Computing the Maximum Volume Inscribed Ellipsoid of a Polytopic Projection

Zhen, J.; den Hertog, D.

Link to publication in Tilburg University Research Portal

COMPUTING THE MAXIMUM VOLUME INSCRIBED ELLIPSOID OF A POLYTOPIC PROJECTION

By

Jianzhe Zhen, Dick den Hertog

Computing the Maximum Volume Inscribed Ellipsoid
of a Polytopic Projection

Jianzhe Zhen*, Dick den Hertog

Department of Econometrics and Operations Research, Tilburg University, 5000 LE, The Netherlands

{j.zhen@uvt.nl, d.denhertog@uvt.nl}

We introduce a novel scheme based on a blending of Fourier-Motzkin elimination (FME) and adjustable robust optimization techniques to compute the maximum volume inscribed ellipsoid (MVE) in a polytopic projection. It is well-known that deriving an explicit description of a projected polytope is NP-hard. Our approach does not require an explicit description of the projection, and can easily be generalized to find a maximally sized convex body of a polytopic projection. Our obtained MVE is an inner approximation of the projected polytope, and its center is a centralized relative interior point of the projection. Since FME may produce many redundant constraints, we apply an LP-based procedure to keep the description of the projected polytopes at its minimal size. Furthermore, we propose an upper bounding scheme to evaluate the quality of the inner approximations. We test our approach on a simple polytope and a color tube design problem, and observe that as more auxiliary variables are eliminated, our inner approximations and upper bounds converge to optimal solutions.

Key words: Fourier-Motzkin elimination; maximum volume inscribed ellipsoid; polytopic projection; chebyshev center; removing redundant constraints; adjustable robust optimization

1. Introduction

Let $\mathcal{P}$ be a polytope in $\mathbb{R}^{n_1+n_2}$ (not necessarily full-dimensional) defined by m linear inequalities

$$\mathcal{P} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n_1+n_2} \mid A \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b} \right\}, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_1}$, $\boldsymbol{y} \in \mathbb{R}^{n_2}$, $A \in \mathbb{R}^{m \times (n_1+n_2)}$, and $\boldsymbol{b} \in \mathbb{R}^m$. The aim of this paper is to propose a novel approach that computes the maximum volume inscribed ellipsoid (MVE) of the projected polytope $\mathcal{P}$ onto the $\boldsymbol{x}$-space. The auxiliary variables $\boldsymbol{y}$ may occur in $\mathcal{P}$ due to the nature of the problem at hand or the result of reformulations to make the problem linear. Finding the MVE of a polytopic projection may arise in many applications.

Application 1. Ellipsoidal approximations to polytopic sets. For polytopes with many constraints and/or facets, ellipsoids are much easier to handle, both theoretically and computationally. For instance, the global minimum of any quadratic functions over an ellipsoid can be located in polynomial time while finding such global minimum over a polytope is generally NP-hard. Since the description of the polytopic feasible set for power system problems often contains prohibitively many constraints, Sarić and Stanković (2008) propose to first simplify the description of the polytopic feasible set to a one-line MVE approximation, and then determine the optimal economic dispatch and locational marginal prices within the MVE.

Application 2. Maximal homothet of a polytopic projection. Aggregation of a large number of responsive loads presents power flexibility for demand response. An effective control and coordination scheme of flexible loads requires an accurate and tractable model that captures their aggregate flexibility. The flexibility of each individual load is a polytope, and their aggregation is the Minkowski sum of these polytopes. The aggregate flexibility can be viewed as the projection of a high-dimensional polytope onto the subspace representing the aggregate power. Zhao et al. (2016) extends our approach to extract the aggregate flexibility of deferrable loads using inner approximation of the polytopic projection.

Application 3. Design centering/tolerance design problems. Consider a manufacturing process in which the characteristics of a product are represented by a vector $\boldsymbol{x}$. Due to unavoidable disturbances, e.g., implementation errors, in the manufacturing process, the realized value of $\boldsymbol{x}$ will deviate from the nominal value in the design. The product is acceptable if $\boldsymbol{x}$ lies in the "region of acceptability" $\mathcal{P}$. Without assuming the structure of the uncertainties, a product design $\boldsymbol{x}$ that is "most interior" in the $\boldsymbol{x}$-space of $\mathcal{P}$ is desired. For problems without auxiliary variables, Graeb (2007) and Hendrix et al. (1996) propose to consider the MVE center as the robust design.

Application 4. Nominal scenario recovery in polytopic uncertainty set. In robust optimization, the uncertainty set contains the scenarios for which one would safeguard. One may be interested to find a centralized nominal scenario that is not far from the (later) realization. For example, the approximated nominal scenario can be used to evaluate the price of robustness (see Bertsimas and Sim (2004)). Due to the existence of auxiliary variables, the MVE center of the projection can be considered as an approximation of the nominal scenario.

Application 5. Robust solutions for system of uncertain linear equations. Given a system of uncertain linear equations $A\boldsymbol{x} = \boldsymbol{b}$, where the coefficient matrix A and right-hand side vector $\boldsymbol{b}$ reside in an polytopic uncertainty set $\mathcal{U}$. The convex representation of the feasible solution set $\{\boldsymbol{x} \mid \exists (A, \boldsymbol{b}) \in \mathcal{U} : A\boldsymbol{x} = \boldsymbol{b}\}$ contains auxiliary variables. Zhen and den Hertog (2015) consider the MVE (and its center) of the feasible solution set as an inner approximation of the set (a centered solution of the system).

The method for finding the MVE of a full-dimensional polytope with no auxiliary variables (e.g., the variables $\boldsymbol{y}$ in $\mathcal{P}$) is well-established, which can be computed in polynomial time by solving a semidefinite programming (SDP) problem (see Boyd and Vandenberghe (2004)). One obvious way of determining the MVE in the $\boldsymbol{x}$-space of a given polytope $\mathcal{P}$ is as follows: firstly, we project $\mathcal{P}$ onto the $\boldsymbol{x}$-space, in other words, derive an explicit description of the $\boldsymbol{x}$-space with no auxiliary variables $\boldsymbol{y}$; then, we can solve an SDP problem to find the MVE of the projected polytope. The projection of $\mathcal{P}$ can be obtained by eliminating the auxiliary variables $\boldsymbol{y}$. The pioneer method Fourier-Motzkin elimination (FME) was first introduced in Fourier (1824), and was rediscovered in Motzkin (1936). In each step of the iterative algorithm the dimension of the polytope is reduced by projecting it onto a hyperplane. Other projection methods, i.e., the Extreme Point Method (see Lassez (1990)) and Convex Hull Method (see Lassez and Lassez (1992)), are evaluated in Huynh et al. (1992). Jones et al. (2004) develop a new algorithm for obtaining the projection of polytopes, which is suited for problems in which the number of vertices far exceeds the number of facets. A more recent development on the method for variable elimination in systems of inequalities can be found in Chaharsooghi et al. (2011). However, Tiwary (2008) shows that deriving an explicit description of a projected polytope is NP-hard in general. The size of the description of the projection and of the intermediate computations can be intractable (see Example 2).

We develop a novel approach for computing the MVE in a polytopic projection. We first eliminate a subset of auxiliary variables via FME, and then compute the MVE of the $\boldsymbol{x}$-space by solving an adjustable robust optimization (ARO) problem. In general, one cannot eliminate all the adjustable variables due to the rapid growth in the number of constraints after FME, otherwise, one can simply solve an SDP problem to find the desired MVE. In order to improve the computability of our approach, we remove the redundant constraints and keep the description of the projected polytope at its minimal size (i.e., no redundant constraints) via an LP-based procedure. Ben-Tal et al. (2004) show that ARO problems are in general NP-hard. Through the lens of FME, we characterize the optimal decision rules for the proposed ARO problems. We further impose linear and quadratic decision rules on the remaining adjustable variables to inner approximate the MVE of the projection. The main advantages of our approach compare to the existing methods are: a) it can deal with general polytopes with auxiliary variables, i.e., it does not require an explicit description of the projection; b) it can be easily extended to find maximally sized convex bodies in polytopic projections; c) it allows users make a trade-off between approximation quality and computational complexity.

In order to evaluate the quality of our lower bounds, we adapt the approach of Hadjiyiannis et al. (2011) (HGK approach) to obtain upper bounds on the optimal solutions of the proposed ARO problem. We show via numerical experiments that the upper bound from HGK approach better

approximates the optimal objective value as more auxiliary variables are eliminated. Using FME to improve the upper bounds from the HGK approach is novel, and can also be easily applied to a broad class of ARO problems.

In this paper, we focus on MVE because it possesses many appealing properties, e.g., it is unique, invariant of the representation of the given convex body, and its center is a centralized (relative) interior of the convex body. In §4.3, we extend our approach to find a largest ball in a polytopic projection, and its center is known as Chebyshev center, which is a point that is farthest from the boundary of the projections.

Our main contributions are as follows:

• We introduce a novel scheme based on a blending of FME and ARO techniques to compute the MVE in a polytopic projection. Firstly, we apply FME to eliminate a subset of auxiliary variables in the given polytope, and then we solve an ARO problem via decision rule approximations, e.g., linear decision rules (LDRs) or quadratic decision rules (QDRs), to compute the desired MVE. For the color tube design problem, the lower bound from QDR approximation is optimal even if no auxiliary variable is eliminated. Our approach can easily be generalized to find a maximally sized convex body in a polytopic projection, and allows users to make a trade-off between solution quality and computational complexity.

• Through the lens of FME, we characterize the optimal decision rules for the proposed ARO problems.

• After eliminating an auxiliary variable via FME, we apply an LP-based removing redundant constraint procedure to keep the description of the projected polytope at its minimal size.

• We further construct an upper bounding scheme based on FME and HGK approach, which can be easily applied to a broad class of ARO problems. From numerical experiments, we observe that, a) as more auxiliary variables are eliminated, the upper bound from HGK approach converge to the optimal solution; b) unexpectedly, the critical scenarios of LDR approximations produce better upper bounds than those from QDR approximations.

The rest of this paper is organized as follows. In §2, we introduce our approach for polytopes that are full-dimensional in the $\boldsymbol{x}$-space. We adapt the approach of Hadjiyiannis et al. (2011) to obtain upper bounds of the optimal MVE in §3. We discuss an LP-based procedure for removing redundant constraints, and extend our approach to find the largest ball in §4. §5 evaluates our approach via numerical experiments. In §6, we conclude with future research directions.

Notations. We use $[n]$, $n \in \mathbb{N}$ to denote the set of running indices, $\{1, \ldots, n\}$. We generally use bold faced characters and capital letters such as $\boldsymbol{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$ to represent vectors and matrices, respectively, i.e., $\boldsymbol{a}_{i\cdot}$ to denote the i-th row of the matrix A, $x_i \in \mathbb{R}$ denotes the i-th

element of $\boldsymbol{x}$, and a_{ij} denotes the j-th element of $\boldsymbol{a}_{i\cdot}$. Special vectors include $\boldsymbol{0}$ and $\boldsymbol{1}$ which are respectively the vector of zeros and the vector of ones. We use $\mathcal{B}^n$ to denote the n-dimensional unit ball $\mathcal{B}^n = \{\boldsymbol{x} \in \mathbb{R}^n : ||\boldsymbol{x}||_2 \leq 1\}$.

2. MVE of Full-dimensional Projected Polytope

Suppose the polytope $\mathcal{P}$ given by (1) is full-dimensional in the $\boldsymbol{x}$-space, where $\boldsymbol{x} \in \mathbb{R}^{n_1}$ and $\boldsymbol{y} \in \mathbb{R}^{n_2}$ are the main variables and auxiliary variables, respectively. We denote $Proj(\mathcal{P})$ as the projection of $\mathcal{P}$ onto the $\boldsymbol{x}$-space, where

$$Proj(\mathcal{P}) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b} \right\}. \tag{2}$$

For a special class of $\mathcal{P}$ where $n_2 = 0$ (i.e., $\mathcal{P} = Proj(\mathcal{P})$), the MVE of $Proj(\mathcal{P})$ can be determined via the following semi-infinite programming problem:

$$\max_{\boldsymbol{x}, E} \left\{ \log \det E \mid \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1} : \quad A(\boldsymbol{x} + E\boldsymbol{\zeta}) \leq \boldsymbol{b} \right\}, \tag{3}$$

where $E \in \mathbb{S}^{n_1}$ with implicit constraint $E \succ 0$, and $\mathbb{S}^{n_1}$ is the set of $n_1 \times n_1$ symmetric matrices. The matrix E models the shape and volume of an n_1-dimensional ellipsoid with the unique center at the optimal $\boldsymbol{x}$ of (3). This is a static linear robust optimization problem under ellipsoidal uncertainty $\mathcal{B}^{n_1}$, which can be reformulated into an SDP problem (see, e.g., Boyd and Vandenberghe (2004), Ben-Tal et al. (2009)):

$$\max_{\boldsymbol{x}, E} \left\{ \log \det E \mid \quad \boldsymbol{a}_{i\cdot}^T \boldsymbol{x} + ||E\boldsymbol{a}_{i\cdot}||_2 \leq b_i \quad \forall i \in [m] \right\}, \tag{4}$$

where $\boldsymbol{a}_{i\cdot}$ denote the i-th row of matrix A. For the rest of this paper, we focus on general polyhedra where $n_2 \in \mathbb{Z}^+$, hence, $\mathcal{P} \neq Proj_{\boldsymbol{x}}(\mathcal{P})$.

2.1. Two Naive Attempts

Two naive extensions of (3) may be as follows:

$$\max_{\boldsymbol{x}, \boldsymbol{y}, E} \left\{ \log \det E \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_2}, \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1+n_2} : \quad A \left[\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} + E\boldsymbol{\zeta} \right] \leq \boldsymbol{b} \right\}, \tag{5}$$

where $E \in \mathbb{S}^{n_1+n_2}$;

$$\max_{\boldsymbol{x}, \boldsymbol{y}, E} \left\{ \log \det E \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_2}, \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1} : \quad A \left[\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} + E\boldsymbol{\zeta} \right] \leq \boldsymbol{b} \right\}, \tag{6}$$

where $E \in \mathbb{S}^{n_1}$. Problem (5) determines the MVE in the $(\boldsymbol{x}, \boldsymbol{y})$-space; Problem (6) finds the MVE of the $\boldsymbol{x}$-space with a fixed optimal vector $\boldsymbol{y}$. The following example shows that due to the existence of auxiliary variables $\boldsymbol{y}$, both (5) and (6) fail to find the MVE of the $\boldsymbol{x}$-space.

Figure 1 The ellipsoids in (7). The triangle corresponds to the polytope $\mathcal{P}$ defined in (7). The dash ellipsoid and the line segment are obtained from (5) and (6), respectively; the thick line segment on x-axis is the optimal MVE in (7).
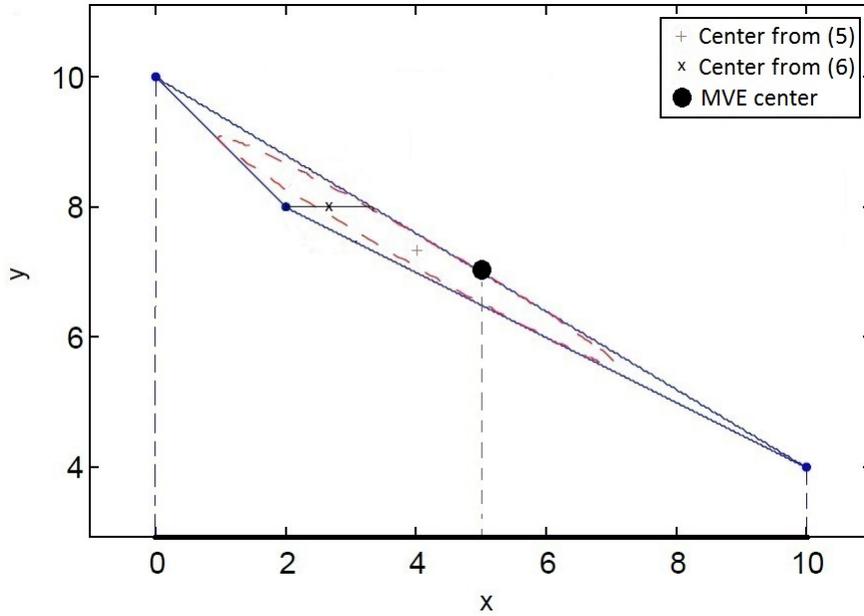
EXAMPLE 1. **Approximating the MVE via (5) and (6).** *Suppose $\mathcal{P}$ is the full-dimensional polytope that is formed by the intersection of three half-planes:*

$$Proj(\mathcal{P}) = \{x \in \mathbb{R} | \ \exists y \in \mathbb{R} : \ -0.5x - y \leq -9, \ 0.6x + y \leq 10, \ -x - y \leq -10\}. \tag{7}$$

Figure 1 depicts the polytope $\mathcal{P}$ and the obtained ellipsoids from (5) and (6). It can readily be seen that the feasible $x \in \mathbb{R}$ lies in the interval $[0, 10]$, i.e., $Proj(\mathcal{P}) = [0, 10]$. Clearly, the MVE of $Proj(\mathcal{P})$ is the interval $[0, 10]$ which is centered at $x = 5$. Problem (5) finds the MVE of $\mathcal{P}$ which is centered at $(4, 7.33)$; Problem (6) finds the longest horizontal line segment in $\mathcal{P}$.

2.2. Fourier-Motzkin Eliminate

Fourier-Motzkin Eliminate (FME) is an old method for deriving polytopic projections. The following algorithm is adapted from (Bertsimas and Tsitsiklis 1997, page 72).

Algorithm 1 *Fourier-Motzkin Elimination for polyhedral projections.*

1. *For some $l \in [n_2]$, rewrite each constraint in $Proj(\mathcal{P})$ into the form:*

$$a_{il+n_1}y_l \leq b_i - \sum_{j \in [n_1]} a_{ij}x_j - \sum_{j \in [n_2] \setminus \{l\}} a_{ij+n_1}y_j \quad \forall i \in [m],$$

if $a_{il+n_1} \neq 0$, divide both sides by a_{il+n_1}. We obtain an equivalent representation of $Proj(\mathcal{P})$ involving the following constraints:

$$y_l \leq \frac{b_i}{a_{il+n_1}} - \sum_{j \in [n_1]} \frac{a_{ij}x_j}{a_{il+n_1}} - \sum_{j \in [n_2] \setminus \{l\}} \frac{a_{ij+n_1}y_j}{a_{il+n_1}} \qquad \forall i \in [m]: a_{il+n_1} > 0,$$

$$y_l \geq \sum_{j \in [n_1]} \frac{a_{ij}x_j}{a_{il+n_1}} + \sum_{j \in [n_2] \setminus \{l\}} \frac{a_{ij+n_1}y_j}{a_{il+n_1}} - \frac{b_i}{a_{il+n_1}} \qquad \forall i \in [m]: a_{il+n_1} < 0,$$

$$0 \leq b_i - \sum_{j \in [n_1]} a_{ij}x_j - \sum_{j \in [n_2] \setminus \{l\}} a_{ij+n_1}y_j \qquad \forall i \in [m]: a_{il+n_1} = 0.$$

2. *Let $Proj(\mathcal{P}_{\setminus \{l\}})$ be the set after y_l is eliminated, and it is defined by the following constraints:*

$$\frac{b_i}{a_{il+n_1}} - \sum_{j \in [n_1]} \frac{a_{ij}x_j}{a_{il+n_1}} - \sum_{j \in [n_2] \setminus \{l\}} \frac{a_{ij+n_1}y_j}{a_{il+n_1}} \geq \qquad \forall i \in [m]: a_{il+n_1} > 0$$

$$\sum_{j \in [n_1]} \frac{a_{kj}x_j}{a_{kl+n_1}} + \sum_{j \in [n_2] \setminus \{l\}} \frac{a_{kj+n_1}y_j}{a_{kl+n_1}} - \frac{b_k}{a_{kl+n_1}} \qquad \forall k \in [m]: a_{kl+n_1} < 0,$$

$$b_i - \sum_{j \in [n_1]} a_{ij}x_j - \sum_{j \in [n_2] \setminus \{l\}} a_{ij+n_1}y_j \geq 0 \qquad \forall i \in [m]: a_{il+n_1} = 0.$$

LEMMA 1. $Proj(\mathcal{P}) = Proj(\mathcal{P}_{\setminus \{l\}})$.

Proof. See Appendix 1.

From Lemma 1, one can repeatedly apply Algorithm 1 to eliminate all the auxiliary variables $\boldsymbol{y}$, which results in an explicit description $Proj(\mathcal{P}_{\setminus [n_2]})$ of $Proj(\mathcal{P})$. We can then compute the MVE of $Proj(\mathcal{P}_{\setminus [n_2]})$ via (4), if the constraints in $Proj(\mathcal{P}_{\setminus [n_2]})$ are not prohibitively many. However, in Step 2 of Algorithm 1, the number of constraints may increase quadratically after each elimination. The complexity of eliminating n_2 adjustable variables from m constraints via Algorithm 1 is $\mathcal{O}(m^{2^{n_2}})$, which is an unfortunate inheritance of FME. In §4.1 we apply an LP-based procedure to remove redundant constraints, and keep the description of the projection at its minimal size. The following example shows that deriving a polytopic projection may lead to an exponential growth in the number of constraints.

EXAMPLE 2. **The complexity of variable eliminations/polytopic projections.** *Let us consider the following two interesting polytopes:*

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \sum_{i=1}^{n_1} |x_i| \leq 1 \right\} \quad and \tag{8}$$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \boldsymbol{\zeta}^T A \boldsymbol{x} \leq 1 \quad \forall \boldsymbol{\zeta} \in [\mathbf{0}, \mathbf{1}] \right\}, \quad where\ A \in \mathbb{R}^{m \times n_1}. \tag{9}$$

We rewrite (8) and (9) into the following representations:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_1} : \sum_{i=1}^{n_1} y_i \leq 1,\ x_i \leq y_i,\ -x_i \leq y_i,\ \forall i \right\} \quad and \tag{10}$$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \exists \boldsymbol{y} \in \mathbb{R}_+^m : \sum_{i=1}^{m} y_i \leq 1,\ \boldsymbol{y} \geq A\boldsymbol{x} \right\}, \quad respectively. \tag{11}$$

If one employs Algorithm 1 to eliminating all the auxiliary variables $\boldsymbol{y}$ in (10) and (11), we have the explicit descriptions with 2^{n_1} constraints and 2^m constraints, respectively:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \sum_{i=1}^{n_1} \max\{x_i, -x_i\} \leq 1 \right\}$$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \sum_{i=1}^{m} \max\left\{ \boldsymbol{a}_{i\cdot}^T \boldsymbol{x}, 0 \right\} \leq 1 \right\},$$

where $\boldsymbol{a}_{i\cdot} \in \mathbb{R}^{n_1}$ denotes the i-th row of A.

Note that polytope (8) is modeled by a piecewise-linear constraint, which is often seen in budget uncertainty sets (e.g., Bertsimas and Sim (2004)); the polytope (9) is modeled by a semi-infinite constraint, which is a typical constraint in robust optimization problems with box uncertainties. The auxiliary variables in the reformulations lift polytopes with many constraints into higher dimensions, so that the reformulations only need few constraints and extra auxiliary variables. However, eliminating the auxiliary variables may lead to an exponential growth in the number of constraints.

2.3. Our Approach

In this subsection, we introduce a novel scheme based on a blending of FME and ARO techniques to compute the MVE in a polytopic projection. Firstly, for $n_2 \in \mathbb{Z}^+$ in $\mathcal{P}$, we propose to solve following variant of Problem (5) and (6) to compute the MVE of the $\boldsymbol{x}$-space of $\mathcal{P}$:

$$\max_{\boldsymbol{x}, \boldsymbol{y}, E} \left\{ \log \det E \mid \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1}, \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A \begin{pmatrix} \boldsymbol{x} + E\boldsymbol{\zeta} \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b} \right\}. \tag{12}$$

This is a two-stage ARO problem with fixed recourse under ellipsoidal uncertainty $\mathcal{B}^{n_1}$. Here, $\boldsymbol{x} \in \mathbb{R}^{n_1}$ and $E \in \mathbb{S}^{n_1}$ are here-and-now decision variables, and $\boldsymbol{y}$ are wait-and-see adjustable variables. The optimal matrix E of (12) models the shape and volume of the n_1-dimensional MVE

with the unique center at the optimal $\boldsymbol{x}$. To the best of our knowledge, finding the MVE in a polytopic projection via ARO is novel. Problem (12) is generally intractable, because the adjustable variables $\boldsymbol{y}$ are decision rules instead of finite vectors of decision variables. The following theorem characterizes optimal decision rules for (12).

THEOREM 1. *There exist polynomials of (at most) degree n_1 and linear in ζ_i, $i \in [n_1]$, that are optimal decision rules for $\boldsymbol{y}$ in (12).*

Proof. See Appendix B. □

In fact, Theorem 1 not only holds for Problem (12), but also for general linear ARO problems with fixed recourse.

Note that Problem (6) is in fact a special case of (12) where the optimal $\boldsymbol{y}$ are constant, a.k.a, static decision rules. More sophisticated yet computationally tractable decision rules can be imposed on $\boldsymbol{y}$, e.g., quadratic decision rules (QDRs):

$$y_j = \boldsymbol{\zeta}^T W_j \boldsymbol{\zeta} + \boldsymbol{v}_{j\cdot}^T \boldsymbol{\zeta} + u_j, \quad \text{for } j = 1, ..., n_2, \tag{13}$$

where $u_j \in \mathbb{R}$, $\boldsymbol{v}_{j\cdot} \in \mathbb{R}^{n_1}$, and $W_j \in \mathbb{R}^{n_1 \times n_1}$. The following theorem provides a computationally tractable reformulation of (12) with quadratic decision rules (13).

THEOREM 2. *The solution $(\boldsymbol{x}^*, \boldsymbol{u}^*, E^*, V^*, W_j^*)$ is optimal for (12) with QDRs (13) if and only if it is an optimal solution of the following SDP problem:*

$$(lbQ) \quad \max_{\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{\lambda}, E, V, W_j} \quad \log \det E$$

$$s.t. \quad \boldsymbol{\tau} \leq \boldsymbol{b}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}$$

$$\begin{bmatrix} \tau_i - \lambda_i - \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{pmatrix} & -\frac{1}{2} \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} E \\ V \end{pmatrix} \\ -\frac{1}{2} \begin{pmatrix} E \\ V \end{pmatrix}^T \boldsymbol{a}_{i\cdot} & \lambda_i I - \sum_{j=1}^{n_2} a_{ij+n_1} W_j \end{bmatrix} \succeq 0 \qquad \forall i \in [m],$$

where $\boldsymbol{a}_{i\cdot}$ is the i-th row of A, $i \in [m]$.

Proof. See Appendix C. □

Problem (lbQ) although computationally tractable, can be difficult to solve in practice, especially when the instance is large. Alternatively, one can impose simpler decision rules on $\boldsymbol{y}$, e.g., linear decision rules (LDRs):

$$\boldsymbol{y} = V \boldsymbol{\zeta} + \boldsymbol{u}, \tag{14}$$

where the coefficients $\boldsymbol{u} \in \mathbb{R}^{n_2}$ and $V \in \mathbb{R}^{n_2 \times n_1}$ are optimization variables. Similarly, the MVE of $Proj(\mathcal{P})$ can be approximated by solving a simpler SDP problem.

COROLLARY 1. *The solution $(\boldsymbol{x}^*, \boldsymbol{u}^*, E^*, V^*)$ is optimal for (12) with LDRs (14) if and only if it is an optimal solution of the following SDP problem:*

$$(lbL) \quad \max_{\boldsymbol{x}, \boldsymbol{u}, E, V} \quad \log \det E$$

$$s.t. \quad \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{pmatrix} + \left\| \begin{pmatrix} E \\ V \end{pmatrix}^T \boldsymbol{a}_{i\cdot} \right\|_2 \leq b_i \qquad \forall i \in [m].$$

Proof. The proof is straightforward, hence it is omitted. □

Due to the imposed structure on $\boldsymbol{y}$, Problem (lbQ) and (lbL) under-approximate the MVE of $Proj(\mathcal{P})$. §5 shows that Problem (lbL) is much faster to solve than (lbQ), but the quality of the solutions from (lbQ) can be much better than (lbL). In the following example, we apply (lbL) to find the MVE of the projected polytope in Example 1.

EXAMPLE 3. **The maximum volume inscribed ellipsoid of a full-dimensional polytope with auxiliary variables.** *Let us again consider the full-dimensional polytope (7). After eliminating y via Algorithm 1, we have $Proj(\mathcal{P}_{\setminus\{1\}}) = \{x \mid x \in [0, 10]\}$. The optimal MVE is centered at $x = 5$ with a radius 5. By solving (lbL) for $Proj(\mathcal{P})$ (i.e., without eliminating y), we find the optimal MVE with $y^* = 7 - 3\zeta$ for $\zeta \in [-1, 1]$. This optimal LDR of y^* coincides with the line that covers all the feasible values of x in $\mathcal{P}$, i.e., $0.6x + y = 10$.*

Zhen et al. (2016) show that in general, the decision rule approximations of ARO problems can simply be improved by eliminating the adjustable variables via FME. Therefore, we propose first (iteratively) eliminate a subset $\mathcal{S} \subseteq [n_2]$ of $\boldsymbol{y}$ via FME till the size of resulting description of

$$Proj(\mathcal{P}_{\setminus\mathcal{S}}) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \exists \boldsymbol{y} \in \mathbb{R}^{\widetilde{n}_2} : \widetilde{A} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \leq \widetilde{\boldsymbol{b}} \right\}, \tag{15}$$

reaches the prescribed limit. Here, $\widetilde{m}$ and $\widetilde{n}_2 = n_2 - |\mathcal{S}|$ denote the number of constraints and remaining auxiliary variables; $\widetilde{A} \in \mathbb{R}^{\widetilde{m} \times (n_1 + \widetilde{n}_2)}$ and $\widetilde{\boldsymbol{b}} \in \mathbb{R}^{\widetilde{m}}$ denote the resulting coefficient matrix and the right-hand-side vector, respectively. Then, we solve the corresponding ARO problem (12) via decision rule approximations to compute the MVE of $Proj(\mathcal{P}_{\setminus\mathcal{S}})$.

FME may lead to many redundant constraints in (15). §4.1 discusses an LP-based procedure to remove **all** the redundant constraints in (15). We observe in §5.2 that this procedure can significantly improve the computability of our approach.

Since Problem (12) have left-hand side uncertainties and non-linear objective function, the performance bounds on LDR approximations in Bertsimas and Goyal (2012) and Bertsimas and Bidkhori (2015) are not applicable here.

3. An Upper Bounding Scheme

3.1. HGK Approach and FME

The scenario counterpart (ub) of Problem (12):

$$(ub) \quad \max_{\boldsymbol{x},Y,E} \quad \log \det E$$

$$\text{s.t.} \quad A \begin{pmatrix} \boldsymbol{x} + E\boldsymbol{\zeta}^j \\ \boldsymbol{y}^j \end{pmatrix} \leq \boldsymbol{b} \qquad \forall j \in [k]$$

produces an upper bound on the optimal value of (12). Here, $\boldsymbol{y}^j$ is the j-th column of $Y \in \mathbb{R}^{n_2 \times k}$, and $\boldsymbol{\zeta}^j \subset \mathcal{B}^{n_1}$, $\forall j \in [k]$, $k \in \mathbb{Z}^+$. Problem (ub) is feasible if there exists a feasible $\boldsymbol{y}^j$ for each scenario $\boldsymbol{\zeta}^j$, $j \in [k]$. In order to obtain a tight upper bound, HGK approach considers the *critically binding scenarios* (CBSs) from decision rule approximations of ARO problems.

A critical scenario of Problem (12) with QDRs can be determined by solving the following problem:

$$\boldsymbol{\zeta}_Q^i = \underset{\boldsymbol{\zeta} \in \mathcal{B}^{n_1}}{\arg\max} \quad \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x}^* \\ \boldsymbol{u}^* \end{pmatrix} + \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} E^* \\ V^* \end{pmatrix} \boldsymbol{\zeta} + \boldsymbol{\zeta}^T \left(\sum_{j=1}^{n_2} a_{ij+n_1} W_j^* \right) \boldsymbol{\zeta} - b_i, \quad i \in [m], \qquad (16)$$

where $(\boldsymbol{x}^*, \boldsymbol{u}^*, E^*, V^*, W^*)$ denotes the optimal solution from (lbQ). From Appendix C, we can reformulate (16) into an SDP problem. The critical scenario $\boldsymbol{\zeta}^k$ is *binding* if the optimal objective value of (16) is 0. Similarly, for (12) with LDRs, the CBSs are obtained from solving the following SOCP problem:

$$\boldsymbol{\zeta}_L^i = \underset{\boldsymbol{\zeta} \in \mathcal{B}^{n_1}}{\arg\max} \quad \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x}^* \\ \boldsymbol{u}^* \end{pmatrix} + \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} E^* \\ V^* \end{pmatrix} \boldsymbol{\zeta} - b_i, \quad i \in [m], \qquad (17)$$

where $(\boldsymbol{x}^*, \boldsymbol{u}^*, E^*, V^*)$ denotes the optimal solution from (lbL).

In §2.3, we use FME to improve the decision rule approximations of Problem (12). FME can also be used to improve the upper bounds obtained from the HGK approach. We propose to first eliminate a subset of auxiliary variables via FME, and then use the CBSs from (16) or (17) to compute the upper bounds of (12). This blending of HGK approach and FME is novel, and can also be applied to a broad class of ARO problems. Via numerical experiments in §5, we observe that as more auxiliary variables are eliminated, the obtained upper bounds converge to optimal solutions.

3.2. A simple iterative procedure

Solving (ub) with a set of many scenarios can be problematic. Therefore, we propose the following iterative procedure. Given the set of critical scenarios $\{\boldsymbol{\zeta}^1, ..., \boldsymbol{\zeta}^m\}$, we first solve an initialization problem

$$(Iub) \quad \max_{\boldsymbol{x}, Y, E} \quad \log \det E$$
$$\text{s.t.} \quad \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x} + E\boldsymbol{\zeta}^i \\ \boldsymbol{y}^i \end{pmatrix} \leq b_i \qquad\qquad \forall i \in [m],$$

where $\boldsymbol{\zeta}^i$, $i \in [m]$, are the critical scenarios from (16) or (17). Then, we check if the i-th constraint, $i \in [m]$, satisfies all the CBSs for the optimal $(\boldsymbol{x}^*, Y^*, E^*)$ from (Iub). If there are violating constraints detected, we add those constraints to (Iub) and solve it again. We repeat this procedure until no violating constraints can be found. This iterative procedure only adds the violating constraints in each iteration. In §5, we observe that this procedure efficiently avoids to include too many redundant constraints, and significantly reduces computation time.

4. Miscellaneous

4.1. Removing Redundant Constraints

It is well-known that FME may produce many redundant constraints and may be very sensitive to those constraints. In order to keep $Proj(\mathcal{P}_{\backslash \mathcal{S}})$ at its minimal size, after eliminating an auxiliary variable via FME, we execute an LP-based procedure to detect and remove the redundant constraints. This removing redundant constraint procedure is first proposed by Caron et al. (1989). We test whether the j-th inequality, $j \in [m]$, in $Proj(\mathcal{P}_{\backslash \mathcal{S}})$ is implied by the rest via the following LP problem:

$$\max_{\boldsymbol{x}, \boldsymbol{y}} \quad \boldsymbol{a}_{j\cdot}^T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} - b_j$$
$$\text{s.t.} \quad \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \leq b_i \qquad i \in [m] \backslash \{j\},$$

where $\boldsymbol{a}_{i\cdot}$ is the i-th row of A, $i \in [m]$, and b_i is the i-th element of the vector $\boldsymbol{b}$. Then the j-th inequality in $Proj(\mathcal{P}_{\backslash \mathcal{S}})$ is redundant if and only if the optimal objective value is less or equal to 0. By successively solving this LP for each untested inequality against the remaining, one would finally obtain a description of $Proj(\mathcal{P}_{\backslash \mathcal{S}})$ at its minimal size. §5.2 shows that this procedure significantly improves the computability of our approach.

4.2. MVE of General Polytope

Suppose the projected polytope $Proj(\mathcal{P})$ defined in (2) is not full-dimensional. There are (hidden) equality constraints in polytope $\mathcal{P}$. We consider two classes of (hidden) equality constraints in $Proj(\mathcal{P})$. The first class consists of equality constraints that contain the auxiliary variables $\boldsymbol{y}$. The second class consists of equality constraints that **only** contain the main variables $\boldsymbol{x}$. The equality constraints formed by inequalities can be detected and separated efficiently (see Fukuda (2013)). For Problem (12), a typical constraint of the first class can be presented as follows:

$$\boldsymbol{a}^T (\boldsymbol{x} + E\boldsymbol{\zeta}) + y = b \quad \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1}, \tag{18}$$

where $\boldsymbol{a} \in \mathbb{R}^{n_1}$ and the variable $y \in \mathbb{R}$ is an auxiliary variable (without loss of generality we assume the coefficient for y is 1). In Gorissen et al. (2015), the authors briefly discuss two ways of dealing with uncertain equality constraints with adjustable variables. Either one can eliminate those equality constraints, or one can apply LDRs. Lemma 2 shows that both procedures are equivalent.

LEMMA 2. *Using an LDR for the auxiliary variable y in (18) is equivalent to eliminating y.*

Proof: See Appendix D. □

Since using decision rules requires extra variables, it is preferred to eliminate the auxiliary variables in the uncertain equality constraints. For simplicity, we assume there is no (hidden) equality constraint containing $\boldsymbol{y}$ in $Proj(\mathcal{P})$. After separating all the hidden equality constraints from the inequalities, we partition the constraints in $Proj(\mathcal{P})$ as follows:

$$Proj(\mathcal{P}) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A_{\boldsymbol{x}}^1 \boldsymbol{x} + A_{\boldsymbol{y}}^1 \boldsymbol{y} \leq \boldsymbol{b}^1, \ A_{\boldsymbol{x}}^2 \boldsymbol{x} = \boldsymbol{b}^2 \right\}. \tag{19}$$

Here, the set without (hidden) equalities $\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A_{\boldsymbol{x}}^1 \boldsymbol{x} + A_{\boldsymbol{y}}^1 \boldsymbol{y} \leq \boldsymbol{b}^1 \right\}$ is full-dimensional. Obviously, $A_{\boldsymbol{x}}^2 \boldsymbol{x} = \boldsymbol{b}^2$ reduce the dimension of $Proj(\mathcal{P})$, and there does not exist a positive definite matrix E such that,

$$A_{\boldsymbol{x}}^2 (\boldsymbol{x} + E\boldsymbol{\zeta}) = \boldsymbol{b}^2 \quad \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1}.$$

Hence, the method introduced in §2 cannot be applied directly. We propose to use the following constraints elimination technique (see Boyd and Vandenberghe (2004)). Let us define the columns of the full rank matrix $F \in \mathcal{R}^{n_1 \times (n_1 - l)}$ to be a basis of the null space of $A_{\boldsymbol{x}}^2$, where $l = rank(A_{\boldsymbol{x}}^2)$. Then, we have

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid A_{\boldsymbol{x}}^2 \boldsymbol{x} = \boldsymbol{b}^2 \right\} = \left\{ F\boldsymbol{z} + \boldsymbol{x}_0 \mid \boldsymbol{z} \in \mathbb{R}^{n_1 - l} \right\}, \tag{20}$$

where $\boldsymbol{x}_0$ is a solution of the linear equations $A_{\boldsymbol{x}}^2 \boldsymbol{x} = \boldsymbol{b}^2$. The new set (on the right hand side of (20)) is full-dimensional in the variable $\boldsymbol{z}$. Hence, $Proj(\mathcal{P})$ can be represented as a full-dimensional set:

$$Proj(\mathcal{Q}) = \left\{ \boldsymbol{z} \in \mathbb{R}^{n_1 - l} \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A^1 \begin{pmatrix} F\boldsymbol{z} + \boldsymbol{x}_0 \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b}^1 \right\},$$

where $A^1 = (A_{\boldsymbol{x}}^1 \ A_{\boldsymbol{y}}^1)$. Note that $Proj(\mathcal{P}) = \{F\boldsymbol{z} + \boldsymbol{x}_0 \mid \boldsymbol{z} \in Proj(\mathcal{Q})\}$. The MVE of the full-dimensional set $Proj(\mathcal{Q})$ can be obtained by solving:

$$\max_{\boldsymbol{x}, \boldsymbol{y}, E} \left\{ \log \det E \mid \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1 - l}, \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A^1 \begin{pmatrix} F(\boldsymbol{z} + E\boldsymbol{\zeta}) + \boldsymbol{x}_0 \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b}^1 \right\}.$$

This is again a two-stage ARO problem. Hence, all the techniques discussed in §2 and §3 can be readily applied. Since the MVE is *affine invariant*, the MVE center of $Proj(\mathcal{P})$ can be recovered from an affine transformation of $\boldsymbol{z}^*$, i.e.,

$$\boldsymbol{x}^* = F\boldsymbol{z}^* + \boldsymbol{x}_0.$$

4.3. Extension to Ball

A largest ball in a full-dimensional $Proj(\mathcal{P})$ can simply be determined by replacing the matrix E with ρI everywhere in (12), where $\rho \in \mathbb{R}_+$ and $I \in \mathbb{R}^{n_1 \times n_1}$ is the identity matrix, the optimal $\boldsymbol{x}$ is a point that is furthest away from the exterior of $Proj(\mathcal{P})$, a.k.a., *Chebyshev center*.

Let us consider a special class of (full dimensional) polyhedra where only a subset of $\boldsymbol{x}$ appears in the constraints containing $\boldsymbol{y}$, e.g., for $\mathcal{S} \subseteq [n_1]$,

$$Proj(\mathcal{P}) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n_1} \mid \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A_{\boldsymbol{x}}^1 \boldsymbol{x}_{\mathcal{S}} + A_{\boldsymbol{y}}^1 \boldsymbol{y} \leq \boldsymbol{b}^1, \ A_{\boldsymbol{x}}^2 \boldsymbol{x} \leq \boldsymbol{b}^2 \right\}. \tag{21}$$

The corresponding problem of finding a largest inscribed ball of (21) can be formulated as follows:

$$\max_{\boldsymbol{x}, \boldsymbol{y}, \rho} \left\{ \rho \mid \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1}, \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A^1 \begin{pmatrix} \boldsymbol{x}_{\mathcal{S}} + \rho \boldsymbol{\zeta}_{\mathcal{S}} \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b}^1, \ A_{\boldsymbol{x}}^2 (\boldsymbol{x} + \rho \boldsymbol{\zeta}) \leq \boldsymbol{0} \right\}. \tag{22}$$

For this special class of polyhedra, the following theorem shows that we can restrict decision rules to be functions in a subset of uncertain parameters $\boldsymbol{\zeta}_{\mathcal{S}}$.

THEOREM 3. *There exist piece-wise affine functions in $\boldsymbol{\zeta}_{\mathcal{S}}$ that are optimal decision rules for $\boldsymbol{y}$ in (22).*

Proof: See Appendix E. □

Since the largest ball and its Chebyshev center is not affine invariant, if $Proj(\mathcal{P})$ is not full dimensional, the constraint eliminate technique discussed in §4.2 cannot be applied. A largest inscribed ball of $Proj(\mathcal{P})$ defined in (19) can be determined by solving the following ARO problem

$$\max_{\boldsymbol{x}, \boldsymbol{y}, \rho} \left\{ \rho \mid \forall \boldsymbol{\zeta} \in \{\boldsymbol{\zeta} \in \mathcal{B}^{n_1} \mid A_{\boldsymbol{x}}^2 \boldsymbol{\zeta} = \boldsymbol{0}\}, \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A^1 \begin{pmatrix} \boldsymbol{x} + \rho \boldsymbol{\zeta} \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b}^1, \ A_{\boldsymbol{x}}^2 \boldsymbol{x} = \boldsymbol{0} \right\}. \tag{23}$$

Here, the constraints $A_{\boldsymbol{x}}^2 \boldsymbol{\zeta} = \boldsymbol{0}$ ensure the ball to be within the same space as $\boldsymbol{x}$. An alternative formulation of Problem (23) can be presented as follows:

$$\max_{\boldsymbol{x},\boldsymbol{y},\rho} \left\{ \rho \mid \forall \boldsymbol{\zeta} \in \mathcal{B}^{n_1}, \exists \boldsymbol{y} \in \mathbb{R}^{n_2} : A^1 \begin{pmatrix} \boldsymbol{x} + \rho P \boldsymbol{\zeta} \\ \boldsymbol{y} \end{pmatrix} \leq \boldsymbol{b}^1, \ A_{\boldsymbol{x}}^2 \boldsymbol{x} = \boldsymbol{0} \right\}, \tag{24}$$

where $P = I - (A_{\boldsymbol{x}}^2)^T [A_{\boldsymbol{x}}^2 (A_{\boldsymbol{x}}^2)^T]^{-1} A_{\boldsymbol{x}}^2$ is the (symmetric) projection matrix that projects vector $\boldsymbol{\zeta} \in \mathbb{R}^{n_1}$ onto the null space of $A_{\boldsymbol{x}}^2$.

COROLLARY 2. *The solution* $(\boldsymbol{x}^*, \boldsymbol{u}^*, \rho^*, V^*, W_j^*)$ *is optimal for (24) with QDRs (13) if and only if it is an optimal solution of the following SDP problem:*

$$(lbQb) \quad \max_{\boldsymbol{x},\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\lambda},\rho,V,W_j} \quad \rho$$

$$s.t. \quad \boldsymbol{\tau} \leq \boldsymbol{b}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}$$

$$A_{\boldsymbol{x}}^2 \boldsymbol{x} = \boldsymbol{0}$$

$$\left[\begin{array}{c|c} \tau_i - \lambda_i - \boldsymbol{a}_{i\cdot}^{1T} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{pmatrix} & -\frac{1}{2} \boldsymbol{a}_{i\cdot}^{1T} \begin{pmatrix} \rho P \\ V P \end{pmatrix} \\ \hline -\frac{1}{2} \begin{pmatrix} \rho P \\ V P \end{pmatrix}^T \boldsymbol{a}_{i\cdot}^1 & \lambda_i I - P \sum_{j=1}^{n_2} a_{ij+n_1}^1 W_j P \end{array} \right] \succeq 0 \qquad \forall i \in [m],$$

where $\boldsymbol{a}_{i\cdot}^1$ *is the i-th row of* A^1, $i \in [m]$, $I \in \mathbb{R}^{n_1 \times n_1}$ *is the identity matrix, and* $P = I - (A_{\boldsymbol{x}}^2)^T [A_{\boldsymbol{x}}^2 (A_{\boldsymbol{x}}^2)^T]^{-1} A_{\boldsymbol{x}}^2$.

Proof. Identical to Theorem 2, hence, it is omitted. $\qquad\square$

Note that the technique discussed in this subsection cannot be applied to find the MVE, since it leads to unbounded objective value if the $\boldsymbol{x}$-space is not full-dimensional. For Problem (24) with LDRs, one can simply replace the matrix E by ρP in all the constraints of (lbL) and maximize ρ instead of logdet$E$.

One can easily extend our framework to find maximally sized convex bodies in polytopic projections, e.g., $\{\boldsymbol{\zeta} \in \mathbb{R}^{n_1} : ||\boldsymbol{\zeta}||_1 \leq 1\}$ and $\{\boldsymbol{\zeta} \in \mathbb{R}^{n_1} : ||\boldsymbol{\zeta}||_\infty \leq 1\}$ instead of $\{\boldsymbol{\zeta} \in \mathbb{R}^{n_1} : ||\boldsymbol{\zeta}||_2 \leq 1\}$ in (12). Zhao et al. (2016) extends our approach to determine the maximum homothet of a given polytope in polytopic projections. In Table 1, we summarize the complexities of Problem (12) with static, linear and quadratic decision rules for different geometries.

5. Numerical Experiments

In this section, we evaluate the performance and applicability of our method. In §5.1, we conduct a simple experiment to examine the performance of our lower and upper bounds. In §5.2, we apply our approach to find a robust temperature profile for the color picture tube design. We denote the

Table 1   The complexities for computing the maximally sized convex bodies via decision rule approximations.

Geometry	Decision rules		
	Static	Linear	Quadratic
Ball	LP	CQP	SDP
MVE	SDP	SDP	SDP
1-box	LP	LP	NP-hard
∞-box	LP	LP	NP-hard

Figure 2   The tetrahedron $\mathcal{P}^1$ defined in (25). The dark shaded square is the projection of $\mathcal{P}^1$ onto the $\boldsymbol{x}$-space.
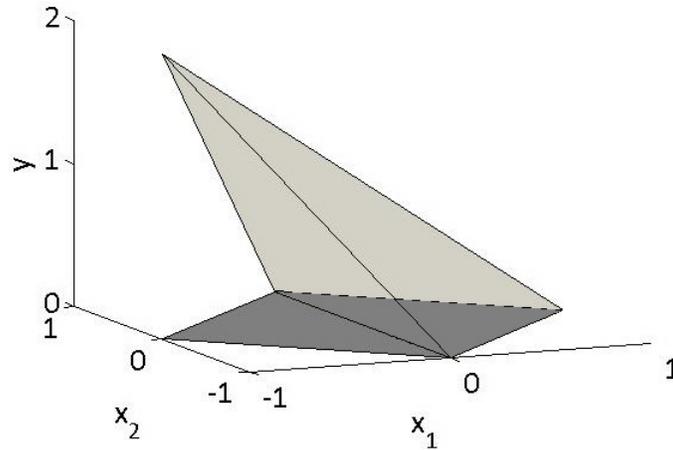
upper bounds from solving (ub) with critical scenarios from (16) and (17) as (ubQ) and (ubL), respectively.

5.1. A Simple Experiment

Let us consider the full-dimensional tetrahedron

$$\mathcal{P}^i = \{(x_1, x_2, y) \mid x_1 + x_2 + y \leq 1, \ x_1 - x_2 + y \leq 1, \ -(1+i)x_1 - iy \leq 0, \ y \geq 0 \}, \qquad (25)$$

where x_1, x_2 are main variables, y is an auxiliary variable, and $i \in [0,8]$. We are interested in the MVE of the projected $\mathcal{P}^i$ onto the $\boldsymbol{x}$-space. When $i = 0$, the projection $Proj(\mathcal{P}^0)$ is an isosceles triangle; when $i = 1$, $Proj(\mathcal{P}^i)$ forms a square (see Figure 2); when $i > 1$, $Proj(\mathcal{P}^i)$ becomes a kite. The explicit description of $Proj(\mathcal{P}^i)$ can be obtained by eliminating y in (25) via FME (see §2):

$$Proj(\mathcal{P}^i) = \{(x_1, x_2) \mid x_1 + x_2 \leq 1, \ x_1 - x_2 \leq 1, \ -x_1 + ix_2 \leq i, \ -x_1 - ix_2 \leq i \}.$$

Note that this is a full-dimensional polyhedron with no auxiliary variables. We can compute the optimal MVE center of $Proj(\mathcal{P}^i)$ via (4).

In Figure 3, the objective values $\log \det E$ (i.e., volumes) of the optimal and approximated MVEs of $Proj(\mathcal{P}^i)$ are plotted. As expected, the lower bounds from (lbQ) consistently outperform the

Figure 3 The volume of optimal and approximated MVEs of $Proj(\mathcal{P}^i)$ for $i \in [0, 8]$. The two solid lines (QDR) are the lower and upper bounds of the volume of approximated MVEs from QDRs (i.e., from (lbQ) and (ubQ)); the two dash lines (LDR) present the lower and upper-bounds of the volume of approximated MVEs from LDRs (i.e., from (lbL) and (ubL)); the shaded area (BestLU) is the region between the best lower and upper bounds from (ubL). The thick line (Optimal) plots the optimal logdet E.
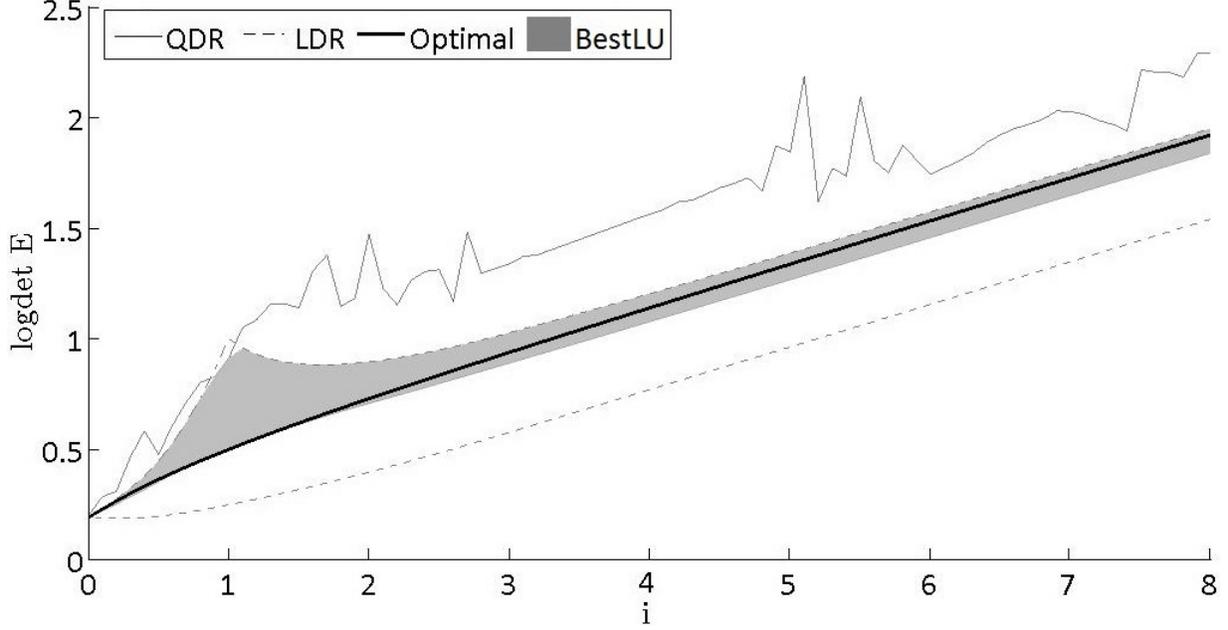

ones from (lbL). On the other hand, despite more computational efforts, the upper bounds from (ubQ) are mostly poorer than the ones from (ubL). The quality of the approximations is also affected by the shape of the tetrahedron $\mathcal{P}^i$. If $i = 0$, all approximations are optimal; when $i = 1$, the solution from (lbQ) is optimal, while the upper bound from (ubL) is at its poorest among all $i \in [0, 8]$.

Similar observations can be obtained from Figure 4. The approximated MVE centers from (lbQ) are the closest to the optimal MVE centers. For $i < 5.5$, the approximated centers from (lbQ) are closer to the optimal ones; for $i > 5.5$, the approximated centers from (ubL) are closer to the optimal ones.

5.2. Robust Color Picture Tube Design

In the manufacturing process of a CRT TV, the *color picture tube* is assembled to the mask-screen and the inner shield by a industrial oven at a high temperature. The oven temperature causes thermal stresses on the tube and if the temperature is too high, the pressure will scrap the tube due to implosion. In den Hertog and Stehouwer (2002), the authors present a temperature profile of a color picture tube, see Figure 5. The profile is characterized by the temperature at 20 locations. To minimize the production cost and hence the number of scraps, an optimal temperature profile that satisfies the following criteria is considered:

Figure 4 The Euclidean distance of the optimal and approximated MVE centers of $Proj(\mathcal{P}^i)$ for $i \in [0,8]$, i.e., the lbL, lbQ, ubL and ubQ plot the Euclidean distance of the optimal MVE centers and approximated MVE centers from (lbL), (lbQ), (ubL) and (ubQ) for $i \in [0,8]$, respectively.
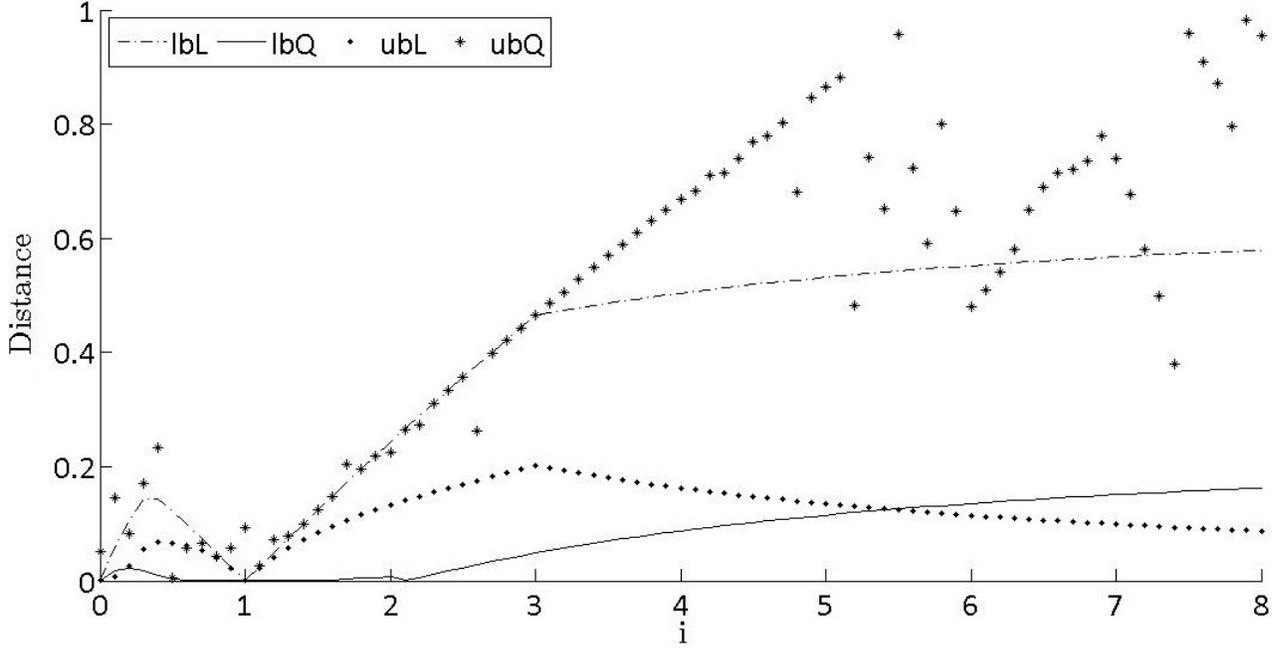

- the maximal stress for the TV tube is minimal
- the temperature differences between near locations are not too high
- the temperatures are in the specified range.

The authors formulate the associated problem as follows:

$$(TV) \quad \min_{\boldsymbol{x}, s_{max}} \quad s_{max}$$
$$\text{s.t.} \quad a_i + \boldsymbol{b}_i^T \boldsymbol{x} \le s_{max} \qquad\qquad \forall i \in [k]$$
$$|A\boldsymbol{x}| \le \Delta T_{max}$$
$$\boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u},$$

where $s_{max} \in \mathbb{R}$ is the maximal stress, $a_i + \boldsymbol{b}_i^T \boldsymbol{x} \in \mathbb{R}$ is the stress at location i of the temperature profile $\boldsymbol{x} \in \mathbb{R}^n$. There are $n = 20$ temperature points on the tube (see Figure 5). Furthermore, these temperatures result in $k = 216$ thermal stresses on different parts of the tube. The 216 response functions of the thermal stresses, $a_i + \boldsymbol{b}_i^T \boldsymbol{x}$, are derived by using the finite element method simulator and regression. The vectors $\boldsymbol{l} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^n$ are the lower and upper bounds of the temperature profile. The parameter $\Delta T_{max} \in \mathbb{R}^d$ represents the maximal allowed temperature on d location combinations; $A \in \mathbb{R}^{d \times n}$ is the coefficient matrix that enforce the specified temperatures do not differ more than ΔT_{max}. For example, the temperatures at locations 2 and 5 in Figure 5 cannot

Figure 5   Temperature profile of a color picture tube.
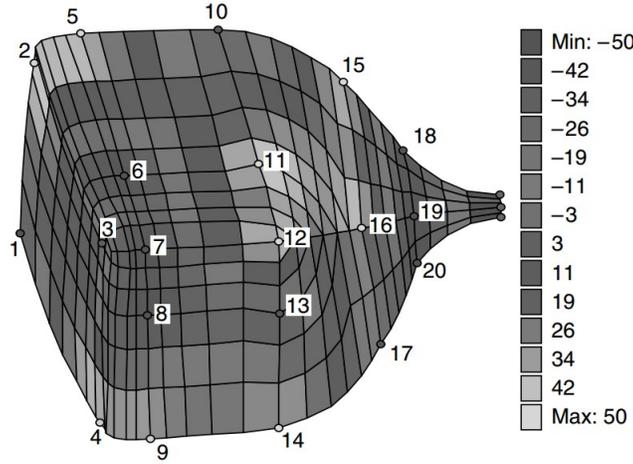
differ more than ΔT_{max}. By solving the nominal problem (TV), the unique minimum $s_{max} = 14.15$ is found. Suppose the maximum tolerance of s_{max} is at most 15. A robust temperature profile that tolerates implementation error and not far from the current profile is desired, say, $||\boldsymbol{x} - \bar{\boldsymbol{x}}||_1 \leq 15$. The feasible temperature profile set is as follows:

$$Proj(\mathcal{P}) = \left\{ \boldsymbol{x} \in [\boldsymbol{l}, \boldsymbol{u}] \mid \exists \boldsymbol{y} \in \mathbb{R}^n : \begin{array}{l} a_i + \boldsymbol{b}_i^T \boldsymbol{x} \leq 15, \ \forall i \in [k], |A\boldsymbol{x}| \leq \Delta T_{max}, \\ \boldsymbol{x} - \bar{\boldsymbol{x}} \leq \boldsymbol{y}, \ \bar{\boldsymbol{x}} - \boldsymbol{x} \leq \boldsymbol{y}, \ \sum_{j=1}^n y_j \leq 15 \end{array} \right\}. \tag{26}$$

We propose the MVE center of (26) as the robust temperature profile and approximate it via our method proposed in §2.

Table 2   The number of constraints and computation times when $i, i \in [20] \cup \{0\}$, auxiliary variables are eliminated. # Elim. denotes the number of auxiliary variables that are eliminated; # Con. gives the resulting number of constraints from FME before and after removing redundant constraints in (26); Time records the computation times needed for solving the corresponding (lbL), (lbQ), (ubL) and (ubQ) after removing the redundant constraints in (26); "$*$" means the computation time exceeds 2 hours.

# Elim.		0	2	4	6	8	10	12	14	16	18	20
# Con.	Before	329	328	336	380	568	1332	4400	16684	65832	262436	1048864
	After	59	58	59	79	171	296	294	1058	1056	1054	1052
Time	lbL	1	1	1	1	3	5	5	31	35	29	28
	lbQ	10	27	67	70	168	397	338	1322	474	345	28
	ubL	43	21	28	65	1178	11399	26777	$*$	$*$	$*$	$*$
	ubQ	87	48	58	139	2585	22984	53660	$*$	$*$	$*$	$*$

Figure 6 shows that, for the four approximations we consider, the more auxiliary variables that are eliminated, the better the solutions become. Surprisingly, the lower bound from (lbQ) is optimal even if no auxiliary variable is eliminated, which significantly outperforms the ones from (lbL). On the other hand, despite more computational efforts are required, the upper bounds from (ubQ) are

Figure 6   The volume of optimal and approximated MVEs of (26) when i, $i \in [20]$, auxiliary variables are elimi-
nated. The two solid lines (QDR) are the lower and upper bounds of the volume of approximated MVEs
from QDRs (i.e., from (lbQ) and (ubQ)); the two dash lines (LDR) present the lower and upper-bounds
of the volume of approximated MVEs from LDRs (i.e., from (lbL) and (ubL)); the shaded area (BestLU)
is the region between the best lower and upper bounds from (ubL). The thick line (Optimal) plots the
optimal logdet E.
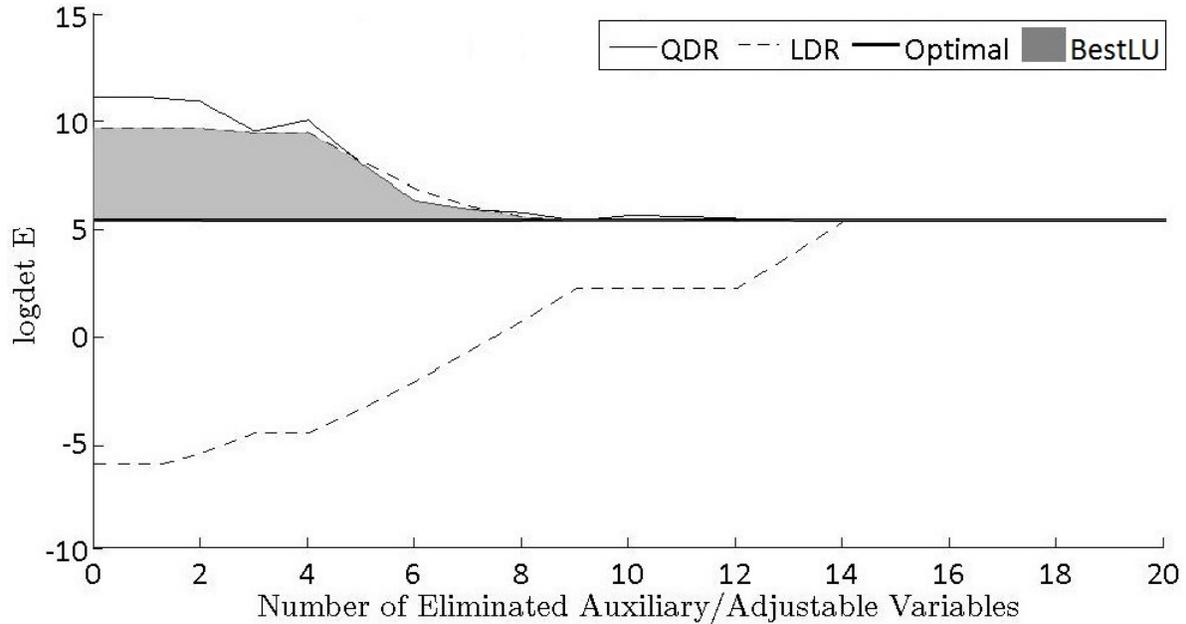
Figure 7   The Euclidean distance of the optimal and approximated MVE centers of (26) when i, $i \in [20]$, auxiliary
variables are eliminated, i.e., the lbL, lbQ, ubL and ubQ plot the Euclidean distance of the optimal
MVE centers and approximated MVE centers from (lbL), (lbQ), (ubL) and (ubQ), respectively.
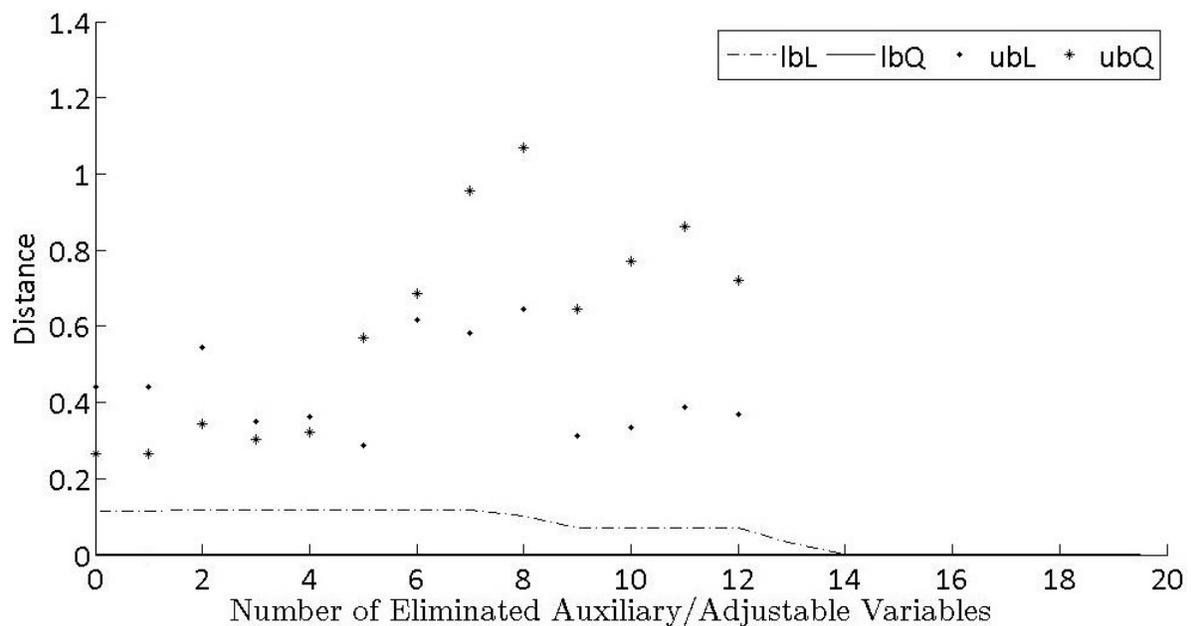
again mostly poorer than the ones from (ubL) (except when 6 auxiliary variables are eliminated). The lower and upper bounds from (lbQ) and (ubL) gives the best bounds for the optimal volumes.

In Figure 7, one can observe that the approximated center from (ubQ) is optimal throughout this experiment; the approximated center from (lbL) gradually converge to the optimal one as more auxiliary variables are eliminated; unexpectedly, the approximated centers from (ubL) and (ubQ) does not seem to converge as more auxiliary variables are eliminated. After eliminating 12 auxiliary variables, the associate upper bound problems become so large that the prescribed computational limit, i.e., 2 hours, is exceeded.

Due to the extra constraints, the set (26) is much smaller than the feasible region of (TV). Hence, there may exist many redundant constraints in (26). Therefore, we first apply the procedure in §4.1 to remove the 270 $(= 329 - 59$, see Table 2$)$ redundant constraints in (26), and solve the four approximations, i.e., (lbL), (lbQ), (ubL) and (ubQ). Similarly, after each auxiliary variable is eliminated, we remove the redundant constraints (if there is any), and solve the four approximations again. From Table 2, one can observe that, without eliminating the redundant constraints in (26) initially, one would end up with an explicit description of (26) with $1,048,864$ constraints while at most 1052 constraints are non-redundant. We also observe that, in this experiment, if the starting polytope does not contain redundant constraints, FME does not produce any redundant constraints after eliminating the auxiliary variables.

The computation times needed for solving (lbL) and (lbQ) increase exponentially at first, then decrease drastically after eliminating 14 auxiliary variables. This is because of the exponential growth in the number of constraints and the reduction in the number of optimization variables after eliminating the auxiliary variables. The upper bounds are computed from (ub) with all critical scenarios from (16) and (17) via the iterative procedure described in §3. We consider all critical scenarios instead of only CBSs, because solving (ub) with only CBSs does not produce finite upper bounds for this experiment. We solve (ub) via the iterative procedure discussed in §3.2. All the upper bounds are obtained within 5 iterates, and only less than 10% of the total constraints are included in (Iub). Hence, at least more than 90% in (ub) appeared to be redundant. All the computations are performed by using cvx 2.1 with MOSEK 7.1 within Matlab R2014a on an Intel Core i5-4590 CPU running at 3.3GHz with 8GB RAM under Windows 8 operating system.

6. Conclusions

By combining FME with techniques from adjustable robust optimization, e.g., decision rule approximations, we construct a computationally tractable approach to find a maximally size convex body

(e.g., ellipsoid, ball, box) of a projected polytope. Furthermore, we use FME to improve the upper bounds from HGK approach, and observe that the critical scenarios of LDRs produce better upper bounds than those from QDRs.

In the recent paper of Jaillet et al. (2016), the authors consider an R-model that determines the most robust solution which would remain feasible in the problems constraints when the uncertain parameters arise over a maximally sized uncertainty set. In contrast with their approach, we propose to use the MVE center as a robust solution without assuming any structure on the uncertainties. One future research direction would be to have a closer investigation on, e.g., the computational complexities, modeling capabilities and solution characteristics of the two models.

On a numerical level, we would like to investigate the performance of our approach on cutting-plane method. The goal of cutting-plane methods is to find a point in a convex set $\mathcal{X}$. The general procedure of cutting-plane methods is as follows. Suppose a polytope $\mathcal{P}$ that contains the set $\mathcal{X}$ is provided, i.e., $\mathcal{X} \subseteq \mathcal{P}$. We first select a point $\boldsymbol{x}^{(0)}$ in $\mathcal{P}$. If $\boldsymbol{x}^{(0)} \in \mathcal{X}$, then we are done; otherwise, a separating hyperplane between $\boldsymbol{x}^{(0)}$ and $\mathcal{X}$ can be constructed. This hyperplane cuts off the halfspace that contains no points of $\mathcal{X}$ in $\mathcal{P}$. Let us denote the resulting set $\mathcal{P}$ after the first cut as $\mathcal{P}_1$. Then, we select another point $\boldsymbol{x}^{(1)}$ in $\mathcal{P}_1$ and repeat the procedures until we locate a point in $\mathcal{X}$. A centralized point $\boldsymbol{x}^{(k)}$ of $\mathcal{P}_k$ is preferred as it leads to a deeper cut of $\mathcal{P}_k$. The location of the selected point $\boldsymbol{x}^{(k)} \in \mathcal{P}_k$ in each iteration determines the convergence rate of the method. Many choices of $\boldsymbol{x}^{(k)} \in \mathcal{P}_k$ are proposed in the literature (see Boyd and Vandenberghe (2007)), e.g., the centroid, the Chebyshev center, the MVE center, the analytic center. Our approach can be applied as an extension of the MVE center cutting-plane method and Chebyshev center cutting-plane method. Suppose the set $\mathcal{X}$ is described by two kinds of variables, i.e., the variables $\boldsymbol{y}$ appear linearly in all the constraints, and the variables $\boldsymbol{x}$ appear nonlinearly in some constraints. The complexity of locating a point in $\mathcal{X}$ arises by the variables $\boldsymbol{x}$, since if all the variables appear linearly in the description of the set $\mathcal{X}$, finding a point in it can be done by solving an LP problem. Therefore, we focus on cutting the projection of $\mathcal{P}$ where the variables $\boldsymbol{x}$ reside in. The cutting-plane can be determined by the Chebyshev center or MVE center of the projected polytope (i.e., the given polytope that is projected onto the space where the nonlinear variables reside in). Our method may significantly speed up the convergence rate of the cutting-plane method in those cases.

Appendix. Proofs of Theorems

A. Fourier-Motzkin Elimination

Proof of Corollary 1. This proof is adapted from (Bertsimas and Tsitsiklis 1997, page 73). If $\overline{\boldsymbol{x}} \in Proj(\mathcal{P})$, there exists some $\boldsymbol{y}$ such that $(\overline{\boldsymbol{x}}, \boldsymbol{y}) \in \mathcal{P}$. In particular, the vector $(\overline{\boldsymbol{x}}, \boldsymbol{y})$ satisfies all the inequalities in Step 1 of Algorithm 1, from which it follows immediately that $(\overline{\boldsymbol{x}}, \boldsymbol{y}_{\backslash \{l\}})$ all the inequalities in Step 2, hence, $\overline{\boldsymbol{x}} \in Proj(\mathcal{P}_{\backslash \{l\}})$. This shows $Proj(\mathcal{P}) \subset Proj(\mathcal{P}_{\backslash \{l\}})$.

We now show that $Proj(\mathcal{P}_{\backslash \{l\}}) \subset Proj(\mathcal{P})$. Let $\overline{\boldsymbol{x}} \in Proj(\mathcal{P}_{\backslash \{l\}})$, there exists some $\boldsymbol{y}_{\backslash \{l\}}$ such that $(\overline{\boldsymbol{x}}, \boldsymbol{y}_{\backslash \{l\}}) \in Proj(\mathcal{P}_{\backslash \{l\}}$. It follows from inequalities in Step 2 that:

$$
\max_{\{k \in [m] \mid a_{kl+n_1} < 0\}} \left\{ \sum_{j \in [n_1]} \frac{a_{kj} x_j}{a_{kl+n_1}} + \sum_{j \in [n_2] \backslash \{l\}} \frac{a_{kj+n_1} y_j}{a_{kl+n_1}} - \frac{b_k}{a_{kl+n_1}} \right\}
$$
$$
\leq \min_{\{i \in [m] \mid a_{il+n_1} > 0\}} \left\{ \frac{b_i}{a_{il+n_1}} - \sum_{j \in [n_1]} \frac{a_{ij} x_j}{a_{il+n_1}} - \sum_{j \in [n_2] \backslash \{l\}} \frac{a_{ij+n_1} y_j}{a_{il+n_1}} \right\}.
$$

Let y_l be any number between the two sides of the above inequality. It then follows that $(\overline{\boldsymbol{x}}, \boldsymbol{y})$ satisfies all the inequalities in Step 1, and therefore, $\overline{\boldsymbol{x}}$ belongs to $Proj(\mathcal{P})$. $\square$

B. Polynomial Decision Rules

Proof of Theorem 1. Suppose the values of all but one element ζ_1 of $\boldsymbol{\zeta}$ are fixed. The ellipsoidal uncertainty set of (12) reduces to a simplex set for which LDRs are optimal decision rules (see Bertsimas and Goyal (2012)). Therefore, there exist optimal decision rules that are linear in each ζ_i, $i \in [n_1]$, for all $\boldsymbol{y}$ in (12). $\square$

C. Tractable Robust Counterpart of (12) with Quadratic Decision Rules

Proof of Theorem 2. This proof is adapted from Ben-Tal et al. (2009). Let us consider the constraint i, $i \in [m]$ in (12), i.e.,

$$
\boldsymbol{a}_{i \cdot}^T \begin{pmatrix} \boldsymbol{x} + E\boldsymbol{\zeta} \\ \boldsymbol{y} \end{pmatrix} \leq b_i \qquad \qquad \forall \boldsymbol{\zeta} : ||\boldsymbol{\zeta}||_2 \leq 1. \tag{27}
$$

Replacing $\boldsymbol{y}$ with quadratic decision rules:

$$
y_j = \boldsymbol{\zeta}^T W_j \boldsymbol{\zeta} + \boldsymbol{v}_{j \cdot}^T \boldsymbol{\zeta} + u_j \quad \text{for } j = 1, ..., n_2,
$$

where $u_j \in \mathbb{R}$, $\boldsymbol{v}_{j\cdot} \in \mathbb{R}^{n_1}$ and $W_j \in \mathbb{R}^{n_1 \times n_1}$, we have

$$\max_{\boldsymbol{\zeta}:||\boldsymbol{\zeta}||_2 \leq 1} \underbrace{\boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{pmatrix}}_{p_i} + \underbrace{\boldsymbol{a}_{i\cdot}^T \begin{pmatrix} E \\ V \end{pmatrix} \boldsymbol{\zeta}}_{2\boldsymbol{q}_i^T} + \boldsymbol{\zeta}^T \underbrace{\left[\sum_{j=1}^{n_2} a_{ij+n_1} W_j \right]}_{R_i} \boldsymbol{\zeta}$$

$$= \max_{\boldsymbol{\zeta}:||\boldsymbol{\zeta}||_2 \leq 1} p_i + 2\boldsymbol{q}_i^T \boldsymbol{\zeta} + \boldsymbol{\zeta}^T R_i \boldsymbol{\zeta}$$

$$= \min_{\tau_i} \left\{ \tau_i \mid \forall(\boldsymbol{\zeta}:||\boldsymbol{\zeta}||_2 \leq 1): \tau_i \geq p_i + 2\boldsymbol{q}_i^T \boldsymbol{\zeta} + \boldsymbol{\zeta}^T R_i \boldsymbol{\zeta} \right\}$$

$$= \min_{\tau_i} \left\{ \tau_i \mid \forall((\boldsymbol{\zeta},t):||\boldsymbol{\zeta}||_2 \leq t^2): (\tau_i - p_i)t^2 - 2t\boldsymbol{q}_i^T \boldsymbol{\zeta} - \boldsymbol{\zeta}^T R_i \boldsymbol{\zeta} \geq 0 \right\}$$

$$= \min_{\tau_i} \left\{ \tau_i \mid \forall(\boldsymbol{\zeta},t), \exists \lambda_i \geq 0: (\tau_i - p_i)t^2 - 2t\boldsymbol{q}_i^T \boldsymbol{\zeta} - \boldsymbol{\zeta}^T R_i \boldsymbol{\zeta} - \lambda_i(t^2 - \boldsymbol{\zeta}^T \boldsymbol{\zeta}) \geq 0 \right\} [\mathcal{S} - Lemma]$$

$$= \min_{\tau_i} \left\{ \tau_i \mid \left[\begin{array}{c|c} \tau_i - p_i - \lambda_i & -\boldsymbol{q}_i^T \\ \hline -\boldsymbol{q}_i & \lambda_i I - R_i \end{array} \right] \succeq 0, \lambda_i \geq 0 \right\}.$$

Hence, the tractable robust counterpart of (27) can be formulated as follows

$$\begin{cases} \tau_i \leq b_i \\ \lambda_i \geq 0 \\ \left[\begin{array}{c|c} \tau_i - \lambda_i - \boldsymbol{a}_{i\cdot}^T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{pmatrix} & -\frac{1}{2}\boldsymbol{a}_{i\cdot}^T \begin{pmatrix} E \\ V \end{pmatrix} \\ \hline -\frac{1}{2} \begin{pmatrix} E \\ V \end{pmatrix}^T \boldsymbol{a}_{i\cdot} & \lambda_i I - \sum_{j=1}^{n_2} a_{ij+n_1} W_j \end{array} \right] \succeq 0. \end{cases}$$

$\square$

D. Uncertain Equality Constraints with Auxiliary Variables

Proof of Theorem 2. Let us consider a general form of (18):

$$\boldsymbol{a}(\boldsymbol{\zeta})^T \boldsymbol{x} + y = b(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{U},$$

where $\boldsymbol{a}(\boldsymbol{\zeta})$ and $b(\boldsymbol{\zeta})$ are affine in $\boldsymbol{\zeta}$, i.e., $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{a}(\boldsymbol{\zeta}) = \boldsymbol{a} + D\boldsymbol{\zeta}$, $b(\boldsymbol{\zeta}) = b + \boldsymbol{c}^T \boldsymbol{\zeta}$, $\boldsymbol{a} \in \mathbb{R}^n$, $D \in \mathbb{R}^{n \times m}$, and $\boldsymbol{c} \in \mathbb{R}^m$, $b \in \mathbb{R}$, and $\mathcal{U}$ is a general uncertainty set. From (14), we have:

$$y = u + \boldsymbol{v}^T \boldsymbol{\zeta},$$

where $u \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^m$. Then we have

$$(\boldsymbol{a} + D\boldsymbol{\zeta})^T \boldsymbol{x} + u + \boldsymbol{v}^T \boldsymbol{\zeta} = b + \boldsymbol{c}^T \boldsymbol{\zeta} \qquad \forall \boldsymbol{\zeta} \in \mathcal{U}$$

$$\Leftrightarrow \boldsymbol{a}^T \boldsymbol{x} + u + (D^T \boldsymbol{x} + \boldsymbol{v} - \boldsymbol{c})^T \boldsymbol{\zeta} = b \qquad \forall \boldsymbol{\zeta} \in \mathcal{U}.$$

The equality constraint is satisfied for all $\boldsymbol{\zeta}$ in $\mathcal{U}$ if and only if

$$\begin{cases} \boldsymbol{a}^T \boldsymbol{x} + u = b \\ (D^T \boldsymbol{x} + \boldsymbol{v} - \boldsymbol{c})^T \boldsymbol{\zeta} = 0 \quad \forall \boldsymbol{\zeta} \in \mathcal{U}. \end{cases}$$

Take $u = b - \boldsymbol{a}^T \boldsymbol{x}$ and $\boldsymbol{v} = \boldsymbol{c} - D^T \boldsymbol{x}$, we have,

$$y = u + \boldsymbol{v}^T \boldsymbol{\zeta} = b - \boldsymbol{a}^T \boldsymbol{x} + \boldsymbol{c}^T \boldsymbol{\zeta} - \boldsymbol{x}^T D \boldsymbol{\zeta} = b(\boldsymbol{\zeta}) - \boldsymbol{a}(\boldsymbol{\zeta})^T \boldsymbol{x}.$$

Hence, substituting $y = b(\boldsymbol{\zeta}) - \boldsymbol{a}(\boldsymbol{\zeta})^T \boldsymbol{x}$ everywhere in the problem is equivalent to using a linear decision rule for y. $\qquad\square$

E. Partially Dependent Optimal Decision Rules

Proof of Theorem 3. Let us consider the corresponding Problem (22) after all auxiliary variables $\boldsymbol{y}$ in (21) are eliminated except for one, say y_1. From the first step of FME for ARO problems (see (Zhen et al. 2016, Algorithm 1)), one can observe that y_1 is upper and lower bounded by piece-wise affine functions in $\boldsymbol{\zeta}_{\mathcal{S}}$. The lower (or upper) bounding piece-wise affine function is an optimal decision rule for y_1. Once the optimal decision rule of y_1 is determined, one can then reverse (Zhen et al. 2016, Algorithm 1)) to recover the optimal decision rules of the eliminated variables, and observe that there exist piecewise affine functions only in $\boldsymbol{\zeta}_{\mathcal{S}}$ that are optimal decision rules for all y_i, $i \in [N_2]$, in Problem (22). $\qquad\square$

References

A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.

A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust Optimization*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2009.

D. Bertsimas and H. Bidkhori. On the performance of affine policies for two-stage adaptive optimization: a geometric perspective. *Mathematical Programming*, 153(2):577–594, 2015.

D. Bertsimas and V. Goyal. On the power and limitations of affine policies in two-stage adaptive optimization. *Mathematical Programming*, 134(2):491–531, 2012.

D. Bertsimas and M. Sim. The price of robustness. *Operations Research*, 52(1):35–53, 2004.

D. Bertsimas and J. Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific, 1997.

S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004.

S. Boyd and L. Vandenberghe. *Localization and Cutting-Plane Methods*. From Stanford EE 364b lecture notes, 2007.

R. Caron, J. McDonald, and C. Ponic. A linear decision-based approximation approach to stochastic programming. *Journal of Optimization Theory*, 62:225–237, 1989.

F.S. Chaharsooghi, M.J. Emadi, M. Zamanighomi, and M.R. Aref. A new method for variable elimination in systems of inequations. *Proceedings IEEE International Symposium on Information Theory (ISIT)*, :1215–1219, 2011.

D. den Hertog and P. Stehouwer. Optimizing color picture tubes by high-cost nonlinear programming. *European Journal of Operational Research*, 140(2):121–197, 2002.

J.B.J. Fourier. reported in: Analyse des travaux de l'Académie Royale des Sciences, pendant l'année 1824, Partie mathématique (1827). *Histoire de l'Academie Royale des Sciences de l'Institut de France*, 7: 47–55, 1824.

K. Fukuda. *Polyhedral Computation*. Lecture notes, ETH Zurich, 2013.

B.L. Gorissen, I. Yanikoglu, and D. den Hertog. A practical guide to robust optimization. *Omega*, 53: 124–137, 2015.

H.E. Graeb. *Analog Design Centering and Sizing*. Springer-Verlag, 2007.

M.J. Hadjiyiannis, P.J. Goulart, and D. Kuhn. A scenario approach for estimating the suboptimality of linear decision rules in two-stage robust optimization. *Proceedings IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, :7386–7391, 2011.

E.M.T. Hendrix, C.J. Mecking, and T.H.B. Hendriks. Finding robust solutions for product design problems. *European Journal of Operational Research*, 92:28–36, 1996.

T. Huynh, C. Lassez, and J.-L. Lassez. Practical issues on the projection of polyhedral sets. *Annals of Mathematics and Artificial Intelligence*, 6:295–316, 1992.

P. Jaillet, S. Jena, T.S. Ng, and M. Sim. Satisficing awakens: Models to mitigate uncertainty. *Optimization Online*, 2016. URL `http://www.optimization-online.org/DB_FILE/2016/01/5310.pdf`.

C.N. Jones, E.C. Kerrigan, and J.M. Maciejowski. Equality set projection: A new algorithm for the projection of polytopes in halfspace representation. CUED Technical Report CUED/F-INFENG/TR.463, 2004.

C. Lassez and J.-L. Lassez. *Symbolic and Numerical Computation for Artificial Intelligence*, chapter 4, pages 103–119. Academic Press, 1992.

J.-L. Lassez. Query constraints. *Proceedings ACM Conference Principles of Database Systems*, 4:288–298, 1990.

T.S. Motzkin. *Beiträge zur Theorie der linearen Ungleichungen*. University Basel Dissertation. Jerusalem, Israel, 1936.

A.T. Sarić and A.M. Stanković. Applications of ellipsoidal approximations to polyhedral sets in power system optimization. *IEEE Transactions on Power Systems*, 23(3):956–965, 2008.

H.R. Tiwary. On computing the shadows and slices of polytopes. *CoRR*, 2008. URL `http://arxiv.org/abs/0804.4150`.

L. Zhao, H. Hao, and W. Zhang. Extracting flexibility of heterogeneous deferrable loads via polytopic projection approximation. *arXiv*, 2016. URL `https://arxiv.org/abs/1609.05966`.

J. Zhen and D. den Hertog. Robust solutions for systems of uncertain linear equations. *CentER Discussion Paper Series*, pages No. 2015–044, 2015.

J. Zhen, D. den Hertog, and M. Sim. Adjustable robust optimization via fourier-motzkin elimination. *Optimization Online*, 2016. URL `http://www.optimization-online.org/DB_FILE/2016/07/5564.pdf`.