens \mathcal{B} -supermodularity, while the latter property is weaker than supermodularity. For an HS-nested set in \mathcal{B} the corresponding marginal vector belongs to the core if the game is half-space \mathcal{B} -supermodular and \mathcal{B} -superadditive. This is a generalization of a core stability result in [9]. For an NT-nested set in \mathcal{B} the corresponding marginal vector belongs to the core if the game is half-space \mathcal{B} -supermodular and 2-superadditive. If the game is totally positive, the GC-solution is the gravity center of the core. This property does not hold for the Myerson value defined in [2].

For a general set system \mathcal{F} on $[n]$, there exists a unique minimal building set $\mathcal{B}(\mathcal{F})$, the so-called building covering of \mathcal{F} , which contains \mathcal{F} . For a game v on \mathcal{F} , we take its M-extension of v with respect to $\mathcal{B}(\mathcal{F})$ to obtain a game $v^{\mathcal{F}}$ on the building covering. The GC-, HS-, NT-, and M-solutions for this latter game we define as the corresponding solutions for the original game v . Core stability of these solutions are provided by the corresponding conditions for the building covering. On the class of convex geometries we compare our solutions with the solution proposed in [4].

The paper is organized as follows. In Section 2, for a given arbitrary set system we introduce the concept of strictly nested set, and for a function on it we define its M-extension. We show that to every maximal strictly nested set there corresponds a rooted tree and define for this tree a marginal value. To each such tree is associated a set of permutations, being the collection of all linear extensions of the poset induced by the rooted tree. We prove that for each such permutation the marginal payoff vector for the M-extension coincides with the marginal vector. In Section 3 we study building sets, maximal strictly nested sets and the M-extension of games on building sets. In Section 4 we introduce the GS-, HS-, NT-solution concepts for building sets and compare them. In Section 5 core stability of the solutions is studied. In Section 6 we consider arbitrary set systems.

2 Strictly nested sets and the M-extension

Let $[n] = \{1, \dots, n\}$ be a finite set and let $\mathcal{F} \subseteq 2^{[n]}$ be a given set system on $[n]$. We assume that both \emptyset and $[n]$ belong to \mathcal{F} and that for any function $v : \mathcal{F} \rightarrow \mathbb{R}$ it holds that $v(\emptyset) = 0$.

For a function $f : 2^{[n]} \rightarrow \mathbb{R}$, let $\mu : 2^{[n]} \rightarrow \mathbb{R}$ be its Möbius inversion, i.e., μ satisfies

$$f(T) = \sum_{T' \subseteq T} \mu(T'), \quad T \in 2^{[n]}.$$

The Möbius inversion of f is given by

$$\mu(T) = \sum_{T' \subseteq T} (-1)^{|T|-|T'|} f(T'), \quad T \in 2^{[n]}.$$

Definition 2.1 *Let $v : \mathcal{F} \rightarrow \mathbb{R}$ be a function, then the M-extension $v^{\mathcal{F}} : 2^{[n]} \rightarrow \mathbb{R}$ of v is given by the following conditions:*

(i) $v^{\mathcal{F}}|_{\mathcal{F}} = v$.

(ii) *For the Möbius inversion of $v^{\mathcal{F}}$, $\mu^{\mathcal{F}}$, it holds that $\mu^{\mathcal{F}}(S) = 0$ for every $S \notin \mathcal{F}$.*

Theorem 2.2 *For a function $v : \mathcal{F} \rightarrow \mathbb{R}$ its M-extension $v^{\mathcal{F}}$ is well defined.*

Proof. Consider the system of linear equations:

$$v(S) = \sum_{T \in \mathcal{F} | T \subseteq S} \mu(T), \quad S \in \mathcal{F}.$$

The matrix for this system corresponds after appropriate reordering of columns and rows by set inclusion to a $(0, 1)$ upper-triangular square matrix with all ones on the diagonal. Therefore the system has a unique solution $\mu(S)$, $S \in \mathcal{F}$. Define $\mu^{\mathcal{F}} : 2^{[n]} \rightarrow \mathbb{R}$ by $\mu^{\mathcal{F}}(S) = \mu(S)$ if $S \in \mathcal{F}$ and $\mu^{\mathcal{F}}(S) = 0$ if $S \notin \mathcal{F}$. Then $v^{\mathcal{F}}$ is the unique function for which $\mu^{\mathcal{F}}$ is its Möbius inverse, i.e.,

$$v^{\mathcal{F}}(S) = \sum_{T \in \mathcal{F} | T \subseteq S} \mu(T), \quad S \in 2^{[n]}.$$

Q.E.D.

We have the following interesting property of the M-extension. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2$ be two set systems and let $v : \mathcal{F}_1 \rightarrow \mathbb{R}$ be a function, then it holds that

$$v^{\mathcal{F}_1} = (v^{\mathcal{F}_1}|_{\mathcal{F}_2})^{\mathcal{F}_2}. \quad (1)$$

In particular this property holds when \mathcal{F}_2 is equal to \mathcal{F}_1 .

Definition 2.3 *A subset \mathcal{N} of \mathcal{F} is a strictly nested set if it satisfies the following conditions:*

(G1) *For any different $S, T \in \mathcal{N}$ it holds that either $S \subset T$ or $T \subset S$ or $S \cap T = \emptyset$.*

(G2) *For any collection of $k, k \geq 2$, disjoint subsets T_1, \dots, T_k in \mathcal{N} it holds that*

$$T'_1 \cup \dots \cup T'_k \notin \mathcal{F}$$

for any nonempty $T'_j \subseteq T_j, j = 1, \dots, k$.

(G3) $[n] \in \mathcal{N}$.

Property (G1) is known under the names *nested sets*, *laminar* or *hierarchy*, see, for example, [16]. Property (G2) is the strength of the nested property, see, for example, [6]. Notice that a *chain* $\{N_1, N_2, \dots, N_{k-1}, [n]\}$ of length k with $N_1 \subset N_2 \subset \dots \subset N_{k-1} \subset [n]$ and $N_j \in \mathcal{F}$ for $j = 1, \dots, k-1$, $1 \leq k \leq n$, is a strictly nested set, since (G2) is automatically fulfilled.

Consider the set system \mathcal{F} consisting of all singletons of $[n]$ and $[n]$ itself. Then any strictly nested set \mathcal{N} consists of some collection of singletons and the set $[n]$. In this example chains are of the form $\{[n]\}$ and $\{\{i\}, [n]\}$ for $i = 1, \dots, n$.

To any strictly nested set \mathcal{N} of \mathcal{F} there corresponds a rooted tree $F^{\mathcal{N}}$, whose vertex set is indexed by a partition of $[n]$, defined as follows. Because of (G1) and (G3) for any strictly nested set \mathcal{N} and $i \in [n]$ there is a unique minimal element in \mathcal{N} , denoted $T^{\mathcal{N}}(i)$, containing i . Let the ordering $\preceq^{\mathcal{N}}$ on $[n]$ be defined by $i \preceq^{\mathcal{N}} j$ if $T^{\mathcal{N}}(i) \subseteq T^{\mathcal{N}}(j)$ and consider the partition of $[n]$ constituted from sets being equivalent elements of $[n]$ with respect to $\preceq^{\mathcal{N}}$. Consider the factor-set $[n]/\sim^{\mathcal{N}}$, that is an element of $[n]/\sim^{\mathcal{N}}$ corresponds to a set of equivalent elements. The ordering $\preceq^{\mathcal{N}}$ induces a poset on $[n]/\sim^{\mathcal{N}}$. The Hasse-diagram of this poset is the rooted tree $F^{\mathcal{N}}$. More precisely, consider all maximal elements of \mathcal{N} different from $[n]$. Let $T^{\mathcal{N}}(i_1), \dots, T^{\mathcal{N}}(i_m)$ be those sets. Because of (G2) their union is not equal to $[n]$. Hence, the set $[n] \setminus (T^{\mathcal{N}}(i_1) \cup \dots \cup T^{\mathcal{N}}(i_m))$ consists of equivalent elements and is the root of the tree $F^{\mathcal{N}}$. The successors of the root are formed by the roots in the subtrees corresponding to the restrictions of \mathcal{N} to each of the sets $T^{\mathcal{N}}(i_1), \dots, T^{\mathcal{N}}(i_m)$. The existence of the tree follows by induction, since the restriction of a strictly nested set to any such set is a strictly nested set with respect to the restriction of \mathcal{F} to the same set.

A strictly nested set \mathcal{N} is *maximal* if it contains n different nonempty sets. Notice that an arbitrary set system may not have a maximal strictly nested set. To every maximal strictly nested set \mathcal{N} in \mathcal{F} there corresponds a rooted tree $F^{\mathcal{N}}$ with vertex set $[n]$. In such a case, the ordering $\preceq^{\mathcal{N}}$ has no multiple equivalent elements, and therefore $([n], \preceq^{\mathcal{N}})$ is a poset. The tree corresponding to a maximal strictly nested set \mathcal{N} describes a particular way how the grand coalition $[n]$ can be formed by letting players join allowable coalitions to form larger allowable coalitions, starting with the empty set. For a maximal strictly nested set \mathcal{N} and $i \in [n]$, let $S^{\mathcal{N}}(i)$ be the set of successors of i in the tree $F^{\mathcal{N}}$ i.e., $j \in S^{\mathcal{N}}(i)$ if $T^{\mathcal{N}}(j)$ is a maximal element of \mathcal{N} in $T^{\mathcal{N}}(i) \setminus \{i\}$. When player i forms the larger coalition $T^{\mathcal{N}}(i)$ in \mathcal{N} he joins simultaneously all allowable coalitions $T^{\mathcal{N}}(j)$, $j \in S^{\mathcal{N}}(i)$, that were formed by his successors. These latter sub-coalitions form a partition of the set $T^{\mathcal{N}}(i) \setminus \{i\}$ of subordinates of i (property G1) and satisfy that their union is not allowable (property G2), i.e., these coalitions or subsets of them are not able to cooperate without player i . At last one player, the root of the tree, forms the grand coalition $[n]$, which is also allowable (property G3), by joining simultaneously all allowable coalitions formed by his successors.

The collection of maximal strictly nested sets in \mathcal{F} describes all different possibilities in which the grand coalition can be formed in this way.

For a maximal strictly nested set \mathcal{N} of \mathcal{F} , denote by $\mathfrak{S}^{\mathcal{N}}$ the set of permutations on $[n]$ which are linear extensions (total orderings of $[n]$) of the poset $([n], \prec^{\mathcal{N}})$. In this way to a set system \mathcal{F} is associated the set of permutations

$$\mathfrak{S}^{\mathcal{F}} := \cup_{\mathcal{N}} \mathfrak{S}^{\mathcal{N}},$$

where the union is taken over the set of all maximal strictly nested sets in \mathcal{F} .

Now we show how this set of permutations is related to the M-extension. For this the following notion is of use, where \mathfrak{S}_n is the set of all permutations of $[n]$.

Definition 2.4 *Let $f : 2^{[n]} \rightarrow \mathbb{R}$ be a function and let $\sigma \in \mathfrak{S}_n$ be a permutation. Then the marginal vector $m^f(\sigma)$ is given by*

$$m_{\sigma^{-1}(i)}^f(\sigma) = f(\{\sigma^{-1}(1), \dots, \sigma^{-1}(i)\}) - f(\{\sigma^{-1}(1), \dots, \sigma^{-1}(i-1)\}), \quad i \in [n].$$

To each permutation σ of $[n]$ corresponds a *chain* of n non-disjoint sets, denoted \mathcal{N}_σ . \mathcal{N}_σ is a maximal strictly nested set in $2^{[n]}$. However, \mathcal{N}_σ may not be a maximal strictly nested set in \mathcal{F} , also not when $\sigma \in \mathfrak{S}^{\mathcal{F}}$.

Theorem 2.5 *Let $v : \mathcal{F} \rightarrow \mathbb{R}$ be a function and let $v^{\mathcal{F}}$ be its M-extension. Then, for any maximal strictly nested set \mathcal{N} of \mathcal{F} and every pair of permutations $\sigma_1, \sigma_2 \in \mathfrak{S}^{\mathcal{N}}$, it holds that*

$$m^{v^{\mathcal{F}}}(\sigma_1) = m^{v^{\mathcal{F}}}(\sigma_2).$$

Since for a game v on \mathcal{F} the marginal vector $m^{v^{\mathcal{F}}}(\sigma)$ is the same for all $\sigma \in \mathfrak{S}^{\mathcal{N}}$, we obtain for every maximal strictly nested set \mathcal{N} of \mathcal{F} a unique marginal vector. This payoff vector is denoted by $m^v(\mathcal{N})$ and called the marginal vector with respect to \mathcal{N} . Notice that

$$m_i^v(\mathcal{N}) = v(T^{\mathcal{N}}(i)) - \sum_{j \in S^{\mathcal{N}}(i)} v(T^{\mathcal{N}}(j)), \quad i \in [n].$$

The payoff $m_i^v(\mathcal{N})$, $i \in [n]$, is the marginal contribution of i when he joins his subordinates in the tree $F^{\mathcal{N}}$. The marginal vector $m^v(\mathcal{N})$ can also be interpreted as follows. Restrict v to \mathcal{N} and consider the Möbius inversion $\mu^{\mathcal{N}}$ of the M-extension of this restricted function. Then

$$m_i^v(\mathcal{N}) = \mu^{\mathcal{N}}(T^{\mathcal{N}}(i)), \quad i = 1, \dots, n.$$

For the proof of Theorem 2.5 we need the following lemma.

Lemma 2.6 *Let $v : \mathcal{F} \rightarrow \mathbb{R}$ be a function and let $v^{\mathcal{F}}$ be its M -extension. Let T_1 and T_2 be two disjoint subsets of $2^{[n]}$ such that for any nonempty $T'_1 \subseteq T_1$ and $T'_2 \subseteq T_2$ it holds that $T'_1 \cup T'_2 \notin \mathcal{F}$. Then*

$$v^{\mathcal{F}}(T_1 \cup T_2) = v^{\mathcal{F}}(T_1) + v^{\mathcal{F}}(T_2).$$

Proof. The proof follows because for the Möbius inversion $\mu^{\mathcal{F}}$ of $v^{\mathcal{F}}$ it holds that $\mu^{\mathcal{F}}(T'_1 \cup T'_2) = 0$ for any $T'_1 \subseteq T_1$ and $T'_2 \subseteq T_2$. Q.E.D.

Proof of Theorem 2.5. If the maximally strictly nested set \mathcal{N} is a chain, then to this chain corresponds only one permutation, and the proposition holds.

Now suppose \mathcal{N} is not a chain and therefore contains disjoint elements. Let σ_1 and σ_2 be two different permutations in $\mathfrak{S}^{\mathcal{N}}$. Then they are different in possibilities to shuffle these disjoint elements, say, T_1, \dots, T_k . Because of property (G2) and the previous lemma, any shuffle yields the same marginal contribution, as for example the shuffle which orders all elements of T_1 above all elements of T_2 , and so on. Q.E.D.

Corollary 2.7 *Let \mathcal{N}_1 and \mathcal{N}_2 be two different maximal strictly nested sets in \mathcal{F} , then*

$$\mathfrak{S}^{\mathcal{N}_1} \cap \mathfrak{S}^{\mathcal{N}_2} = \emptyset.$$

Proof. Since the union of \mathcal{N}_1 and \mathcal{N}_2 is not a strictly nested set, there exists a function v on the set system \mathcal{F} which has different marginal vectors, i.e., $m^v(\mathcal{N}_1) \neq m^v(\mathcal{N}_2)$. Therefore $\mathfrak{S}^{\mathcal{N}_1} \cap \mathfrak{S}^{\mathcal{N}_2} = \emptyset$. Q.E.D.

From this corollary it follows that $\mathfrak{S}^{\mathcal{F}}$ is partitioned into $\mathfrak{S}^{\mathcal{N}}$ over all maximal strictly nested sets \mathcal{N} in \mathcal{F} . The collection $\mathcal{N}(\mathcal{F})$ of strictly nested sets in \mathcal{F} forms a poset with respect to inclusion, that is for $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$, we set $\mathcal{A} \leq \mathcal{B}$ if $\mathcal{A} \supseteq \mathcal{B}$, and one can define a corresponding cell complex $C(\mathcal{F})$ such that the open cells of the complex are indexed by the strictly nested sets and the closure of a cell contains all cells which are dominated by it in the poset. The vertices of $C(\mathcal{F})$ correspond to the maximal strictly nested sets in \mathcal{F} .

We also may consider the dual collection of simplicial cones. Namely, to a strictly nested set \mathcal{N} in \mathcal{F} is associated a cone $K(\mathcal{N})$ in \mathbb{R}^n spanned by vectors ξ_T , $T \in \mathcal{N}$, where ξ_T is the characteristic function of T , that is $\xi_{T,i} = 1$ if $i \in T$ and zero otherwise, and the vectors $\pm \xi_{[n]}$. We denote $\Sigma(\mathcal{F})$ the collection of these simplicial cones.

In the next section we consider an important class of set systems such that, for any set system \mathcal{F} of the class, $\Sigma(\mathcal{F})$ is a normal fan to a polytope and the complex of the faces of that polytope is the cell complex $C(\mathcal{F})$.

3 Building sets

By Algaba, Bilbao, and López, see [3], the concept of partition system is introduced as a combinatorial abstraction of connected subgraphs of a graph. Partition systems are equivalent to the following concept introduced by Postnikov, see [19].

Definition 3.1 *A set system \mathcal{B} on $[n]$ is a building set if it satisfies the following conditions:*

- (B1) *For any $S, T \in \mathcal{B}$ such that $S \cap T \neq \emptyset$ it holds that $S \cup T \in \mathcal{B}$.*
- (B2) *\mathcal{B} contains all singletons $\{i\}$, $i \in [n]$.*

Condition (B1) is known as union stability and is introduced in [1]. We assume again that any building set contains $[n]$.

Example 3.2 *Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G) = [n]$ and edge set $E(G) \subseteq \{\{i, j\} \subset [n] \mid i \neq j\}$. Then the set system consisting of the vertex sets of all connected subgraphs of G forms a building set, called the graphical building $\mathcal{B}(G)$ of G .*

Postnikov, see [19], defines \mathcal{B} -nested sets for a building set \mathcal{B} as follows.

Definition 3.3 *A subset \mathcal{N} of a building set \mathcal{B} on $[n]$ is a \mathcal{B} -nested set if it satisfies the following conditions:*

- (N1) *For any different $S, T \in \mathcal{N}$, it holds that either $S \subset T$ or $T \subset S$ or $S \cap T = \emptyset$.*
- (N2) *For any collection of k , $k \geq 2$, disjoint proper subsets T_1, \dots, T_k in \mathcal{N} it holds that $T_1 \cup \dots \cup T_k \notin \mathcal{B}$.*
- (N3) *$[n] \in \mathcal{N}$.*

By definition, any strictly nested set in \mathcal{B} is a \mathcal{B} -nested set. The converse is also true.

Lemma 3.4 *Let \mathcal{B} be a building set on $[n]$ and let \mathcal{N} be a \mathcal{B} -nested set. Then \mathcal{N} is a strictly nested set in \mathcal{B} .*

Proof. For simplicity suppose that $k = 2$ and for disjoint T_1, T_2 in \mathcal{N} there exist nonempty $T'_1 \subseteq T_1$ and $T'_2 \subseteq T_2$ such that $T'_1 \cup T'_2 \in \mathcal{B}$. Because of property (B1) it holds that $T_1 \cup T'_2 \in \mathcal{B}$ and again by (B1) $T_1 \cup T_2 \in \mathcal{B}$, which contradicts (N2). Q.E.D.

From the lemma it follows that for a building set \mathcal{B} the set of \mathcal{B} -nested sets coincides with the set of strictly nested sets in \mathcal{B} .

Theorem 3.5 *Let \mathcal{B} be a building set on $[n]$. Then for any permutation $\sigma \in \mathfrak{S}_n$ there exists a maximal strictly nested set \mathcal{N} in \mathcal{B} such that $\sigma \in \mathfrak{S}^{\mathcal{N}}$.*

Proof. Let $\sigma \in \mathfrak{S}_n$ be a permutation. We construct a maximal strictly nested set \mathcal{N} such that $\sigma \in \mathfrak{S}^{\mathcal{N}}$ as follows. Step 1: $\{\sigma^{-1}(1)\}$ is an element of \mathcal{B} , thus we set $\mathcal{N} := \{\{\sigma^{-1}(1)\}\}$. Step $k = 2, \dots, n$: Let $N(k)$ be the maximal element of \mathcal{B} which contains $\sigma^{-1}(k)$ and is a subset of $\{\sigma^{-1}(1), \dots, \sigma^{-1}(k)\}$. Such a set exists, due to the definition of a building set and since $\{\sigma^{-1}(k)\} \in \mathcal{B}$. We set $\mathcal{N} := \mathcal{N} \cup \{N(k)\}$. After n steps, we will add $[n]$ and since $[n] \in \mathcal{B}$ we obtain a collection \mathcal{N} of n nested sets containing also $[n]$. We have to check the validity of (N2). Suppose (N2) is not valid, then there exists disjoint sets $N(k_1)$ and $N(k_2)$ such that $N(k_1) \cup N(k_2) \in \mathcal{B}$. Let $k_2 > k_1$, then $N(k_1) \cup N(k_2) \in \mathcal{B}$ and $N(k_1) \cup N(k_2) \in \{\sigma^{-1}(1), \dots, \sigma^{-1}(k_2)\}$, which contradicts maximality of $N(k_2)$. Q.E.D.

From this theorem and the previous lemma it follows that any building set contains maximal strictly nested sets. Another interesting property of a building set is that any $T \in \mathcal{B}$ is included in some maximal strictly nested set.

Lemma 3.6 *Let \mathcal{B} be a building set on $[n]$ and $T \in \mathcal{B}$. Then there exists a maximal strictly nested set in \mathcal{B} which contains T among its elements.*

Proof. The restriction of \mathcal{B} to T is a building set, denoted by $\mathcal{B}|_T$. By the previous lemma, we have a maximal $\mathcal{B}|_T$ -nested set on T . Let us extend this set to a maximal \mathcal{B} -nested set on $[n]$. Pick an element $i \in [n] \setminus T$ and we choose a maximal element of \mathcal{B} in $T \cup \{i\}$ which contains i . We add this set to the $\mathcal{B}|_T$ -nested set on T . On the next step, we pick an element $i' \in [n] \setminus (T \cup \{i\})$ and consider a maximal element of \mathcal{B} in $T \cup \{i\} \cup \{i'\}$ which contains i' and we add this set, and so on. On the last step we will add $[n]$ since $[n] \in \mathcal{B}$, and we will end up with a maximal \mathcal{B} -nested set. Q.E.D.

Lemma 3.7 *Let \mathcal{B} be a building set on $[n]$ and let $v : \mathcal{B} \rightarrow \mathbb{R}$ be a function, then the M -extension $v^{\mathcal{B}}$ of v is given by, for any $T \in 2^{[n]}$,*

$$v^{\mathcal{B}}(T) = \sum_{j=1}^k v(T_j),$$

where $\{T_1, \dots, T_k\}$ is the unique partition of T into maximal elements of \mathcal{B} .

Proof. For any set $T \subset [n]$, there exists a unique maximal partition of T in sets from \mathcal{B} as follows by induction. Take any $t \in T$ and consider a maximal set of \mathcal{B} which contains t and is contained in T . Such a set exists and denote this set by T_1 . Then T_1 together with the partition of $T \setminus T_1$, which exists by induction, form the desired partition of T . Because of property (B1), any subset S , $S \subset T$, having a non-empty intersection with more than one element of the partition does not belong to \mathcal{B} , and hence $\mu^{\mathcal{B}}(S) = 0$. This implies the proposition. Q.E.D.

Because of this lemma and Theorem 2.5 we may construct a maximal strictly nested set corresponding to a permutation σ as follows. To the set $\{\sigma^{-1}(1), \dots, \sigma^{-1}(k)\}$, $k = 1, \dots, n$, of the chain \mathcal{N}_σ , we associate a partition of this set that corresponds to the value $v^{\mathcal{B}}(\{\sigma^{-1}(1), \dots, \sigma^{-1}(k)\})$. Then the collection of such sets of \mathcal{B} constitutes the same \mathcal{B} -nested sets as constructed in the proof of Theorem 3.5.

An important property of building sets is that, for any given building set \mathcal{B} , the poset of a strictly nested set is a face poset of a simple polyhedron $\Delta_{\mathcal{B}}$, or equivalently the collection of simplicial cones, $\Sigma(\mathcal{B})$, is a normal fan to a polytope. We will prove this in Section 5.

For a building set \mathcal{B} , let us describe which maximal strictly nested sets are being joined by an edge of the cell complex $C(\mathcal{B})$ to a given maximal strictly nested set \mathcal{N} . This gives also the characterization of the cones that are adjacent to cone $K(\mathcal{N})$ in the fan $\Sigma(\mathcal{B})$.

Let \mathcal{N} be a maximal strictly nested set in \mathcal{B} . Pick an $i \in [n]$ and consider $u(i) \in [n]$ such that $i \in S^{\mathcal{N}}(u(i))$. Denote by $\hat{T}(u(i))$ a maximal set of \mathcal{B} which contains $u(i)$, does not contain i , and is contained in $T(u(i))$. Then $(\mathcal{N} \setminus \{T^{\mathcal{N}}(i)\}) \cup \{\hat{T}(u(i))\}$ is also a maximal strictly nested set, and this set and \mathcal{N} are the two endpoints of the edge corresponding to the strictly nested set $\mathcal{N} \setminus \{T^{\mathcal{N}}(i)\}$.

For an arbitrary set system \mathcal{F} on $[n]$ there is a minimum building set $\mathcal{B}(\mathcal{F})$ containing \mathcal{F} , called the *building covering* of \mathcal{F} . Since the intersection of two buildings sets is also a building set, $\mathcal{B}(\mathcal{F})$ is uniquely defined.

The following proposition shows that the cell complex corresponding to \mathcal{F} is a sub-complex of the cell complex of $\mathcal{B}(\mathcal{F})$.

Proposition 3.8 *Let \mathcal{F} be a set system and let $\mathcal{B}(\mathcal{F})$ be the building covering of \mathcal{F} . Then every strictly nested set in \mathcal{F} is a $\mathcal{B}(\mathcal{F})$ -nested set.*

Proof. Let \mathcal{N} be a strictly nested set in \mathcal{F} . Let T_1, \dots, T_k be disjoint elements in \mathcal{N} . Then we have to check that the union $T_1 \cup \dots \cup T_k$ cannot be implemented of the form $A_1 \cup \dots \cup A_t$, where A_1, \dots, A_t is a non-disjoint family in \mathcal{F} . Suppose not, then such a non-disjoint family A_1, \dots, A_t exists. Then there exists at least one element of this family that has a non-empty intersection with at least two elements of the family T_1, \dots, T_k . This contradicts with property (G2). Q.E.D.

4 Solution concepts for building sets

Let \mathcal{B} be a building set system on $[n]$ and $v : \mathcal{B} \rightarrow \mathbb{R}$ a function. We consider \mathcal{B} as coalition structure on the set of n players and v as characteristic function with $v(T)$, $T \in \mathcal{B}$, the

worth of coalition T . Denote by $\mathcal{V}(\mathcal{B})$ the set of cooperative games on \mathcal{B} . A *solution* is a mapping from $\mathcal{V}(\mathcal{B})$ to \mathbb{R}^n . In [2], a Myerson-type value was defined for building sets. Such a solution coincides with the M-value defined in Section 2.

We introduce new solutions. We need some notions. For a set system \mathcal{F} an element $S \in \mathcal{F}$ is called a *half-space* if $[n] \setminus S \in \mathcal{F}$.

Definition 4.1 *For a set system \mathcal{F} a maximal strictly nested set \mathcal{N} is a half-space nested set (HS-nested set) if for every $i \in [n]$ with $S^{\mathcal{N}}(i) \neq \emptyset$, the set $T^{\mathcal{N}}(i)$ is a half-space.*

Since for a maximal strictly nested set \mathcal{N} every set $T^{\mathcal{N}}(i)$, $i \in [n]$, belongs to \mathcal{F} , we have that \mathcal{N} is an HS-nested set if for every $i \in [n]$ the complement to a non-singleton $T^{\mathcal{N}}(i)$ also belongs to \mathcal{F} . This means that in the corresponding tree $F^{\mathcal{N}}$, for every node it holds that after contracting all subordinates of the node and the node itself to the unique predecessor of the node, the resulting set of nodes is an allowable coalition. This restricts the collection of maximal strictly nested sets.

Given a graph G , we call a tree an *HS-tree* on G if it corresponds to an HS-nested set in the graphical building $\mathcal{B}(G)$.

Let us describe how to construct HS-trees by induction on the number of vertices of graphs. Suppose, for all graphs with less than n nodes, HS-trees are listed. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = [n]$. Pick a vertex $i \in V(G)$ and delete all edges in $E(G)$ having i as an endpoint. Let $E(i)$ denote this set of edges and let $\{i\}, G_1, \dots, G_m$ be the components of the graph $G = ([n], E(G) \setminus E(i))$. For each component G_j pick an HS-tree F_j with respect to the building set $\mathcal{B}(G_j)$, $j = 1, \dots, m$, and let r_j be the root of F_j . Since, for any $j_1 \neq j_2$, G_{j_1} and G_{j_2} are not connected, every $\mathcal{B}(G_j)$ -nested set is a $\mathcal{B}(G)$ -nested set. Now, we join i and r_j , $j = 1, \dots, m$, and obtain a tree F with root i and as its successors r_1, \dots, r_m . This tree is an HS-tree if and only if, for each $j = 1, \dots, m$, one of the following conditions is fulfilled

- $\{i, r_j\} \in E(G)$;
- $\{i, r_j\} \notin E(G)$ and for every component K of $F_j \setminus \{r_j\}$ there exists a node $w \in K$ such that $\{i, w\} \in E(G)$;
- $\{i, r_j\} \notin E(G)$ and there exists a singleton component $\{w\}$ of $F_j \setminus \{r_j\}$ such that $\{i, w\} \in E(G)$.

Definition 4.2 *For a set system \mathcal{F} a maximal strictly nested set \mathcal{N} is an NT-nested set if, for every $i \in [n]$ and $j \in S^{\mathcal{N}}(i)$, it holds that $\{i, j\}$ is an element of \mathcal{F} .*

A maximal strictly nested set \mathcal{N} is an NT-nested set if every player is able to cooperate with each of his successors in the corresponding tree $F^{\mathcal{N}}$. This restricts the collection of maximal strictly nested sets.

Example 4.3 Let $\mathcal{B}(G)$ be a graphical building for a connected graph G on a vertex set $[n]$. Then there is a bijection between the collection of maximal NT-nested sets in $\mathcal{B}(G)$ and the set of normal trees on G . The latter form a subset of rooted spanning trees of G (see [11]). However, not every rooted spanning tree of a graph G corresponds to an NT-tree.

We proceed to construct NT-trees by induction on the cardinality of the vertex set. Suppose for all graphs G on $[k]$, $k < n$, the corresponding NT-trees have been constructed. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = [n]$. Pick a vertex $i \in V(G)$ and delete all edges in $E(G)$ having i as an endpoint. Let $E(i)$ denote this set of edges. Then consider the components $\{i\}, G_1, \dots, G_m$ of the graph $G = ([n], E(G) \setminus E(i))$. For each component G_j there is an edge $\{i, r_j\} \in E(G)$ with $r_j \in V(G_j)$, $j = 1, \dots, m$. Let F_j be an NT-tree in G_j with root r_j , $j = 1, \dots, m$. Then an NT-tree in G with root i is obtained by joining each F_j to i by edge $\{i, r_j\}$.

From the constructions above it follows that for a building set every NT-nested set is an HS-nested set.

Next is an example of maximal strictly nested sets, NT-nested sets, and HS-nested sets for a graphical building with the graph being a line-tree on $[n]$.

Example 4.4 Let $A_n = ([n], E)$, where $E = \{\{i, i+1\} \mid i = 1, \dots, n-1\}$, be a line-tree on $[n]$. Then there is a bijection between maximal $\mathcal{B}(A_n)$ -nested sets and plane binary trees on $[n]$. We proceed by induction on n . Suppose that for $k < n$ such a bijection between $\mathcal{B}(A_k)$ -nested trees and plane binary trees on $[k]$ exist. Note that $\mathcal{B}(A_k)$ consists of intervals, that is sets of the form $\{a, a+1, \dots, a+b\}$ with $a, a+b \in [k]$. Pick a vertex $i \in [n]$. Then, in any $\mathcal{B}(A_n)$ -nested set, there are exactly two successors of i , since these subsets are intervals and because of condition (N2), $\{1, \dots, i-1\}$ and $\{i+1, \dots, n\}$. Then any $\mathcal{B}(A_{i-1})$ -nested set in $\{1, \dots, i-1\}$ is a binary tree as well as is any $\mathcal{B}(A_{n-i})$ -nested set in $\{i+1, \dots, n\}$. This provides the required bijection. The number of such trees is the Catalan number $C_n := \frac{(2n)!}{(n+1)n!}$. HS-nested sets in $\mathcal{B}(A_k)$ are constructed by the above construction and only the first and third items are possible. In Fig. 1 we depict an HS-tree.

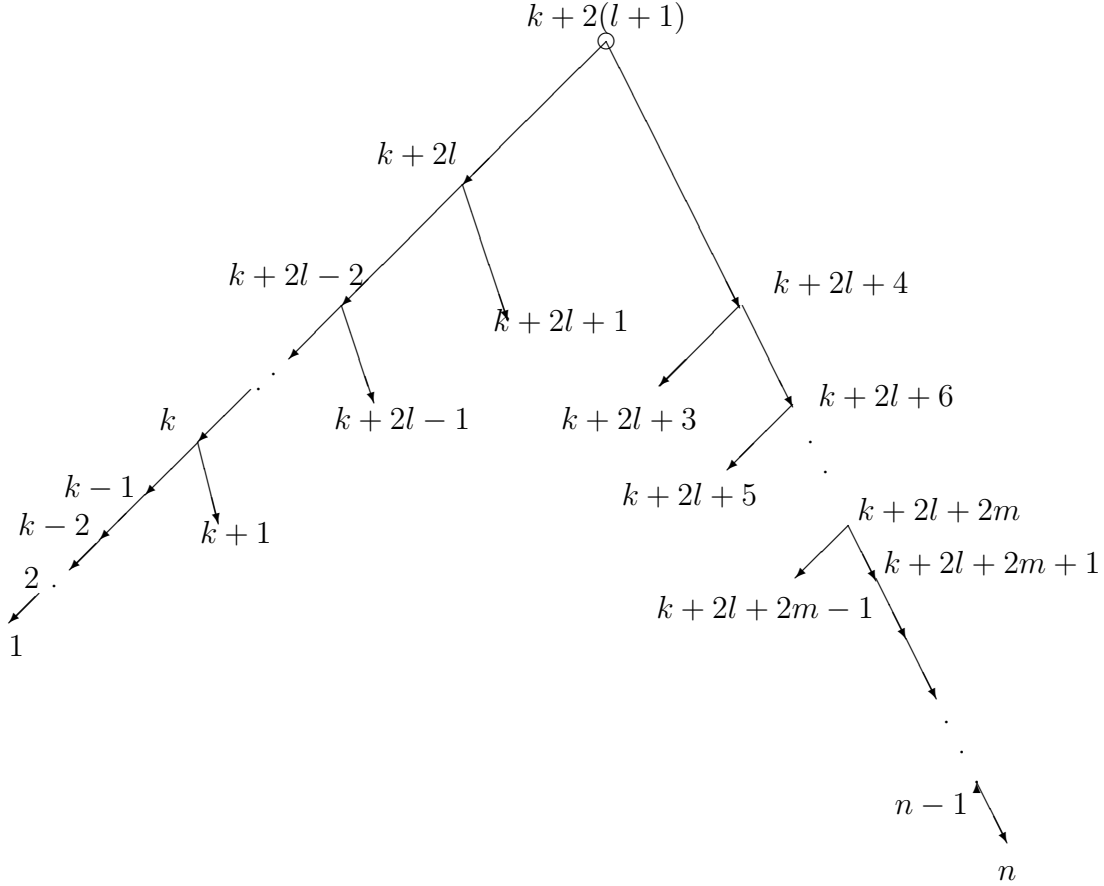


Figure 1. An HS-tree in case of a line-tree.

There are $\sum_{i=1}^n F(i + 1)F(n - i + 2)$ HS-trees, where $F(k)$, $k = 1, 2, \dots$, denotes the Fibonacci sequence. Any NT-tree takes the form: the root r has as successors vertices $r - 1$ and $r + 1$, and vertex i , $i \neq r$, has as successor vertex $i - 1$ if $i < r$ and vertex $i + 1$ if $i > r$. In total there are n NT-trees, because for each $i \in [n]$ there is one NT-tree having i as root.

The next example shows that in non-graphical building sets HS- and NT-nested sets may not exist.

Example 4.5 Let the building set \mathcal{B} on $[n]$ consist of all singletons and the set $[n]$. This building set has n maximal strictly nested sets, each consisting of $[n]$ and all singletons but one. Each of these maximal nested sets is neither an NT-nested nor an HS-nested set.

Definition 4.6 Let a building set \mathcal{B} on $[n]$ be given. For a game $v \in \mathcal{V}(\mathcal{B})$ the following solutions are given:

- The GC-solution is the average of the marginal vectors $m^v(\mathcal{N})$ over all maximal strictly nested sets \mathcal{N} in \mathcal{B} .
- The HS-solution is the average of the marginal vectors $m^v(\mathcal{N})$ over all maximal HS-nested sets \mathcal{N} in \mathcal{B} .
- The NT-solution is the average of the marginal vectors $m^v(\mathcal{N})$ over all maximal NT-nested sets \mathcal{N} in \mathcal{B} .

The GC-solution exists for any building set, while for a graphical building set all three solutions exist. The GC-solution differs from the Myerson-value as introduced in [2] on the class of union stable systems.

In case the graph is the line-tree A_n , to compute the GC-solution we have to take the average over $C_n := \frac{(2n)!}{(n+1)!n!}$ marginal vectors for binary trees, to compute the NT-solution we have to take the average over n NT-trees, and to compute the Myerson value we have to take the average over all $n!$ permutations.¹

Consider first the case of a graphical building set system $\mathcal{B}(G)$ for a connected graph G on $[n]$. If G is the complete graph, all three solutions coincide and are equal to the Shapley and the Myerson value. Denote $n(G)$ as the number of maximal strictly nested sets in $\mathcal{B}(G)$, and, for a maximal nested set \mathcal{N} in $\mathcal{B}(G)$, denote $p(\mathcal{N})$ as the cardinality of the corresponding set $\mathfrak{S}^{\mathcal{N}}$ of permutations. Then the difference between the GC-solution and the Myerson value is that

$$GC(v) = \frac{1}{n(G)} \sum_{\mathcal{N}} m^v(\mathcal{N})$$

and the Myerson value is equal to

$$M(v) = \frac{1}{n!} \sum_{\mathcal{N}} p(\mathcal{N}) m^v(\mathcal{N}),$$

where both summations are over all maximal strictly nested sets in $\mathcal{B}(G)$. The GC-solution is just the average of all different marginal vectors, while the Myerson value is a weighted average of these vectors with the weights determined by the number of permutations that correspond to the maximal strictly nested sets. The HS-solution is a new solution concept and takes the average of a specific set of marginal vectors, whereas the NT-solution takes the average of a more specific set of marginal vectors. For graphical building sets the NT-solution coincides with the average tree solution introduced in [17].

Example 4.7 Let $G = ([n], E)$ be a circular graph with $n = 4$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$. Then the Myerson value is the average of $(4!) = 24$ marginal vectors, four of them

¹One may compare the complexities of $n! \cong (n/e)^n$ and $C_n \cong 4^n$.

showing up twice, the GC-solution is the average of all 20 different marginal vectors, the HS-solution is the average of 16 of these marginal vectors, and the NT-solution is the average of 8 of these latter marginal vectors.

Example 4.8 Let $G = ([n], E)$ be a graph with $n = 4$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$. Then the Myerson value is the average of $(4! =)24$ marginal vectors, two of them twice, the GC-solution is the average of all 22 different marginal vectors, the HS-solution is the average of 18 of these marginal vectors, and the NT-solution is the average of 14 of these latter marginal vectors. There are five trees corresponding to maximal strictly nested sets that have node 1 (or node 3) as root and six such trees that have node 2 (or node 4) as root, there are five HS-trees that have node 1 as root and four HS-trees that have node 2 as root, and there are three normal trees that have node 1 as root and four normal trees that have node 2 as root.

The following is an example of a building set which possesses neither an HS-nested nor an NT-nested set.

Example 4.9 Let the building set \mathcal{B} on $[n]$ consist of all singletons and the set $[n]$. Then only the GC-solution exists and is equal to

$$GC_j(v) = v(\{j\}) + \frac{1}{n}(v([n]) - \sum_{i \in [n]} v(\{i\})), \quad j \in [n].$$

5 Core stability

In this section we present conditions under which the solutions defined in the previous section are elements of the core. Our main tool is to establish such conditions for building sets. By using Proposition 3.8 one may formulate corresponding conditions for arbitrary set systems as conditions for the building covering of the set system.

Definition 5.1 Let v be a game on the set system \mathcal{F} , then the core $C(v)$ is given by

$$C(v) = \{x \in \mathbb{R}^n \mid x([n]) = v([n]), x(S) \geq v(S), S \in \mathcal{F}\}.$$

Obviously it holds that $C(v^{\mathcal{F}}) \subseteq C(v)$, where $v^{\mathcal{F}}$ is the M-extension of v . For a building set \mathcal{B} and game $v \in \mathcal{V}(\mathcal{B})$ it holds that $C(v) = C(v^{\mathcal{B}})$. Let \mathcal{B} be a building set.

Definition 5.2 A function $f : \mathcal{B} \rightarrow \mathbb{R}$ is \mathcal{B} -supermodular if

$$f(A) + f(B) \leq f(A \cup B) + f^{\mathcal{B}}(A \cap B)$$

for any $A, B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$.

Definition 5.3 A function $f : \mathcal{B} \rightarrow \mathbb{R}$ is \mathcal{B} -superadditive if for any disjoint family T_1, \dots, T_k in \mathcal{B} such that $T_1 \cup \dots \cup T_k \in \mathcal{B}$ it holds

$$f(T_1 \cup \dots \cup T_k) \geq f(T_1) + \dots + f(T_k).$$

Definition 5.4 A function $f : \mathcal{B} \rightarrow \mathbb{R}$ is 2-superadditive if for any disjoint pair of sets T_1 and T_2 in \mathcal{B} such that $T_1 \cup T_2 \in \mathcal{B}$ it holds

$$f(T_1 \cup T_2) \geq f(T_1) + f(T_2).$$

In case of $\mathcal{B} = 2^{[n]}$, the conjunction of \mathcal{B} -supermodularity and 2-superadditivity is equivalent to the usual supermodularity condition that $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq [n]$. Remark that in this case \mathcal{B} -superadditivity follows from 2-superadditivity, which is not the case in general.

Theorem 5.5 Let $v \in \mathcal{V}(\mathcal{B})$ be a \mathcal{B} -supermodular and \mathcal{B} -superadditive game on a building set \mathcal{B} . Then the core $C(v)$ contains the GC-solution and is equal to the convex hull of the marginal vectors $m^{\mathcal{N}}(v)$ over all maximal strictly nested sets \mathcal{N} in \mathcal{B} .

For the proof of this theorem we need the following proposition.

Proposition 5.6 The collection of cones, $\Sigma(\mathcal{B})$, is a simplicial cone complex, that is for any maximal strictly nested sets \mathcal{N}_1 and \mathcal{N}_2 it holds that $K(\mathcal{N}_1) \cap K(\mathcal{N}_2) = K(\mathcal{N}_1 \cap \mathcal{N}_2)$.

Proof. We have to prove that $K(\mathcal{N}_1 \cap \mathcal{N}_2) \supset K(\mathcal{N}_1) \cap K(\mathcal{N}_2)$. This follows from the fact that a cone $K(\mathcal{N})$ does not contain vectors ξ_T with $T \in \mathcal{B}$ and $T \notin \mathcal{N}$. If this is not the case and such a vector ξ_T exists, then ξ_T is equal to the sum of some $\xi_{N_1}, \dots, \xi_{N_k}$ with N_1, \dots, N_k disjoint elements of \mathcal{N} . Then $N_1 \cup \dots \cup N_k = T \in \mathcal{B}$, which contradicts property (N2) of \mathcal{B} -nested sets. Q.E.D.

Because of Theorem 3.5, the simplicial cone collection $\Sigma(\mathcal{B})$ of a building set \mathcal{B} is a full fan, that is $\cup_{\mathcal{N}} K(\mathcal{N}) = \mathbb{R}^n$.

Proof of Theorem 5.5. A function $f : \mathcal{B} \rightarrow \mathbb{R}$ is $\Sigma(\mathcal{B})$ -supermodular if the extension of f by affinity on each cone $K(\mathcal{N})$, \mathcal{N} being a maximal strictly nested set in \mathcal{B} , yields a concave function. Because of Theorem 3.5 each cone $K(\mathcal{N})$ is the union of cones corresponding to permutations, which implies that, for a $\Sigma(\mathcal{B})$ -supermodular function f , the M-extension $f^{\mathcal{B}}$ is a submodular function on $2^{[n]}$, and therefore due to Theorem 2.5 and to Edmonds theorem in [12] (see also Shapley [20]), $C(f^{\mathcal{B}})$ and therefore also $C(f)$ is equal to the convex hull of the marginal vectors $m^f(\mathcal{N})$ over all maximal strictly nested sets \mathcal{N} in \mathcal{B} .

We still have to check that v is $\Sigma(\mathcal{B})$ -supermodular. One-dimensional cones of the fan $\Sigma(\mathcal{B})$ correspond to strictly nested sets $\{A, [n]\}$, $A \in \mathcal{B}$. Consider two full-dimensional

adjacent cones corresponding to maximal \mathcal{B} -nested sets \mathcal{N} and $(\mathcal{N} \setminus \{T^{\mathcal{N}}(i)\}) \cup \{\hat{T}(u(i))\}$ as described before sharing the facet corresponding to the \mathcal{B} -nested set $\mathcal{N} \setminus \{T^{\mathcal{N}}(i)\}$, for some $i \in [n]$. The linear relation between the spanning vectors of the one-dimensional cones corresponding to the nested sets $\{T^{\mathcal{N}}(i), [n]\}$ and $\{\hat{T}(u(i)), [n]\}$ on the one hand and the spanning vectors of the facet on the other hand is of the form

$$\xi_{T^{\mathcal{N}}(i)} + \xi_{\hat{T}(u(i))} = \xi_{T^{\mathcal{N}}(u(i))} - \sum_{j \in U} \xi_{T^{\mathcal{N}}(j)} + \sum_{k \in M} \xi_{T^{\mathcal{N}}(k)},$$

where $j \in U$ if $T^{\mathcal{N}}(j)$ is a maximal element of \mathcal{N} in $T^{\mathcal{N}}(u(i)) \setminus (T^{\mathcal{N}}(i) \cup \hat{T}(u(i)))$, and $k \in M$ if $T^{\mathcal{N}}(k)$ is a maximal element of \mathcal{N} in $T^{\mathcal{N}}(i) \cap \hat{T}(u(i))$. Because of \mathcal{B} -supermodularity we have

$$v(T^{\mathcal{N}}(i)) + v(\hat{T}(u(i))) \leq v(T^{\mathcal{N}}(i) \cup \hat{T}(u(i))) + v^{\mathcal{B}}(T^{\mathcal{N}}(i) \cap \hat{T}(u(i))). \quad (2)$$

Then \mathcal{B} -superadditivity implies

$$v(T^{\mathcal{N}}(i) \cup \hat{T}(u(i))) + \sum_{j \in U} v(T^{\mathcal{N}}(j)) \leq v(T^{\mathcal{N}}(u(i))), \quad (3)$$

and due to property (N2) of \mathcal{B} -nested sets we have

$$v^{\mathcal{B}}(T^{\mathcal{N}}(i) \cap \hat{T}(u(i))) = \sum_{k \in M} v(T^{\mathcal{N}}(k)).$$

Summing up the inequalities (2) and (3) and the last equality, we get

$$v(T^{\mathcal{N}}(i)) + v(\hat{T}(u(i))) + \sum_{j \in U} v(T^{\mathcal{N}}(j)) \leq v(T^{\mathcal{N}}(u(i))) + \sum_{k \in M} v(T^{\mathcal{N}}(k)),$$

which implies that v is $\Sigma(\mathcal{B})$ -supermodular. Q.E.D.

As a consequence of the proof of this theorem we obtain the following result proven by Postnikov ([19], Proposition 7.5).

Proposition 5.7 *For any function $\mu : \mathcal{B} \rightarrow \mathbb{R}_{++}$ the normal fan of the polytope*

$$\sum_{T \in \mathcal{B}} \mu(T) \Delta_T$$

is the fan $\Sigma(\mathcal{B})$.

Proof. Given a function $\mu : \mathcal{B} \rightarrow \mathbb{R}_{++}$, define a game $f^\mu : \mathcal{B} \rightarrow \mathbb{R}_{++}$ by

$$f^\mu(S) = \sum_{T \in \mathcal{B} | T \subset S} \mu(T), \quad S \in \mathcal{B}.$$

Due to Danilov and Koshevoy theorem in [7] the core of the game f^μ has the form of the Minkowski sum of simplices

$$\sum_{T \in \mathcal{B}} \mu(T) \Delta_T.$$

It is easy to see that, because $\mu(T) > 0$ for any $T \in \mathcal{B}$, for the function f^μ the \mathcal{B} -supermodularity and \mathcal{B} -superadditivity inequalities are strict. Hence the inequalities (2) and (3) are also strict and this implies that the normal fan to $C(f^\mu)$ coincides with the fan $\Sigma(\mathcal{B})$. Q.E.D.

From this proposition it follows that the GC-solution is the gravity center of the core $C(v)$ in case the characteristic function v is totally positive. Recall, that a function $f : \mathcal{B} \rightarrow \mathbb{R}$ is said to be *totally positive* if the linear system

$$\sum_{T \in \mathcal{B} | T \subseteq S} \mu(T) = f(S), \quad S \in \mathcal{B}$$

has a positive solution.

If a building set \mathcal{B} contains at least one maximal HS-nested set or maximal NT-nested set, we can ensure that the marginal vector corresponding to such a maximal strictly nested set belongs to the core under weaker conditions than \mathcal{B} -supermodularity.

Definition 5.8 *A function $f : \mathcal{B} \rightarrow \mathbb{R}$ is half-space \mathcal{B} -supermodular if, for any $S, T \in \mathcal{B}$ such that $S \cap T \neq \emptyset$ and at least one of the sets S or T is a half-space in \mathcal{B} , it holds that*

$$f(S) + f(T) \leq f(S \cup T) + f^{\mathcal{B}}(S \cap T). \quad (4)$$

For $\mathcal{B} = 2^{[n]}$, half-space $2^{[n]}$ -supermodularity coincides with $2^{[n]}$ -supermodularity, since any subset of $[n]$ is a half-space in $2^{[n]}$. For other buildings sets \mathcal{B} half-space \mathcal{B} -supermodularity is weaker than \mathcal{B} -supermodularity, since we do not require validity (4) for two non-half-spaces.

Theorem 5.9 *Let $v : \mathcal{B} \rightarrow \mathbb{R}$ be a half-space \mathcal{B} -supermodular and \mathcal{B} -superadditive game on building set \mathcal{B} . Then, for any maximal HS-nested set \mathcal{N} in \mathcal{B} , the marginal vector $m^v(\mathcal{N})$ belongs to the core $C(v)$.*

Proof. Let \mathcal{N} be a maximal HS-nested set in \mathcal{B} . We have to prove that for every $Q \in \mathcal{B}$ it holds that

$$\sum_{j \in Q} m_j^v(\mathcal{N}) \geq v(Q). \quad (5)$$

For a set $Q \in \mathcal{N}$ denote by $S^{\mathcal{N}}(Q)$ the set of successors of Q in the tree $F^{\mathcal{N}}$, i.e., $i \in S^{\mathcal{N}}(Q)$ if $i \notin Q$ and $i \in S^{\mathcal{N}}(j)$ for some $j \in Q$. We proceed by induction on the

number of components of Q in the tree $F^{\mathcal{N}}$. When Q is connected in $F^{\mathcal{N}}$, (5) takes the form

$$v(Q \cup (\cup_{j \in S^{\mathcal{N}}(Q)} T^{\mathcal{N}}(j))) \geq v(Q) + \sum_{j \in S^{\mathcal{N}}(Q)} v(T^{\mathcal{N}}(j)). \quad (6)$$

This inequality holds because of \mathcal{B} -superadditivity.

Suppose (5) holds for any $Q \in \mathcal{B}$ having at most l components in $F^{\mathcal{N}}$. Consider any $Q \in \mathcal{B}$ having $l + 1$ components in $F^{\mathcal{N}}$. Denote by Q_0, Q_1, \dots, Q_l these components.

One can easily check that inequality (5) is equivalent to

$$\sum_{k=0}^l v(Q_k \cup (\cup_{j \in S^{\mathcal{N}}(Q_k)} T^{\mathcal{N}}(j))) \geq v(Q) + \sum_{k=0}^l \sum_{j \in S^{\mathcal{N}}(Q_k)} v(T^{\mathcal{N}}(j)). \quad (7)$$

Because of property (G2) of strictly nested sets, we can order Q_0, Q_1, \dots, Q_l such that, for any $k \neq 0$, Q_k belongs to $T^{\mathcal{N}}(j)$ for some $j \in S^{\mathcal{N}}(Q_0)$, and all such $T^{\mathcal{N}}(j)$ can not be singletons.

Pick $j \in S^{\mathcal{N}}(Q_0)$ such that $T^{\mathcal{N}}(j) \cap Q \neq \emptyset$. Denote by Q_1^j, \dots, Q_t^j the components of $Q \cap T^{\mathcal{N}}(j)$.

Due to half-space \mathcal{B} -supermodularity we have

$$v(Q \cup T^{\mathcal{N}}(j)) \geq v(Q) + v(T^{\mathcal{N}}(j)) - v^{\mathcal{B}}(Q_1^j \cup \dots \cup Q_t^j). \quad (8)$$

Because Q_1^j, \dots, Q_t^j are disjoint subsets of sets of the NT-nested set \mathcal{N} , there exists a partition $\{P_1, \dots, P_s\}$ of the set $[t]$, such that

$$v^{\mathcal{B}}(Q_1^j \cup \dots \cup Q_t^j) = \sum_{h=1}^s v(\cup_{u \in P_h} Q_u^j),$$

where, for each $h = 1, \dots, s$, the union $\cup_{u \in P_h} Q_u^j$ belongs to \mathcal{B} .

Because of the induction, we have validity of (7) for each set $\cup_{u \in P_h} Q_u^j$, $h = 1, \dots, s$, that is

$$\sum_{u \in P_h} v(Q_u^j \cup (\cup_{k \in S^{\mathcal{N}}(Q_u^j)} T^{\mathcal{N}}(k))) \geq v(\cup_{u \in P_h} Q_u^j) + \sum_{u \in P_h} \sum_{k \in S^{\mathcal{N}}(Q_u^j)} v(T^{\mathcal{N}}(k)). \quad (9)$$

Summing up these inequalities over $h = 1, \dots, s$ together with inequality (8), we get

$$v(Q \cup T^{\mathcal{N}}(j)) + \sum_{u=1}^t v(Q_u^j \cup (\cup_{h \in S^{\mathcal{N}}(Q_u^j)} T^{\mathcal{N}}(h))) \geq v(Q) + v(T^{\mathcal{N}}(j)) + \sum_{u=1}^t \sum_{h \in S^{\mathcal{N}}(Q_u^j)} v(T^{\mathcal{N}}(h)).$$

Continue this procedure by adding every $T^{\mathcal{N}}(j)$, $j \in S^{\mathcal{N}}(Q_0)$, and we get inequality (7), which proves the theorem. Q.E.D.

As a consequence of this theorem, we obtain that if it exists the HS -solution belongs to the core under half-space \mathcal{B} -supermodularity and \mathcal{B} -superadditivity of the game. Furthermore, from the proof of Theorem 5.9 the next proposition immediately follows.

Proposition 5.10 *Let \mathcal{B} be a building set having a maximal HS-nested set \mathcal{N} such that every $Q \in \mathcal{B}$ induces a subtree on $F^{\mathcal{N}}$, and let $v : \mathcal{B} \rightarrow \mathbb{R}$ be a \mathcal{B} -superadditive function. Then the marginal vector $m^{\mathcal{N}}(v)$ belongs to the core $C(v)$.*

If the graph G is a tree on $[n]$, then for the graphical building $\mathcal{B}(G)$ every NT-nested set satisfies the assumption of Proposition 5.10. Therefore for a superadditive function $v : \mathcal{B}(G) \rightarrow \mathbb{R}$ it holds that the NT-solution belongs to the core. Such a specification of Proposition 5.10 was proven in [9], see also [17].

For the case when \mathcal{B} contains an NT-nested set, we can weaken the \mathcal{B} -superadditivity requirement in Theorem 5.9 to 2-superadditivity.

Theorem 5.11 *Let $v : \mathcal{B} \rightarrow \mathbb{R}$ be a half-space \mathcal{B} -supermodular and 2-superadditive game on building set \mathcal{B} . Then, for any maximal NT-nested set \mathcal{N} in \mathcal{B} , the marginal vector $m^v(\mathcal{N})$ belongs to the core $C(v)$.*

Proof. We have to prove that for every $Q \in \mathcal{B}$ it holds that

$$\sum_{j \in Q} m_j^v(\mathcal{N}) \geq v(Q). \quad (10)$$

Because \mathcal{B} is a building set it holds for every $i \in [n]$ that the complement of $T^{\mathcal{N}}(i)$ belongs to \mathcal{B} and therefore $T^{\mathcal{N}}(i)$ is a half-space.

We again proceed by induction on the number of components of Q in the tree $F^{\mathcal{N}}$. First consider the case when Q is one component in $F^{\mathcal{N}}$. For such Q we have to establish inequality (10). For any $K \subset S^{\mathcal{N}}(Q)$, $k' \in S^{\mathcal{N}}(Q)$, $k' \notin K$, the sets $Q \cup (\cup_{k \in K} T^{\mathcal{N}}(k)) \cup T^{\mathcal{N}}(k')$, $Q \cup (\cup_{k \in K} T^{\mathcal{N}}(k))$ and $T^{\mathcal{N}}(k')$ belong to \mathcal{B} and the latter set is a half-space in \mathcal{B} . Therefore, by half-space \mathcal{B} -supermodularity the following inequality holds

$$v(Q \cup (\cup_{k \in K} T^{\mathcal{N}}(k)) \cup T^{\mathcal{N}}(k')) \geq v(Q \cup (\cup_{k \in K} T^{\mathcal{N}}(k))) + v(T^{\mathcal{N}}(k')). \quad (11)$$

Gradually taking off $F^{\mathcal{N}}(j)$, $j \in S^{\mathcal{N}}(Q)$, one by one from the set $Q \cup (\cup_{j \in S^{\mathcal{N}}(Q)} T^{\mathcal{N}}(j))$ and applying (11), we obtain (10).

Now, by repeating the proof of the previous theorem for the case when the intersection Q contains more than one component in the tree $F^{\mathcal{N}}$ we obtain the validity of inequality (10). Q.E.D.

From this theorem it follows that the NT-solution belongs to the core under half-space \mathcal{B} -supermodularity and 2-superadditivity.

Remark. For a connected graph G , $\mathcal{B}(G)$ -superadditivity is equivalent to 2-superadditivity, because according to ([21], Proposition 7.3) a building set \mathcal{B} is a graphical building if and only if for any $I, J_1, \dots, J_k \in \mathcal{B}$ such that $I \cup J_1 \cup \dots \cup J_k \in \mathcal{B}$ it holds that there exists i satisfying $I \cup J_i \in \mathcal{B}$. Due to this characterization, for a graphical building the assumptions in Theorem 5.9 boil down to the assumptions in Theorem 5.11.

6 Solutions for arbitrary set systems

For arbitrary set systems we have the following analogues of the above defined solutions and core stability theorems.

Let \mathcal{F} be a set system on $[n]$ and $v : \mathcal{F} \rightarrow \mathbb{R}$ a function. We consider the building covering $\mathcal{B}(\mathcal{F})$ and the (restricted) M-extension function $v^{\mathcal{F}} : \mathcal{B}(\mathcal{F}) \rightarrow \mathbb{R}$.

Definition 6.1 *For a game $v \in \mathcal{V}(\mathcal{F})$ there are the following solutions:*

- *The GC-solution (gravity-center solution) is the average of the marginal vectors $m^{v^{\mathcal{F}}}(\mathcal{N})$ over all maximal $\mathcal{B}(\mathcal{F})$ -nested sets \mathcal{N} .*
- *The HS-solution is the average of the marginal vectors $m^{v^{\mathcal{F}}}(\mathcal{N})$ over all maximal HS-nested sets \mathcal{N} in $\mathcal{B}(\mathcal{F})$.*
- *The NT-solution is the average of the marginal vectors $m^{v^{\mathcal{F}}}(\mathcal{N})$ over all maximal NT-nested sets \mathcal{N} in $\mathcal{B}(\mathcal{F})$.*

First we state the core stability theorems and then give some examples.

Theorem 6.2 *Let \mathcal{F} be a set system and let $v : \mathcal{F} \rightarrow \mathbb{R}$ be a characteristic function.*

- *For a maximal strictly nested set \mathcal{N} in $\mathcal{B}(\mathcal{F})$, the marginal vector $m^{v^{\mathcal{F}}}(\mathcal{N})$ belongs to the core $C(v)$ if the M-extension $v^{\mathcal{F}}$ of v is $\mathcal{B}(\mathcal{F})$ -supermodular and $\mathcal{B}(\mathcal{F})$ -superadditive.*
- *For a maximal HS-nested set \mathcal{N} in $\mathcal{B}(\mathcal{F})$, the marginal vector $m^{v^{\mathcal{F}}}(\mathcal{N})$ belongs to the core $C(v)$ if the M-extension $v^{\mathcal{F}}$ of v is half-space $\mathcal{B}(\mathcal{F})$ -supermodular and $\mathcal{B}(\mathcal{F})$ -superadditive.*
- *For a maximal NT-nested set \mathcal{N} in $\mathcal{B}(\mathcal{F})$, the marginal vector $m^{v^{\mathcal{F}}}(\mathcal{N})$ belongs to the core $C(v)$ if the M-extension $v^{\mathcal{F}}$ of v is half-space $\mathcal{B}(\mathcal{F})$ -supermodular and 2-superadditive.*

Proof. We present a proof of the second item: From Proposition 3.8 and Theorem 5.9 it holds that the marginal vector $m^v(\mathcal{N})$ is a vertex of the core $C(v^{\mathcal{B}(\mathcal{F})})$ because according to (1) the M-extension of the restriction $v^{\mathcal{F}}|_{\mathcal{B}(\mathcal{F})}$ is equal to the M-extension $v^{\mathcal{F}}$.

Because $C(v^{\mathcal{B}(\mathcal{F})}) \subset C(v)$, it holds that $m^v(\mathcal{N}) \in C(v)$. For a point x in the core $C(v)$ it holds that $v(T^{\mathcal{N}}(i)) \leq \sum_{j \in T^{\mathcal{N}}(i)} x_j$ for all $i \in [n]$. We have n independent inequalities because for any two of the sets $T^{\mathcal{N}}(i)$, $i \in [n]$, it holds that either one is a

subset of the other or the two sets do not intersect. Because of this $m^v(\mathcal{N})$ is a vertex of $C(v)$. Q.E.D.

We may also define a generalization of the Myerson value for a game on an arbitrary set system.

Definition 6.3 *For a function $v : \mathcal{F} \rightarrow \mathbb{R}$ let $v^{\mathcal{F}} : 2^{[n]} \rightarrow \mathbb{R}$ be its M-extension. Then the M-solution, denoted $M(v)$, is the average of the marginal vectors $m^{v^{\mathcal{F}}}(\sigma)$ over all permutations $\sigma \in \mathfrak{S}_n$.*

We have the following result.

Theorem 6.4 *Let \mathcal{F} be a set system and let $\mathcal{B}(\mathcal{F})$ be the building covering of \mathcal{F} , and let $v : \mathcal{F} \rightarrow \mathbb{R}$ be a characteristic function. Suppose that the M-extension $v^{\mathcal{F}}$ is $\mathcal{B}(\mathcal{F})$ -supermodular and $\mathcal{B}(\mathcal{F})$ -superadditive. Then the M-solution belongs to the core $C(v)$.*

As example consider first the case of $\mathcal{F} = \{[n]\}$. This set system has no maximal strictly nested set. Its building covering is equal to $\{[n], \{1\}, \dots, \{n\}\}$. The M-extension of $v([n])$ is a function which equals zero on all proper subsets of $[n]$. The GC-solution coincides with the M-solution and is equal to the egalitarian solution

$$GC_j(v) = \frac{1}{n}v([n]), \quad j \in [n].$$

For this example the HS- and NT-solution do not exist.

Let us consider the case of convex geometries, set systems considered in [4].

Definition 6.5 *A set system \mathcal{F} on $[n]$ is a convex geometry if \mathcal{F} is stable under intersection and the following shelling property holds. For any $T \in \mathcal{F}$, including $T = \emptyset$, it holds that there exists $i \in [n] \setminus T$ such that $T \cup \{i\}$ is an element of \mathcal{F} .*

To a convex geometry \mathcal{F} is associated a collection $Bas(\mathcal{F})$ of linear orders, a subset of \mathfrak{S}_n , see [8]. These permutations correspond to all maximal chains in \mathcal{F} . In [4] Bilbao defines the Shapley value for a convex geometry \mathcal{F} by the *B-solution*, $B(v)$, given by

$$B(v) := \frac{1}{|Bas(\mathcal{F})|} \sum_{\sigma \in Bas(\mathcal{F})} m^v(\mathcal{N}_\sigma).$$

Example 6.6 *Let G be a tree, then the graphical building $\mathcal{B}(G)$ is a convex geometry. In this case the four solutions GC-, HS-, NT-, and M-solutions all differ from the B-solution. Remark, that for the line-tree A_n , there are only two permutations, the identical and the reverse to the identical, which define HS-trees.*

This is an interesting example, which shows that different view points on the same coalition structure yield different view points on solutions.

In contrast to Theorem 3.5 for building sets, in an arbitrary system the union of permutations that correspond to the maximally strictly nested sets may not coincide with the set of all permutations. For example, this is the case for convex geometries which do not contain all singletons.

Example 6.7 *Let $([n], \prec)$ be a poset, then the set of ideals form a convex geometry. Recall, that a subset $I \subset [n]$ is an ideal, if, for any $i \in I$ and $j \prec i$, it follows that $j \in I$. Denote by $\mathcal{I}(\prec)$ the set of all ideals. This set is stable under union and intersection. Therefore, the building covering of it is equal to $\mathcal{I}(\prec) \cup \{\{1\}, \dots, \{n\}\}$, that is we have to add all missing singletons. Given a function $v : \mathcal{I}(\prec) \rightarrow \mathbb{R}$, the B-solution is the average of $m^v(\mathcal{N}_\sigma)$ over all permutations σ being linear extensions of \prec . This solution is different from the GC-, HS-, NT-, and M-solutions.*

Example 6.8 *In the previous example let $n = 3$ and $3 \prec 1$ and $3 \prec 2$. Then the set of ideals consists of the sets $\{3\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. This is a convex geometry with building covering the set of ideals plus the singletons $\{1\}$ and $\{2\}$. This graphical building set corresponds to a line-tree with node 3 connected to both 1 and 2. The M-extension of a game v is defined by setting $v(\{1\}) = v(\{2\}) = 0$. Then the B-solution is $(\frac{1}{2}(v(123) - v(23) + v(13) - v(3)), \frac{1}{2}(v(123) + v(23) - v(13) - v(3)), v(3))$, and the NT-solution and the HS-solution are both equal to $(\frac{1}{3}(v(\{1, 2, 3\}) - v(\{2, 3\})), \frac{1}{3}(v(\{1, 2, 3\}) - v(\{1, 3\})), \frac{1}{3}(v(\{1, 2, 3\}) + v(\{1, 3\}) + v(\{3\}))$. The M-solution is $(\frac{1}{6}(2v(\{1, 2, 3\}) - 2v(\{2, 3\}) + v(\{1, 3\}) - v(\{3\})), \frac{1}{6}(2v(\{1, 2, 3\}) - 2v(\{1, 3\}) + v(\{2, 3\}) - v(\{3\})), \frac{1}{6}(2v(\{1, 2, 3\}) + v(\{1, 3\}) + v(\{2, 3\}) - 2v(\{3\})))$,*

Next is an example of another important class of convex geometries.

Example 6.9 *Let $X = \{x^1, \dots, x^n\}$ be a set in \mathbb{R}^k . Then define the convex geometry \mathcal{F} as follows: A is in \mathcal{F} if the convex hull of the points x^a , $a \in A$, contains no x^b , $b \notin A$, i.e., $\text{co}(\{x^a \mid a \in A\}) \cap X = \{x^a \mid a \in A\}$.*

Example 6.10 *Consider in the previous example the points $x^1 = (0, 0)$, $x^2 = (1, 0)$, $x^3 = (2, 0)$, and $x^4 = (1, 1)$. The corresponding convex geometry \mathcal{F} consists of all subsets of $\{1, 2, 3, 4\}$ except $\{1, 3\}$ and $\{1, 3, 4\}$. The building covering is equal to $\mathcal{F} \cup \{1, 3, 4\}$. For a function $v : \mathcal{F} \rightarrow \mathbb{R}$, the M-extension has to be specified at $\{1, 3\}$ and $\{1, 3, 4\}$: $v^{\mathcal{F}}(\{1, 3\}) = v(\{1\}) + v(\{3\})$ and $v^{\mathcal{F}}(\{1, 3, 4\}) = v(\{1, 4\}) + v(\{3, 4\}) - v(\{4\})$. The B-solution is different from the other solutions.*

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