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Longevity Risk and Natural Hedge Potential in Portfolios of Life Insurance Products: The Effect of Investment Risk

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Abstract

Payments of life insurance products depend on the uncertain future evolution of survival probabilities. This uncertainty is referred to as longevity risk. Existing literature shows that the effect of longevity risk on single life annuities can be substantial, and that there exists a (natural) hedge potential from combining single life annuities with death benefits or from investing in survivor swaps. The effect of financial risk on these hedge effects is typically ignored. The aim of this paper is to quantify longevity risk in portfolios of mortality-linked assets and liabilities, taking into account the effect of financial risk. We find that investment risk significantly affects the impact of longevity risk in life insurance products. It also significantly affects the hedge potential that arises from combining life insurance products, or from investing in longevity-linked assets. For example, our results suggest that ignoring the effect of financial risk can lead to severe overestimation of the natural hedge potential from death benefits, and underestimation of the hedge effects of survivor swaps.

Keywords: Life insurance, life annuities, death benefits, survivor swaps, risk management, financial risk, longevity risk, insolvency risk, capital adequacy.

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1 Introduction

Our goal in this paper is to quantify longevity risk in portfolios of life insurance products, taking into account the potential effect of investment risk on the impact of longevity risk. Specifically, our focus is on potential interactions between liability mix effects and asset mix effects.

Existing literature suggests that uncertainty regarding the future development of human life expectancy potentially imposes significant risk on pension funds and insurers (see, for example, Olivieri and Pitacco, 2001; Brouhns, Denuit, and Vermunt, 2002; Cossette, Delwarde, Denuit, Guillot, and Marceau, 2007; Dowd, Cairns, and Blake, 2006; Hári, De Waegenaere, Melenberg, and Nijman, 2008). Existing literature also shows that the natural hedge potential that arises from combining life annuities and death benefits may be substantial (see, for example, Cox and Lin, 2007; Wang, Huang, Yang, and Tsai, 2010; Tsai, Wang, and Tzeng, 2010). These analyses quantify longevity risk in annuity portfolios by determining its effect on the probability distribution of the *present value* of all future payments, for a given, deterministic, and constant term structure of interest rates. A drawback of this approach is that it does not allow to take into account the possible interaction between longevity risk and financial risk, i.e., it is a "liability-only" approach. Hári et al. (2008) quantify longevity risk in portfolios of single life annuities in the presence of financial risk by determining its effect on the volatility of the *funding ratio*. The funding ratio is defined as the ratio of the value of the assets over the value of the liabilities. They find that financial risk can significantly affect the impact of longevity risk on funding ratio volatility. However, a drawback of a funding ratio approach is that it requires specifying the probability distribution of the value of the liabilities at a future date. Determining the value of longevity-linked liabilities is still a contentious issue. Although in recent years there has been considerable interest in developing pricing models for longevity-linked assets and liabilities (see, for example, Blake and Burrows, 2001; Dahl, 2004; Lin and Cox, 2005; Denuit, Devolder, Goderniaux, 2007; Bauer, Boerger, and Russ, 2010), the lack of liquidity for trade in longevity-linked assets and/or liabilities makes it very difficult to calibrate these models. As long as this remains the case, it is unclear to what extent a funding ratio approach accurately reflects the effect of longevity risk.

Our goal in this paper is threefold. First, we quantify the impact of longevity risk in portfolios of life insurance products, taking into account potential interactions between financial risk and longevity risk. To avoid making any assumptions regarding the value at which longevity-linked liabilities can be sold, we quantify risk by means of the probability of ruin in a run-off approach. Specifically, for any given investment strategy, we determine the minimal required buffer (i.e., the asset value in excess of the best estimate value of the liabilities), such that the probability that the insurer or pension fund will be able to pay all future liabilities is sufficiently high (see, for example, Olivieri and Pitacco, 2003). The size of the buffer will be affected by longevity risk, which arises due to uncertain deviations in the future liability payments from their current best estimates, and by financial risk, which arises due to uncertainty in future returns on assets. Part of the financial risk arises due to uncertain returns on the assets needed to cover *unexpected* deviations of the liabilities from their expected values, and, therefore, cannot be fully hedged. We find that the effect of this unhedgeable financial risk on the required solvency buffer depends significantly on the type of liability. This suggests important interactions between financial risk and longevity risk.

Second, we quantify the effect of unhedgeable financial risk on the natural hedge potential, i.e., the risk reduction, that arises from combining liabilities with different sensitivities to longevity risk. Whereas financial risk is typically hedgeable for a deterministic stream of liabilities, the unhedgeable financial risk arises from the uncertainty in the stream of future payments. Life insurers and pension funds often hold several types of longevity-linked liabilities, such as single life annuities, last survivor annuities, and death benefit insurance.[1] Because the payments of these different life insurance products typically have different sensitivities to changes in mortality rates, insurers with a "diversified" portfolio of liabilities may be less sensitive to longevity risk.[2] The existing literature on such liability mix effects focuses on the natural hedge potential, i.e., risk reduction, of death benefits in portfolios of life annuities, and uses a liability-only approach to quantify the risk reduction.[3] We quantify the effect of investment risk on the natural hedge potential from combining life insurance products with different sensitivities to longevity risk. We find, for example, that ignoring unhedgeable investment risk may lead to significant overestimation of the hedge potential from death benefits in portfolios of single life annuities. The extent to which the hedge potential is overestimated depends nontrivially on the liability mix.

Third, we quantify the effect of potential interactions between liability mix effects and asset mix effects on the risk reduction from investing in survivor swaps. Because the payments of survivor swaps are based on actual survival of a reference population, they may be used to partially hedge longevity risk. Existing literature shows that the hedge potential can be affected by basis risk, i.e., residual risk due to differences in characteristics of the insured population and the reference population (see, for example, Dowd, Cairns, and Blake, 2006). In this paper we show that, in addition to basis risk, the hedge potential of survivor swaps also depends nontrivially on both the asset mix and the liability mix. Depending on the liability mix, the hedge potential of survivor swaps may either increase or decrease when investment risk is higher.

The paper is organized as follows. In Section 2 we define the life insurance liabilities that we consider, and discuss how they are affected by longevity risk. Section 3 gives a formal definition of the risk measure. Section 4 shows how investment risk affects the

[1] Many defined benefit pension funds offer both old-age pension insurance and partner pension insurance. The latter consists of a survivor annuity that yields periodic payments if the partner of the insured person is alive and the insured person has passed away. The Retirement Equity Act of 1984 (REA) amended the Employee Retirement Income Security Act of 1974 (ERISA) to introduce mandatory spousal rights in pension plans.

[2] Cox and Lin (2007) show empirically that a life insurer who has 95% of its business in annuities and 5% in death benefits prices its annuities on average 3% higher than an insurer who has 50% of its business in annuities and 50% of its business in death benefits. This indicates that insurers with death benefit liabilities have a competitive advantage.

[3] Wang et al. (2010) and Tsai et al. (2010) quantify the natural hedge potential of death benefits in portfolios of life annuities, and determine the optimal liability mix.

impact of longevity risk in single life annuities, survivor annuities, and death benefits, respectively. In Section 5, we quantify the effect of the interaction between liability mix effects and asset mix effects. Section 6 deals with the effect of liability and asset mix on the hedge potential of survivor swaps. Section 7 concludes.

2 Life insurance liabilities and longevity risk

In this section we introduce the life insurance liabilities that we consider, and discuss how they are affected by systematic and non-systematic longevity risk.

In addition to traditional old-age pensions, which take the form of a single life annuity, pension funds and insurers typically also offer other types of life insurance products, such as partner pensions and death benefits. A partner pension consists of a survivor annuity. It provides the partner of a deceased participant with a life long annuity payment. The death benefit consists of a single payment at the moment the insured person dies. Formally, we consider the following three types of liabilities:

(i) A *single life annuity*, which yields a nominal yearly payment of 1, with a last payment in the year the insured person dies;

(ii) A *survivor annuity*, which yields a nominal yearly payment of 1 in every year that the spouse outlives the insured person;

(iii) A *death benefit*, which yields a nominal single payment of 1 in the year that the participant dies.

We let $\mathcal{P} = \{sl, surv, db\}$ denote the set of life insurance products, and we denote a product by $p \in \mathcal{P}$, where $p = sl$ refers to a singe life annuity, $p = surv$ refers to a survivor annuity, and $p = db$ refers to a death benefit. These liabilities consist of (a stream of) payments in future periods. Because in any future period, the level of the payment depends on whether the insured person is alive, and, in case of survivor annuities, whether the partner is alive, the net cash outflow of these life insurance products is affected by two types of longevity risk:

- *non-systematic longevity risk:* conditional on given survival probabilities, whether an individual survives an additional year is a random variables;
- *systematic longevity risk:* the survival probabilities for future dates are uncertain.

While non-systematic longevity risk is diversifiable (i.e., the risk becomes negligible when portfolio size is large, see, for example, Olivieri and Pitacco, 2001), this is not the case for systematic longevity risk. Therefore, throughout the paper we assume that portfolios are large enough for non-systematic longevity risk to be negligible, and focus on the impact of systematic longevity risk. Because survival rates depend significantly on age and gender, we characterize an insured/participant by a vector $(\overline{x}, \overline{g})$, where

$$\begin{array}{lllllll} \overline{x} & = & x, & \overline{g} & = & g, & \text{if } p \in \{sl, db\}, \\ \overline{x} & = & (x, y), & \overline{g} & = & (g, g'), & \text{if } p = surv, \end{array}$$

where x denotes the age of the insured, $g \in \{m, f\}$ denotes the gender of the insured, and, in case of survivor annuities, y denotes the age of the partner, and $g' \in \{m, f\}$ denotes her/his gender. Then, for any given year t, the liability payments in a future year $t+\tau$, $\tau \geq 0$ for a single life annuity, a survivor annuity, and a death benefit insurance for an individual characterized by $(\overline{x}, \overline{g})$ in year t, are given by (see, for example, Gerber 1997):[4]

$$\begin{aligned}
\widetilde{L}_{p,\tau,t}(\overline{x}, \overline{g}) &= {}_{\tau}p_{x,t}^{(g)}, && \text{for } p = sl \text{ (singe life annuity)},\\
&= \left(1 - {}_{\tau}p_{x,t}^{(g)}\right) \cdot {}_{\tau}p_{y,t}^{(g')}, && \text{for } p = surv \text{ (survivor annuity)},\\
&= {}_{\tau-1}p_{x,t}^{(g)} - {}_{\tau}p_{x,t}^{(g)}, && \text{for } p = db \text{ (death benefit)},
\end{aligned} \tag{1}$$

where, following Cairns, Blake and Dowd (2006), we let:

- $p_{x+s,t+s}^{(g)}$ for $s \geq 0$ denote the *future* one-year survival probabilities of the cohort aged x in year t, given by $p_{x+s,t+s}^{(g)} = \mathbb{P}\left(T_{x,t} \geqslant s+1 | T_{x,t} \geqslant s, \mathcal{F}_{\infty}\right)$, where $T_{x,t}$ denotes the random remaining lifetime of an individual aged x at time t, and $\mathcal{F}_{\infty}$ denotes the set that contains all information regarding mortality rates at all (future) dates;
- ${}_{\tau}p_{x,t}^{(g)} = p_{x,t}^{(g)} \cdot p_{x+1,t+1}^{(g)} \cdot \dots \cdot p_{x+\tau-1,t+\tau-1}^{(g)}$ denotes the *future* τ-years survival probability of the cohort aged x in year t.

We consider a given and fixed date t, and quantify the risk in the liabilities in a run-off approach in which there are no new entrants in the portfolio, and no premiums are paid after date t. Without loss of generality, we let $t = 0$ and suppress the dependence on t unless it is required for clarity. Because our focus is on the interaction between liability mix and asset mix effects, we will consider portfolios consisting of several products, with varying weights, and with insureds with varying characteristics. Specifically, let $\mathcal{I}$ denote the set of insureds. The total payment in year τ is of the form

$$\widetilde{L}_{\tau} = \sum_{i \in \mathcal{I}} \sum_{p \in \mathcal{P}} \overline{\delta}_{i,p} \cdot \widetilde{L}_{p,\tau}(\overline{x}_i, \overline{g}_i), \tag{2}$$

where $\overline{\delta}_{i,p}$ denotes the insured right of insured i for pension product p. Throughout the paper, we denote BEL for the current (i.e., date-0) best estimate value of the liabilities, which is defined as the market value of the expected liabilities, i.e.,

$$BEL = \sum_{\tau=1}^{\infty} \mathbb{E}\left[\widetilde{L}_{\tau}\right] \cdot P^{(\tau)}, \tag{3}$$

where $P^{(\tau)}$ denotes the current market value of a zero-coupon bond with maturity τ. In Subsection 3.2 we discus the calculation of he expectation in (3)

[4] Existing literature shows that there exists dependence between the remaining lifetimes of a participant and his (her) partner at micro-level, e.g., due to the fact that partners have similar lifestyles, or that the passing away of a partner affects the surviving relative's quality of life. Because our focus in this paper is on systematic longevity risk, we ignore this dependence and assume that the remaining lifetimes of the spouses, conditional on the survival probabilities, are independent.

3 Quantifying risk

In this section we discuss how we quantify risk in portfolios that are sensitive to both longevity risk and financial risk. In Subsection 3.1 we formally define the risk measure. In Subsection 3.2, we provide a brief discussion of the models according to which the risk in the death rates, interest rates, and asset returns are generated. A complete description of these models can be found in Appendices A and B.

3.1 Risk measure

We quantify risk in portfolios of life insurance products by determining, for any given investment strategy, the minimal initial asset value such that the probability that the *terminal asset value* is positive is sufficiently large. The terminal asset value is defined as the remaining asset value after the last payment has been made. Without loss of generality, we express the initial asset value A_0 as the best estimate value of the liabilities, BEL, plus a buffer that is a percentage of the best estimate value, i.e.,

$$A_0 = (1 + c) \cdot BEL. \tag{4}$$

Then, for a given $\varepsilon > 0$, we determine the minimum value of the buffer percentage c such that:

$$\mathbb{P}\left(A_T < 0 \mid A_0 = (1 + c) \cdot BEL\right) \leqslant \varepsilon, \tag{5}$$

where T denotes the last period in which a payment needs to be made, and A_T denotes the corresponding terminal asset value.

The minimal required buffer percentage c depends on the probability distribution of the terminal asset value, A_T, which in turn depends on the initial asset value A_0, the liability payments, $\widetilde{L}_\tau$ (as defined in (2)), and the investment strategy. Specifically, the asset dynamics is given by:

$$A_\tau = (1 + r_\tau) \cdot A_{\tau-1} - \widetilde{L}_\tau, \qquad \tau = 1, \cdots, T,$$

where A_τ denotes the net asset value at the end of period τ, r_τ denotes the return on assets during period τ, and $\widetilde{L}_\tau$ denotes the liabilities paid at the end of period τ. Because we want to be able to distinguish between hedgeable and unhedgeable financial risk, we allow for the case where the insurer uses a different investment strategy for the best estimate value (BEL) and for the buffer ($c \cdot BEL$). Specifically, we define the following strategies.

Definition 1 *An investment strategy consists of:*

- *for every duration $\tau = 1, \cdots, T$: an asset mix for the best estimate value corresponding to duration τ (i.e., the amount $\mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)}$); the corresponding return in periods $s = 0, \cdots, \tau$ is denoted $r_s^{be,(\tau)}$;*

- *an asset mix for the buffer portfolio; the corresponding return in periods $\tau = 0, \cdots, T$ is denoted r_τ^{bu}.*

In every period τ, the accumulated value of the best estimate portfolio corresponding to duration τ is used to pay the liabilities in period τ; any shortage or excess is taken from, or reinvested in, the buffer portfolio.

Whereas the value of the buffer portfolio is affected by both longevity risk and investment risk, the value of the best estimate portfolio is only affected by investment risk. For example, when the buffer portfolio is invested in equity and the best estimate portfolio in zero-coupon bonds, a lower return on the assets, or a higher than expected realization of the liabilities, leads to a smaller proportion of assets invested in equity.

With the above defined investment strategy, we obtain the following result.

Proposition 2 *The minimum required buffer percentage to satisfy (5) is given by:*

$$c = \frac{Q_{1-\epsilon}(L)}{BEL} - 1, \tag{6}$$

where

$$L = BEL + \sum_{\tau=1}^{T} \left(\frac{\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)} \cdot \prod_{s=1}^{\tau} \left(1 + r_s^{be,(\tau)}\right)}{\prod_{s=1}^{\tau} \left(1 + r_s^{bu}\right)} \right), \tag{7}$$

and $Q_{1-\varepsilon}(L)$ denotes the $(1-\varepsilon)-$quantile of L.

Proof. The date-τ value of the best estimate portfolio corresponding to duration τ is given by $\mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)} \cdot \prod_{s=0}^{\tau} \left(1 + r_s^{be,(\tau)}\right)$. Combined with (3), this implies that the terminal asset value is given by:

$$\begin{aligned} A_T &= c \cdot BEL \prod_{\tau=1}^{T}\left(1 + r_\tau^{bu}\right) + \sum_{\tau=1}^{T} \left(\mathbb{E}\left[\widetilde{L}_\tau\right] P^{(\tau)} \prod_{s=1}^{\tau}\left(1 + r_s^{be,(\tau)}\right) - \widetilde{L}_\tau\right) \prod_{s=\tau}^{T}\left(1 + r_s^{bu}\right) \\ &= \left[(1+c) \cdot BEL - L\right] \cdot \prod_{\tau=1}^{T}\left(1 + r_\tau^{bu}\right), \end{aligned} \tag{8}$$

with L as defined in (7). Therefore, the terminal asset value A_T is nonnegative if

$$(1+c) \cdot BEL \geqslant L, \tag{9}$$

The result now follows immediately from (5). ■

The above proposition shows that the required buffer percentage follows from determining the $1 - \varepsilon$ quantile of the random variable L. The random variable L can be interpreted as follows. Conditional on any given future asset returns (r_s^{bu} and $r_s^{be,(\tau)}$), and cash flows ($\widetilde{L}_\tau$ as defined in (2)), L represents the value of the assets needed at date 0 to pay all future liability payments. For the sake of intuition, consider for example the case where all assets yield a deterministic and constant annual return, i.e., $r_s^{bu} = r_s^{be,(\tau)} = r$ for some $r > 0$, and $P^{(\tau)} = 1/(1+r)^\tau$. Then it follows immediately from (3) and (7) that L simplifies to:

$$L = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau}{(1+r)^\tau}, \tag{10}$$

i.e., L equals the discounted present value of all future liability payments. Thus, the standard approach in which longevity risk is quantified by determining its effect on the probability distribution of the present value of liabilities can be seen as a special case of our model. The more general case in (7), however, allows to take into account interactions between financial risk and longevity risk.

3.2 Modeling mortality rates and asset returns

To determine the minimum required buffer from (6), we simulate 15,000 scenarios for death rates and asset returns, and on the basis of these scenarios we calculate the $1-\varepsilon$ quantile of L. In this subsection we briefly describe the models we use to generate these scenarios.

To model asset returns, we use a Vasicek model for the term structure of interest rates, combined with a Geometric Brownian motion with time-varying drift for stock prices. We include both process risk (i.e., risk given estimated parameter values) and parameter risk (i.e., risk due to estimation inaccuracy). To estimate the parameters, we use the daily instantaneous short rate, the daily interest rate on a 10 years Dutch government bond, and the daily return on the Dutch stock index "AEX", obtained from Datastream. For a more detailed description of the models and the estimation technique, and for parameter estimates, we refer to Appendix A. We use these models to generate 15000 scenarios for asset returns.

For the probability distribution of the future survival probabilities we include process risk, parameter risk, and model risk. To incorporate model risk, we estimate three classes of survival probability models, namely the Lee-Carter (1992) class of models, the Cairns-Blake-Dowd (2006) class of models, and the P-Splines model (Currie, Durbin, and Eilers; 2004). We generate 5,000 scenarios for future survival rates from each class of models: 5,000 scenarios from Lee-Carter (1992)-type models with three different specifications, namely the Lee-Carter (1992) model (1,666 scenarios), the Brouhns, Denuit, and Vermunt (2002) model (1667 scenarios), and the Cossete et al. (2007) model (1,667 scenarios); 5,000 scenarios from Cairns-Blake-Dowd (2006) models with four different specifications, allowing for a quadratic term in the age effect, and/or constant/diminishing age effects in the cohort effects (each specification 1,250 scenarios); and, 5,000 scenarios from the P-Splines model with one specification. To estimate the parameters in each model, we use age-, gender-, and time-specific number of deaths and exposures to death for the Netherlands, obtained from the Human Mortality Database. For a detailed description of the models and the estimation techniques, and for parameter estimates, we refer to Appendix B.

4 Hedgeable and unhedgeable investment risk

In this section we investigate how investment risk affects the impact of longevity risk in single life annuities, survivor annuities, and death benefits, respectively. To do so, we decompose L from (7) into three components, i.e.,

$$L = BEL + L^{long} + L^{invest},$$

where

(i) $BEL = \sum_{\tau=1}^{\infty} \mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)}$ is the *deterministic component.* BEL is the market value of the *expected* liabilities. It represents the asset value that is needed on date 0 to pay all future expected liabilities, given that the expected liabilities are cash flow matched. Expected liabilities are cash flow matched iff for every duration τ, the amount $\mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)}$ is invested in (default-free) zero-coupon bonds with maturity τ.

(ii) L^{invest} is the *pure investment risk component*:

$$L^{invest} = \sum_{\tau=1}^{T} \frac{\mathbb{E}\left[\widetilde{L}_\tau\right] - \mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)} \cdot \prod_{s=1}^{\tau}\left(1 + r_s^{be,(\tau)}\right)}{\prod_{s=1}^{\tau}\left(1 + r_s^{bu}\right)}.$$

This component represents the asset value that, conditional on given asset returns, is needed on date 0 in addition to BEL to pay all future *expected* liabilities when expected liabilities are not cash flow matched. This component is affected by risk that arises due to uncertain deviations of the τ-years return on the best estimate portfolio ($\prod_{s=1}^{\tau}\left(1 + r_s^{be,(\tau)}\right)$) from the cash-flow matching return ($\frac{1}{P^{(\tau)}}$).

(iii) L^{long} is the *longevity risk component*:

$$L^{long} = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]}{\prod_{s=1}^{\tau}\left(1 + r_s^{bu}\right)}.$$

This component represents the asset value that, conditional on given asset returns, is needed at date 0 in addition to BEL and L^{invest} to pay all future *unexpected* liability payments (i.e., payments in excess of the expected value). This component is affected by two sources of longevity risk: *pure longevity risk* that arises due to deviations of the liabilities from their expected values ($\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]$), and *longevity induced investment risk* that arises due to uncertain returns on these deviations.

Thus, the present value variable L can be decomposed in a deterministic term that reflects the required asset value in absence of both longevity risk and financial risk (BEL), a term that reflects the required additional asset value in absence of longevity risk, but with financial risk (L^{invest}), and a term that reflects the required additional asset value due to longevity risk (L^{long}). Both L^{invest} and L^{long} are affected by investment risk, but only L^{long} is affected by longevity risk. Moreover, whereas L^{invest} reflects hedgeable risk (L^{invest} reduces to zero when the expected liabilities are cash flow matched), L^{long} reflects unhedgeable risk that arises due to uncertainty in the returns on assets required to cover unexpected liabilities.

Throughout the paper, we will use the following terminology:

- *hedgeable investment risk* as the risk due to uncertainty in the pure investment risk component L^{invest};
- *unhedgeable investment risk* as the risk due to uncertainty in the longevity risk component L^{long} that arises from uncertainty in the buffer returns r_τ^{bu};
- *longevity risk* as the the risk due to uncertainty in the longevity risk component L^{long} that arises from the uncertainty in the liability payments $\widetilde{L}_\tau$;
- *natural hedge* potential as the risk reduction in the longevity risk component L^{long} from combining different life insurance products.

We emphasize that our goal in this paper is not to determine the optimal investment portfolio, but rather to investigate to what extent the effect of longevity risk depends the investment strategy.

In the remainder of this section we illustrate the effect of financial risk on the impact of longevity risk by comparing the benchmark case, when L is defined as in (10), to the case where investment returns are uncertain. To quantify the effect of both hedgeable and unhedgeable interest rate risk, we compare two investment strategies. The first investment strategy is a "risky" one in which *all* assets are (re)invested in a risky portfolio. The second investment strategy is one in which the best estimate value is invested in bonds, and the buffer portfolio is invested in risky assets. Specifically, we consider the following two investment strategies:

- A *risky investment strategy* in which both the best estimate value BEL, and the buffer $c \cdot BEL$ are (re)invested in a portfolio that yields returns r_s^{bu}, i.e.,

  $$r_s^{be,(\tau)} = r_s^{bu}, \text{ for all } s = 0, \cdots, T, \text{ and } \tau = 1, \cdots, T.$$

  It then follows from Proposition 2 that the minimal required buffer percentage c is given by (6) with:

  $$L = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau}{\prod_{s=1}^{\tau} (1 + r_s^{bu})}, \tag{11}$$

- A *best estimate hedge strategy* in which the best estimate value BEL is cash flow matched, i.e.,

$$\prod_{s=1}^{\tau}\left(1+r_s^{be,(\tau)}\right)=\tfrac{1}{P^{(\tau)}},\quad \text{for all } s=0,\cdots,T, \text{ and } \tau=1,\cdots,T, \tag{12}$$

  and the buffer $c \cdot BEL$ is (re)invested in a portfolio that yields random returns r_s^{bu} in periods $s=0,\cdots,T$. It then follows from Proposition 2 that the minimal required buffer percentage c is given by (6) with:

$$L=BEL+\sum_{\tau=1}^{T}\frac{\widetilde{L}_\tau-\mathbb{E}\left[\widetilde{L}_\tau\right]}{\prod_{s=1}^{\tau}\left(1+r_s^{bu}\right)}. \tag{13}$$

  This strategy eliminates hedgeable investment risk, i.e., $L^{invest}=0$. Investment risk arises only due to uncertain deviations of $\widetilde{L}_\tau$ from its expectation $\mathbb{E}\left[\widetilde{L}_\tau\right]$. These (uncertain) deviations affect the value of the buffer portfolio, generating unhedgeable investment risk.

To investigate whether the impact of financial risk depends strongly on the type of liability, we consider two types of insured individuals, i.e., male insureds and female insureds aged $x=65$, and three types of liabilities, i.e., single life annuities (i.e., $\widetilde{L}_\tau=\widetilde{L}_{sl,\tau}(x,g)$), survivor annuities (i.e., $\widetilde{L}_\tau=\widetilde{L}_{surv,\tau}(\overline{x},\overline{g})$), and death benefits (i.e., $\widetilde{L}_\tau=\widetilde{L}_{db,\tau}(x,g)$). In case of survivor annuities, the partner of a male insured is a female aged $y=62$; the partner of a female insured is a male aged $y=68$. Regarding asset returns, we consider the case where the buffer (and thus also the best estimate value in case of the risky strategy) is invested in one-year bonds.

We use the models described in the Appendix to simulate future investment returns and survival probabilities. We then use these simulated distributions to determine the minimum required buffer percentage c to reduce the probability of ruin to 2.5%, using (6) and (7) with $\varepsilon=0.025$. Table 1 displays the minimal required buffer percentage c for the risky investment strategy (c^{risky}; second column), for the best estimate hedge strategy as defined in (12) (c^{BEh}; third column), and for the benchmark liability-only case with a deterministic return of $r=4\%$ (c^{LO}; last column). Because it is intuitively clear that the effect of longevity risk as well as of financial risk on the required buffer may depend substantially on the duration of the liabilities, the first column displays the duration of the expected liabilities, which is given by:

$$\text{Duration}=\frac{\sum_{\tau=1}^{T}\tau\cdot P^{(\tau)}\cdot\mathbb{E}\left[\widetilde{L}_\tau\right]}{\sum_{\tau=1}^{T}P^{(\tau)}\cdot\mathbb{E}\left[\widetilde{L}_\tau\right]}.$$

Table 1 shows that the effect of investment risk on the minimal required buffer percentage depends heavily on the type of liability. First, compared to the liability-only

Table 1: Minimal required buffer percentages

Product	Duration	c^{risky}	c^{BEh}	c^{LO}
Male single life annuity	8.2	26.5%	5.9%	4.9%
Female single life annuity	8.9	30.6%	7.4%	6.1%
Male survivor annuity	16.3	79.3%	23.5%	15.5%
Female survivor annuity	13.6	55.4%	38.2%	29.6%
Male death benefit	14.4	66.9%	10.6%	7.3%
Female death benefit	16.7	87.1%	16.5%	10.1%

approach (c^{LO}), the required buffer percentage under the risky investment strategy (c^{risky}) increases by a factor ranging from 2.5 (for female survivor annuities) to more than 9 (for female death benefits). These huge differences are partly due to the fact that under the naive investment strategy, there is a mismatch between the duration of the investments (one year) and the duration of the liabilities; this mismatch induces significant reinvestment risk. Second, compared to the risky strategy, the best estimate hedge strategy (c^{BEh}) leads to significant reductions in the required buffer percentages. However, even with this conservative investment strategy in which all hedgeable investment risk is eliminated, the required buffer percentages are still significantly larger than under the liability-only approach. The extent to which the required buffer percentage is underestimated with the liability-only approach depends nontrivially on the type of liability. It varies from 20% for male single life annuities to 63% for female death benefits.

5 Effect of unhedgeable investment risk

The results of the previous section suggest that there are nontrivial interactions between longevity risk and investment risk; the effect of investment risk on the required buffer depends strongly on the type of liability. In this section we quantify the effect of these interactions on the impact of longevity risk in portfolios of life insurance products. To focus on longevity risk, we consider a best estimate hedge strategy as defined in (12). This ensures that all hedgeable investment risk is eliminated (i.e., L^{invest} is deterministic), and investment risk arises only due to uncertain returns on the buffer portfolio, which cannot be fully hedged because of the longevity uncertainty in the stream of the payments $\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]$.

Compared to the benchmark liability-only approach, taking into account investment risk implies that (comparing (13) to (10)):

(i) the expected liabilities are valued at market value, i.e., using a term structure of interest rates instead of a flat discount rate (i.e., BEL instead of $\sum_{\tau=1}^{T} \mathbb{E}\left[\widetilde{L}_\tau\right] / (1+r)^\tau$)),

(ii) deviations from the expected value ($\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]$) are subject to uncertain returns (i.e., r_s^{bu} instead of r).

The first effect is deterministic, but the second is stochastic and can therefore nontrivially affect required buffer percentages. Specifically, uncertain buffer returns imply that L is affected by simultaneous deviations of the liabilities from their expected value (i.e., $\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right] \neq 0$), and of the returns from the flat rate (i.e., $r_s^{bu} \neq r$). The effect of uncertain deviations of the liabilities from their expected values is aggravated (weakened) when these deviations are accompanied by lower (higher) than expected returns on the buffer portfolio. Therefore, changes in the liability mix will not only affect the "pure longevity risk" component, i.e., the risk given known future investment returns, but also the interactions between longevity risk and investment risk. Ignoring these interactions may lead to inaccurate quantification of the hedge potential that arises from combining different types of liabilities (for example, the natural hedge potential of death benefits).

In this section we investigate the effect of interactions between unhedgeable financial risk and longevity risk in portfolios with single life annuities, survivor annuities, and death benefits. To do so, we determine the buffer percentage c from (6) and (13) for various asset and liability mixes, and compare the results to the buffers resulting from a liability-only approach in (10) with $r = 4\%$. To quantify the impact of unhedgeable financial risk, we consider four different investment strategies for the buffer portfolio: 100% one-year zero-coupon bonds; 67% one-year zero-coupon bonds, 33% equity; 33% one-year zero-coupon bonds, 67% equity; and 100% equity. With regard to the liability mix, we consider portfolios that differ in terms of *gender mix* (ratios of male insured rights over total insured rights for each product) and in terms of *product mix* (ratios of insured rights for the different life insurance products) for each gender. Gender mix nontrivially affects the required buffer percentage because male and female mortality trends are not perfectly correlated. Product mix nontrivially affects the required buffer percentage because survivor annuity payments and single life annuity payments are negatively correlated. Therefore, we consider two types of insured individuals, male insureds and female insureds aged 65, who each may hold insured rights ($\overline{\delta}_{i,p}$, see (2)) for three different types of liabilities: single life annuities ($p = sl$), survivor annuities ($p = surv$), and death benefits ($p = db$). The partner of a male insured (if present) is aged 62; the partner of a female insured (if present) is aged 68. It is verified easily that the minimum required buffer percentage c is then given by (6) and (13) with:[5]

$$\begin{aligned}\widetilde{L}_\tau = (1-\gamma) \cdot & \left[\widetilde{L}_{sl,\tau}(65, f) + w_f \cdot \widetilde{L}_{surv,\tau}(65, 68, f, m) + d_f \cdot \widetilde{L}_{db,\tau}(65, f)\right] \\ & + \gamma \cdot \left[\widetilde{L}_{sl,\tau}(65, m) + w_m \cdot \widetilde{L}_{surv,\tau}(65, 62, m, f) + d_m \cdot \widetilde{L}_{db,\tau}(65, m)\right], \end{aligned} \tag{14}$$

where $\widetilde{L}_{sl,\tau}(\cdot), \widetilde{L}_{surv,\tau}(\cdot)$, and $\widetilde{L}_{db,\tau}(\cdot)$ are as defined in (1), and where

- γ is the fraction of male single life annuities rights relative to the total single life annuities rights,

[5] Straightforward algebra shows that the aggregate liability payment in year τ in (2) is given by (14) multiplied by $\sum_{i \in \mathcal{I}} \overline{\delta}_{i,sl}$, the total insured rights for single life annuities. It follows immediately from Proposition 2 that the minimum required buffer percentage c is unaffected when all liability payments are divided by $\sum_{i \in \mathcal{I}} \overline{\delta}_{i,sl} > 0$.

- w_g for $g \in \{m, f\}$ is the ratio of survivor annuity rights for gender g over single life annuities rights for gender g, and,
- d_g for $g \in \{m, f\}$ is the ratio of death benefit rights for gender g over single life annuities rights for gender g.

In Subsection 5.1 we investigate interactions between longevity risk and investment risk in portfolios of single life and survivor annuities ($d_g = 0$). In Subsection 5.2 we quantify the effect of unhedgeable investment risk on the hedge potential from including death benefits ($d_g \neq 0$).

5.1 Interaction effects in annuity portfolios

In this section we consider portfolios of single life and survivor annuities, and quantify the effect of unhedgeable investment risk on: (i) the required buffer percentage for a given liability mix, and, (ii), the hedge potential that arises from the liability mix. Without death benefits, it follows from (14) that the effect of liability mix is fully characterized by the gender mix γ, and by the ratios w_m and w_f of insured rights for survivor annuities over insured rights for single life annuities for males and females, respectively.

Figure 1 displays the minimum required buffer percentage c as a function of gender mix and product mix in portfolios of single life and survivor annuities. To limit the number of parameters, we consider the case where the product mix is equal for both genders, i.e., $w_m = w_f = w$.

The left panels in Figure 1 display the minimum required buffer percentage c as a function of gender mix (i.e., γ), for three different product mixes:

- top panel: portfolios with only single life annuities, i.e., with $w = 0$;
- middle panel: portfolios with both single life and survivor annuities where the insured right for survivor annuities is 35% of the insured right for single life annuities, i.e., with $w = 0.35$,
- bottom panel: portfolios with both single life and survivor annuities where the insured right for survivor annuities is 70% of the insured right for single life annuities, i.e., with $w = 0.7$.

The right panels display the minimal required buffer percentage c as a function of product mix (i.e., w), for three different gender mixes:

- top panel: portfolios with only male insureds, i.e., with $\gamma = 1$;
- middle panel: portfolios with only female insureds, i.e., with $\gamma = 0$;
- bottom panel: portfolios with 50% male insured rights and 50% female insured rights, i.e., with $\gamma = 0.5$.

In each case we consider four different asset mixes for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$.
The figure shows that there are important interactions between longevity risk and investment risk. First, the effect of unhedgeable financial risk depends strongly on the liability mix. Second, the effect of liability mix depends nontrivially on the asset mix. Specifically, we observe the following.

Liability mix effects (i.e., effects of gender mix and product mix). For any given asset mix, both gender mix and product mix can significantly affect the required buffer percentage, because different types of liabilities have different sensitivities to changes in mortality rates. Specifically:

- For each product mix w, portfolios with exclusively male liabilities ($\gamma = 1$) require lower buffer percentages than portfolios with exclusively female liabilities ($\gamma = 0$). However, in portfolios with only single life annuities (i.e., $w = 0$, top panel), risk is minimized with a mixture of female and male liabilities. This occurs because male and female liabilities are imperfectly correlated, so that there is some diversification effect from combining these liabilities.[6] Including survivor annuities (middle and lower panels) increases the correlation between male and female liabilities and thus reduces the diversification effect. As a consequence, mixing male and female liabilities does not yield significant risk reduction in these cases.

- Combining single life with survivor annuities (right panels) may either increase or decrease the required buffer percentage. This occurs because there are two opposite effects. On the one hand survivor annuities can reduce required buffers because survivor annuity payments are negatively correlated with single life annuity payments.[7] On the other hand, survivor annuity payments are more affected by the uncertainty in future survival probabilities because they have a longer duration (see Table 1). For portfolios with predominantly female rights (middle panel),

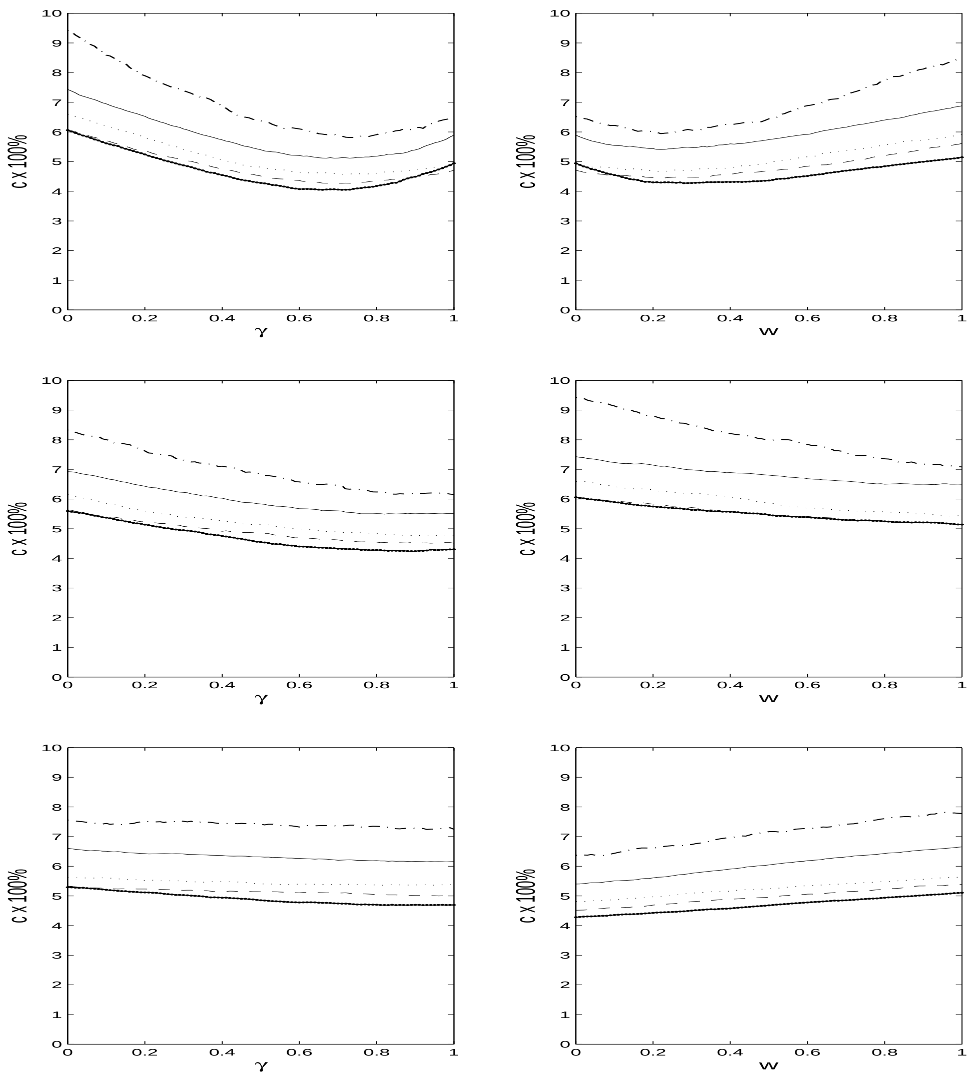
[6] The underlying intuition in both cases is as follows. In each case, the random variable of interest can be written as a convex combination $L = \alpha L_1 + (1-\alpha)L_2$ of two present value variables L_1 and L_2. It holds that

$$\mathrm{Var}\{L\} = \alpha^2 \cdot \mathrm{Var}\{L_1\} + (1-\alpha)^2 \cdot \mathrm{Var}\{L_2\} + 2\alpha(1-\alpha) \cdot \mathrm{Cov}\{L_1, L_2\}.$$

Thus, the variance is minimized with an unbalanced portfolio that puts all weight on the liability with the lowest variance if $\mathrm{Cov}\{L_1, L_2\} > \min\{\mathrm{Var}\{L_1\}, \mathrm{Var}\{L_2\}\}$, but the variance is minimized at an internal $\alpha \in (0,1)$ if $\mathrm{Cov}\{L_1, L_2\} < \min\{\mathrm{Var}\{L_1\}, \mathrm{Var}\{L_2\}\}$. Thus, shifting more weight to the higher risk liability is beneficial if the covariance is sufficiently low.

[7] This occurs for two reasons. First, an increase in life expectancy of the insured delays the onset of payments of the survivor annuity, so that they are more heavily discounted. Second, the difference between male and female life expectancies decreases, so that the duration of survivor annuity payments decreases.

Figure 1: Required buffer percentage for portfolios of single life and survivor annuities.



The left panels display the required buffer percentage as a function of γ (gender mix). The upper panel represents a fund with only single life annuities ($w = 0$), the middle panel one with single life annuities and survivor annuities with $w = 0.35$, and the bottom panel one with single life annuities and survivor annuities with $w = 0.7$. The right panels display the required buffer percentage as a function of w (product mix). The upper panel represents a fund with only males ($\gamma = 1$), the middle one a fund with only females ($\gamma = 0$), and the bottom one a fund with 50% male rights and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves: 100% one-year zero-coupon bonds; dashed curves: 67% one-year zero-coupon bonds and 33% equity; dotted curves: 33% one-year zero-coupon bonds and 67% equity; dashed-dotted curves: 100% equity. The bold solid curves correspond to the liability-only approach.

the former effect dominates; for portfolios with half male and half female rights (bottom panel), the latter effect dominates.

- *Accurate quantification of liability mix effects requires specification of the asset mix.* For example, the middle right panel shows that the potential risk reduction from combining single life annuities with survivor annuities is significantly larger when the buffer portfolio is fully invested in equity than for the other asset mixes that we consider.

Impact of unhedgeable investment risk. We observe two effects:

- For every liability mix, the required buffer percentage is significantly affected by *unhedgeable* investment risk. An increase in equity leads to a higher expected return, but it also yields a higher probability that the realized return is lower than expected. The impact of unhedgeable financial risk is minimized when 1/3 of the buffer is invested in equity.

- *Accurate quantification of the effect of unhedgeable financial risk requires specification of the liability mix.* For example, for portfolios with predominantly female rights, unhedgeable investment risk affects the required buffer more strongly when the fraction of survivor annuity rights is high. The opposite holds for portfolios with half male and half female rights. Although we are not interested in the best equity portfolio, we do observe that the reserve requirement is lower if the insurer invests some of his buffer portfolio in equities. This is due to the use of a particular quantile of L in the determination of the reserve requirements and the equity risk premium. Hence, the investment strategy which reduces the ruin probability will depend on the quantile used in the ruin probability.

These results suggest that separately quantifying investment risk and longevity risk, as is proposed by the Dutch regulator, likely leads to inaccurate quantifications of the impact of longevity risk. Second, ignoring the impact of unhedgeable financial risk may lead to inaccurate quantification of the risk reduction that arises from combining different types of longevity-linked liabilities.

5.2 Natural hedge potential of death benefits

In this subsection we investigate the effect of unhedgeable investment risk on the natural hedge potential from death benefits in portfolios of life annuities. To do so, we determine the minimum required buffer percentage c as a function of both asset and liability mix. We then compare the results to the benchmark case considered in the existing literature (for example, Wang et al. 2010, and Tsai et al. 2010), where: (i) longevity risk is quantified with a liability-only approach (i.e., ignoring unhedgeable financial risk), and, (ii) longevity-linked liabilities other than single life annuities and death benefits (such as, for example, survivor annuities) are ignored.

The following proposition shows that in the benchmark case longevity risk in single life annuities can be fully hedged by death benefits.

Proposition 3 *Let* $r_s^{be,(\tau)} = r_s^{bu} = r$, *for* $s = 0, \cdots, T$, *and* $\tau = 1, \cdots, T$, *and for some (non-random)* $r > 0$. *Then, for portfolios of single life annuities and death benefits with*

$$d_m = d_f = \frac{1+r}{r}, \tag{15}$$

it holds that the terminal asset value A_T *is unaffected by longevity risk, and is nonnegative for any* $c \geqslant 0$.

Proof. It follows from the proof of Proposition 2 that $A_T = [(1+c) \cdot BEL - L] \cdot (1+r)^T$, with $L = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau}{(1+r)^\tau}$. Moreover, it follows from (14), (15), and the fact that the portfolio does not contain survivor annuities (i.e., $w_m = w_f = 0$) that:

$$\widetilde{L}_\tau = (1-\gamma) \left[\widetilde{L}_{sl,\tau}(65, m) + \delta \widetilde{L}_{db,\tau}(65, m) \right] + \gamma \cdot \left[\widetilde{L}_{sl,\tau}(65, f) + \delta \widetilde{L}_{db,\tau}(65, f) \right],$$

where $\delta = \frac{1+r}{r}$. Therefore,

$$L = (1-\gamma) \cdot \overline{L}(m) + \gamma \cdot \overline{L}(f),$$

where for $g \in \{m, f\}$, it holds that:

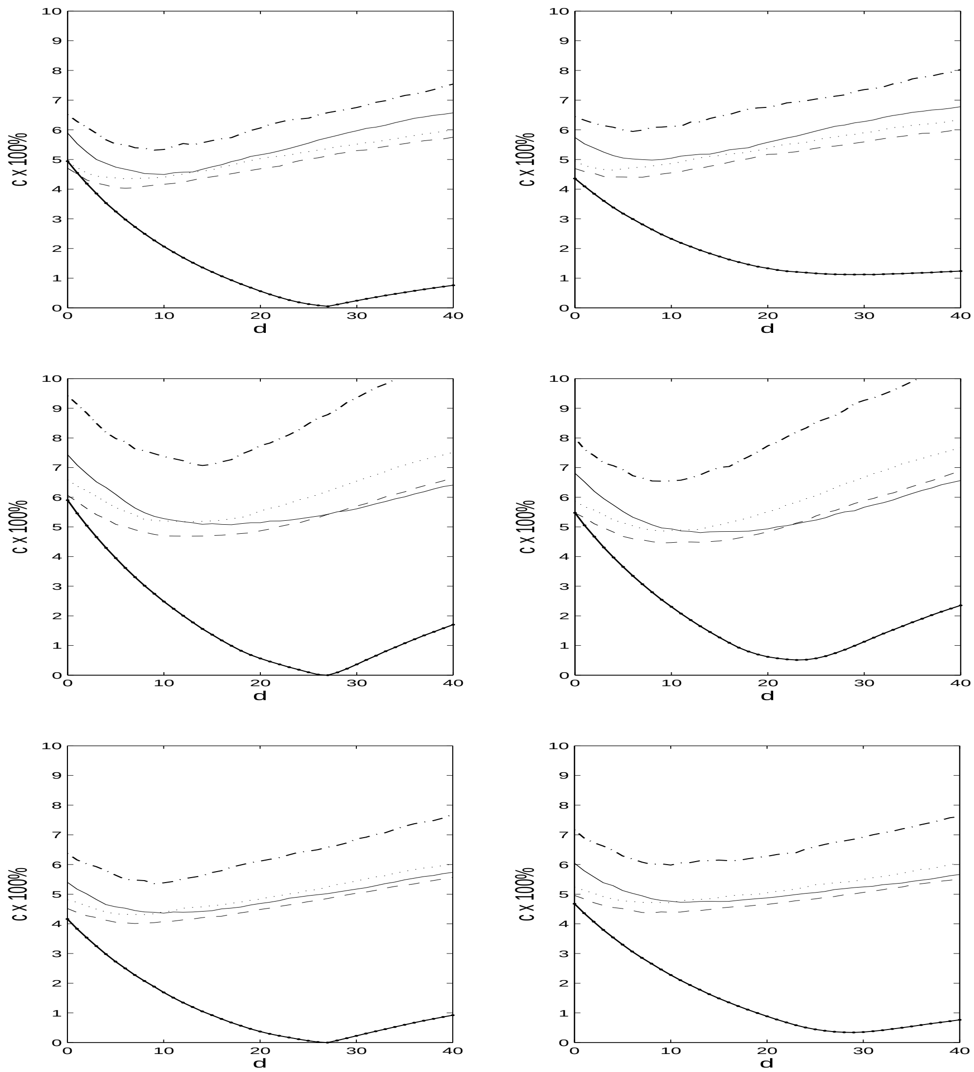
$$\begin{aligned}
\overline{L}(g) &:= \sum_{\tau=1}^{T} \left[\frac{{}_\tau p_{65}^{(g)} + \delta \cdot \left({}_{\tau-1}p_{65}^{(g)} - {}_\tau p_{65}^{(g)} \right)}{(1+r)^\tau} \right] \\
&= \sum_{\tau=1}^{T-1} \left(\frac{\left(1 - \delta + \frac{\delta}{1+r}\right)}{(1+r)^\tau} \right) {}_\tau p_{65}^{(g)} - (1-\delta) \cdot \frac{{}_T p_{65}^{(g)}}{(1+r)^T} + \delta \cdot {}_0 p_{65}^{(g)} \\
&= \delta.
\end{aligned}$$

The last equality follows from $\delta = \frac{1+r}{r}$, ${}_0 p_x^{(g)} = 1$, and ${}_T p_x^{(g)} = 0$. Therefore, $BEL = L = \delta$, and the terminal asset value is given by $A_T = c \cdot \delta \cdot (1+r)^T$, which is deterministic and nonnegative for any $c \geqslant 0$. ■

Proposition 3 shows that in the benchmark liability-only case that is typically examined in the literature, longevity risk in single life annuities can be fully hedged with death benefits. In the remainder of this section we show that unhedgeable investment risk can significantly reduce the hedge potential from death benefits in portfolios of life annuities.

Figure 2 displays the effect of death benefits on the required buffer percentage c for portfolios of single life and survivor annuities, and for given investment strategies. It considers a case where product mix is identical for both genders, i.e., $w = w_m = w_f$ and $d = d_m = d_f$. The left panels in Figure 2 display the minimum required buffer percentage c as a function of d, the ratio of the insured rights for death benefits over single life annuities, in portfolios with only single life annuities, i.e., with $w = 0$. The right panels display the minimum required buffer as a function of d, for portfolios of

Figure 2: Required buffer percentage for portfolios of single life and survivor annuities and death benefits



The graphs present the required buffer percentage as a function of d (ratio of death benefits) in portfolios of life insurance products. The left panels correspond to portfolios with only single life annuities, the right panels correspond to portfolios with single life annuities and survivor annuities with $w = 0.5$. The upper panels correspond to a fund with only males ($\gamma = 1$), the middle panels correspond to a fund with only females ($\gamma = 0$), and the lower panels correspond to a fund with 50% male and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves: 100% one-year zero-coupon bonds; dashed curves: 67% one-year zero-coupon bonds and 33% equity; dotted curves: 33% one-year zero-coupon bonds and 67% equity; dashed-dotted curves: 100% equity. The bold solid curves correspond to the liability-only approach.

single life annuities and survivor annuities with $w = 0.5$. The top panel corresponds to males (i.e., $\gamma = 1$), the middle panel to females (i.e., $\gamma = 0$), and the bottom panel to portfolios with 50% male rights and 50% female rights ($\gamma = 0.5$). In each case we consider four different investment strategies for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$.

In line with results reported in, for example, Wang et al. (2010) and Tsai et al. (2010), we find that death benefits can significantly reduce the required buffer percentages in portfolios of life annuities. However, we find that the risk reduction can be significantly affected by unhedgeable investment risk. Specifically,

- *Ignoring the effect of unhedgeable financial risk leads to significant overestimation of the hedge potential.* Whereas with a liability-only approach to quantify longevity risk (bold solid lines)), the minimum required buffer percentage under the optimal hedge is zero (see Proposition 3), it varies from around 4% to more than 9%, depending on the asset mix when we take into account the effect of unhedgeable financial risk.

- *Accurate quantification of the hedge potential requires specification of both the existing liability mix and the asset mix.* While the hedge potential from death benefits is generally different for female liabilities (middle row) and for male liabilities (upper row), the difference is much more significant for the risky investment strategy (100% stocks) than for the other strategies that we consider. Also, comparing the left and right panels shows that, depending on the investment strategy, the hedge potential from death benefits may, but need not, decrease significantly when the portfolio also contains survivor annuities.

6 Hedge effects of survivor swaps

In this section we investigate the hedge potential from investing in survivor swaps. Dowd, Blake, Cairns, and Dawson (2006) discuss the mechanism and use of survivor swaps as an instruments for managing, hedging, and trading mortality-dependent risks. A survivor swap can be defined as a swap involving at least one future (stochastic) mortality-dependent payment. Given this definition, the most basic case of a survivor swap is an exchange of a single fixed payment for a single mortality-dependent payment. More precisely, let ref denote a reference population. Then, at time $t = 0$, party A agrees with party B that A pays to B at time $\tau > 0$ the amount $K(\tau, ref)$ known at time 0, and B pays to A at the amount $S(\tau, ref)$ which depends on realized mortality until date τ in the reference population, and is thus currently stochastic. The payments made in this agreement are that party B pays A the amount $S(\tau, ref) - K(\tau, ref)$, if $K(\tau, ref) < S(\tau, ref)$, and party A pays B the amount $K(\tau, ref) - S(\tau, ref)$, if

$K(\tau, ref) > S(\tau, ref)$. Hence, the payment from party B to party A equals:

$$SS(\tau, ref) = S(\tau, ref) - K(\tau, ref), \tag{16}$$

where $S(\tau, ref)$ is the random mortality-dependent payment and $K(\tau, ref)$ is the fixed payment.

The survivor swaps we consider in this paper is one where the floating leg $S(\tau, ref)$ is the realized survival rate for the 65-year old cohort in the underlying reference population, i.e., $S(\tau, ref) = {}_{\tau}p_{65}^{(ref)}$. Typically, the fixed leg $K(\tau, ref)$ is determined such that there is no cash transfer at the time of the issue. However, there is currently no publicly traded market in longevity-linked products and hence we do not observe the market price of longevity risk.[8] To avoid making assumptions regarding the price of the swap, we set $K(\tau, ref)$ equal to the current expected value of $S(\tau, ref)$. Then, the payment in period τ of the survivor swap is given by:

$$SS(\tau, ref) = {}_{\tau}p_{65}^{(ref)} - \mathbb{E}\left[{}_{\tau}p_{65}^{(ref)} \right], \tag{17}$$

and there is a cash transfer at the time of issue which equals the (over the counter) price of the survivor swap. We consider a vanilla survivor swap $VSS(ref)$ that consists of a portfolio of survivor swaps with maturities $\tau = 1, \cdots, T$.

It now remains to specify a reference population. A natural reference group from the point of view of the insurer (party B) is the population of the insurer. However, the insurer may then have more information about the population than the seller (party A) of the survivor swap. Since the insurer may have this private information, buying a survivor swap can be interpreted as a signal that the reference group has low mortality probability, and hence the price of the survivor swaps would be high, see Biffis and Blake (2010). Another problem with the natural reference group from the point of view of the insurer is the tradeability of the survivor swaps; when every life insurer has a different reference group, many different survivor swaps are needed. This would lead to much higher transaction costs for the seller of the survivor swap, since he has to put extra efforts in estimating the size of longevity risk in the survivor swaps (Blake, Cairns, Dowd, and McMinn, 2006). In order to eliminate the private information problem and to increase the tradeability, the whole population of a country is often chosen as reference group, since the information on this reference group is the same for the issuer and buyer of the swap. An example is the first longevity bond[9] issued by European Investment Bank/Bank National de Paris announced in November 2004, which had as reference population the English and Welsh males at age 65 in 2003.

In this section we investigate the effect on solvency capital requirement of vanilla survivor swaps with reference population the Dutch aged 65 in 2006. We use two different

[8] For an excellent discussion on issues related to pricing of longevity-linked assets or liabilities, see Bauer, Boerger, and Russ (2010).

[9] The longevity bond was issued by the EIB and managed by BNP Paribas. The face value was £540 million, and was primarily intended for purchase by U.K. pension funds. The survivor swap involved yearly coupon payments that were tied to an initial annuity payment of £50 million indexed to the survivor rates of English and Welsh males aged 65 years in 2003. The longevity bond was withdrawn prior to issue (Mitchell, Piggott, Sherris, and Yow, 2006).

vanilla survivor swaps, one with reference group the whole male population aged 65 (i.e., $ref = m$), and another with reference group the whole female population aged 65 (i.e., where $ref = f$). Let s_m (s_f) be the number of vanilla survivor swaps with reference population males (females). Then, the liability payment in year τ, net of payoff from longevity swaps, is given by:

$$\widetilde{L}_\tau = \widetilde{L}_\tau - s_m \cdot SS(\tau, m) - s_f \cdot SS(\tau, f). \tag{18}$$

Let $V_{VSS}(s_m, s_f)$ denote the date-0 (over the counter) price of the vanilla survivor swap. Then, it follows from Proposition 2 and (6) that the minimal required initial asset value in order to limit the probability of ruin to ε is given by:

$$A_0 = BEL + \overline{c}(s_m, s_f) \cdot BEL + V_{VSS}(s_m, s_f),$$

where

$$\overline{c}(s_m, s_f) = \frac{Q_{1-\varepsilon}(L(s_m, s_f))}{BEL} - 1,$$

with

$$L(s_m, s_f) = BEL + \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right] - s_m \cdot SS(\tau, m) - s_f \cdot SS(\tau, f)}{\prod_{s=1}^{\tau}\left(1 + r_s^{(bu)}\right)}.$$

Note that $\overline{c}(s_m, s_f) \cdot BEL$ now represents the required buffer in excess of the best estimate of the liabilities *and* the price of the vanilla survivor swap. Note also that a change in the portfolio of swaps not only affects the required buffer, but also the price of the portfolio, $V_{VSS}(s_m, s_f)$. Because we choose not to make assumptions regarding the price of the survivor swaps, we cannot determine the "optimal" fraction of survivor swaps, i.e., the fraction that minimizes the required asset value A_0. However, for any given portfolio of survivor swaps (s_m, s_f), we can determine the relative attractiveness of the vanilla survivor swaps for different liability mixes and asset mixes. Moreover, for any given asset mix, we can determine the maximum price of the portfolio of survivor swaps under which a lower asset value, i.e., A_0, is sufficient to cover all future liabilities with probability at least $1 - \varepsilon$ with survivor swaps than without survivor swaps. This maximum price is given by:

$$V_{VSS}^{\max}(s_m, s_f) = [\overline{c}(0, 0) - \overline{c}(s_m, s_f)] \cdot BEL. \tag{19}$$

In Subsection 6.1 we investigate how the hedge effect of survivor swaps depends on the liability and asset mix in a benchmark case without basis risk, i.e., in a setting in which the survival rates of the insured population are identical to those of the reference population. In Subsection 6.2 we investigate how these effects are affected by basis risk that arises from differences in the mortality experience in the reference group of the survivor swap and the population of the insurer. In order to focus on the effect of unhedgeable financial risk on the reduction in longevity risk, we consider the investment strategies defined in Section 5.

6.1 Vanilla survivor swaps and product mix

We now investigate the potential hedge effects of survivor swaps for portfolios of life insurance products with different product and gender mixes, and different investment strategies. We also determine the maximum price under which investing in survivor swaps leads to lower capital requirements in each case. In order to reduce the number of parameters, we let $s_m = \gamma \cdot s$, and $s_f = (1 - \gamma) \cdot s$. It then follows immediately from (1), (14), and (17), and from the fact that there is no basis risk, that longevity risk in a fraction s of the single life annuity rights for both males and females is fully hedged.[10]

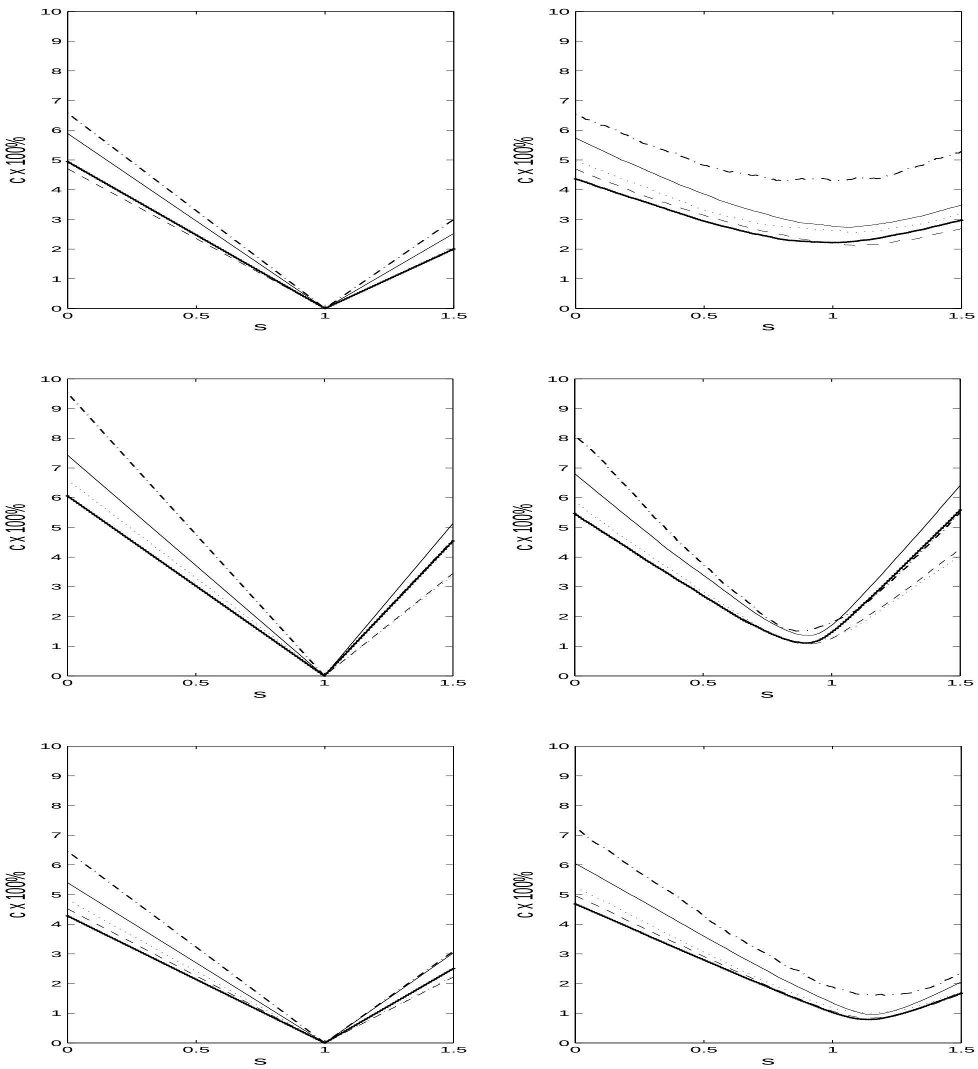
Figures 3 and 4 display the minimum required buffer, and the maximum price as defined in (19), respectively, as a function of s for different asset and liability mixes, i.e., in portfolios of single life annuities (left panels), and in portfolios of single life and survivor annuities with $w = 0.5$ (right panels), for males (top panel), females (middle panel), and $\gamma = 0.5$ (bottom panel). In each case we consider four different investment strategies for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$.

From Figure 3 we observe that survivor swaps can lead to significant reductions in the required solvency buffer. However, the effect depends strongly on both liability mix and asset mix. Because there is no basis risk, longevity risk in portfolios with only single life annuities (left panels) can be fully eliminated by survivor swaps (with $s = 1$). For portfolios with also survivor annuities, the maximal risk reduction is attained by buying either strictly more or strictly less survivor swaps than the face value of the single life annuities, i.e., with $s < 1$ or $s > 1$. This occurs because survivor annuities to some extent can provide a natural hedge for single life annuities, but on the other hand are also affected more strongly by longevity risk because they have longer duration. The first effect dominates for portfolios with only female insureds, whereas the second effect dominates for portfolios with half male and half female insured rights. Comparing the top left and right panels shows that for male insureds, the hedge potential of survivor swaps reduces dramatically when the portfolio also contains survivor annuities. Comparing the right top and middle panels shows that the hedge potential of survivor swaps in portfolios with both single life annuities and survivor annuities is sufficiently weaker in portfolios with predominantly male insureds.

With regard to the interaction between longevity risk and investment risk, we observe that ignoring the effect of unhedgeable financial risk may lead to both over- or underestimation of the hedge potential of survivor swaps, depending on the investment strategy.

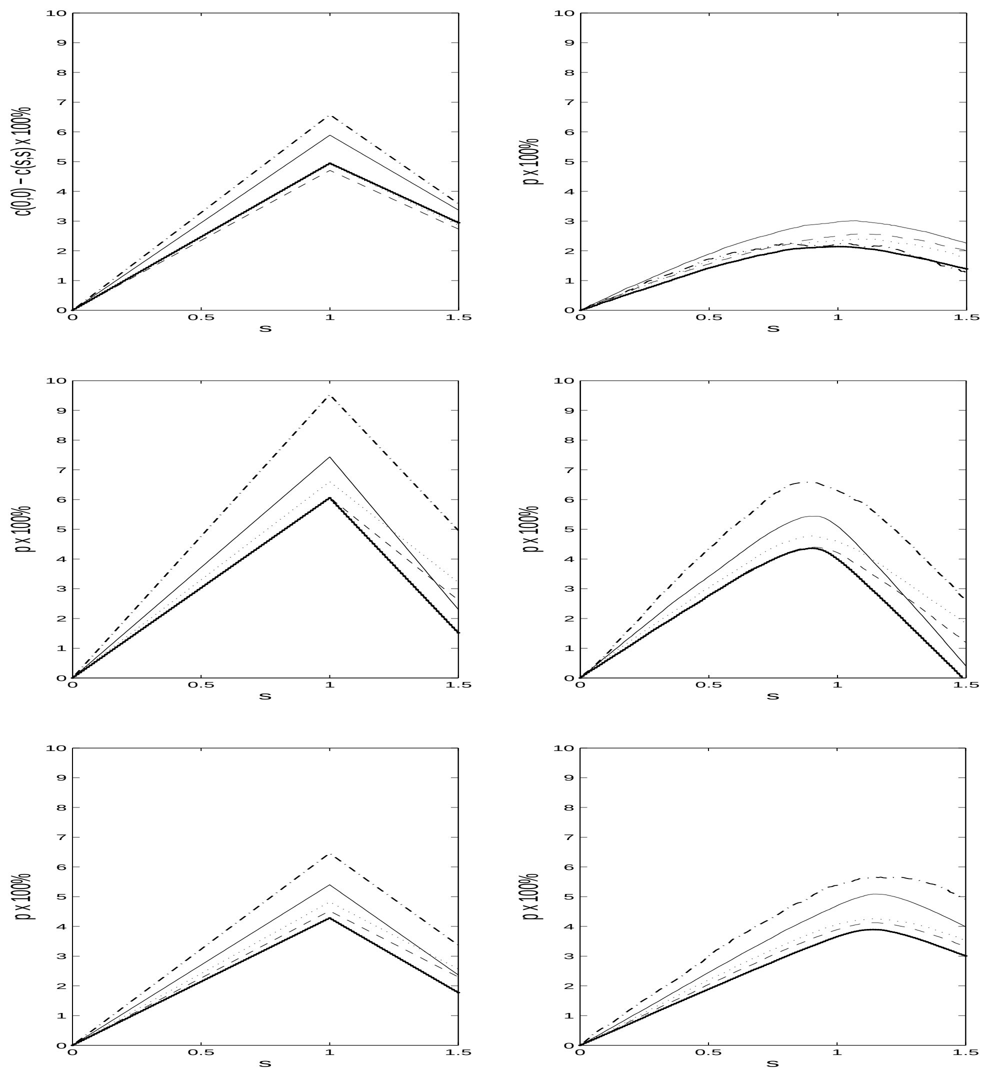
[10] It follows from (1), (14), and (17) that when $s_m = \gamma \cdot s$, and $s_f = (1 - \gamma) \cdot s$, a fraction s of the single life annuity payments in year τ, $\gamma \cdot \widetilde{L}_{sl,\tau}(65, m) + (1 - \gamma) \cdot \widetilde{L}_{sl,\tau}(65, f)$, is effectively replaced by its expected value, $\gamma \cdot \mathbb{E}\left[\widetilde{L}_{sl,\tau}(65, m)\right] + (1 - \gamma) \cdot \mathbb{E}\left[\widetilde{L}_{sl,\tau}(65, f)\right]$.

Figure 3: Required buffer percentage for portfolios of annuities and vanilla survivor swaps without basis risk



The figure displays the required buffer percentage, $\overline{c}(s, s)$ as a function of s, for a fund with only single life annuities ($w = 0$) (left panels) and for a fund with single life and survivor annuities with $w = 0.5$ (right panels). The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves: 100% one-year zero-coupon bonds; dashed curves: 67% one-year zero-coupon bonds and 33% equity; dotted curves: 33% one-year zero-coupon bonds and 67% equity; dashed-dotted curves: 100% equity. The bold solid curves correspond to the liability-only approach.

Figure 4: Maximum price of vanilla survivor swaps without basis risk



The figure displays the maximum price, $p = \overline{c}(0,0) - \overline{c}(s,s)$, as a function of s, for a fund with only single life annuities ($w = 0$) (left panels) and for a fund with single life and survivor annuities with $w = 0.5$ (right panels). The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves: 100% one-year zero-coupon bonds; dashed curves: 67% one-year zero-coupon bonds and 33% equity; dotted curves: 33% one-year zero-coupon bonds and 67% equity; dashed-dotted curves: 100% equity. The bold solid curves correspond to the liability-only approach.

6.2 Vanilla survivor swaps with basis risk

In the previous section we showed that vanilla survivor swaps can substantially reduce reserve requirements in portfolios of life insurance products. For portfolios consisting of only single life annuities, they can even eliminate all longevity risk. However, in these calculations we have ignored the impact of *basis risk*, i.e., the mortality rates of the individuals in the reference group for the vanilla survivor swap are assumed to be equal to the mortality rates of the insured population. There is ample empirical evidence, however, that survival rates of insured populations can differ significantly from those of the general population. As discussed above, there are important hurdles to create a liquid market in survivor swaps without basis risk, because that would require fine tuning the survivor swap to the population of the insurer.

Dowd, Cairns, and Blake (2006) investigate the hedge effectiveness of a longevity bond with basis risk that arises because the longevity bond is based on the mortality experience of the cohort of 60-year-old males, and the insured population consists of 65-year-old males. They find that the hedge potential is not significantly affected by this basis risk. In this paper we quantify the effect of basis risk that arises due to differences in survival probabilities for insured individuals compared to those of the whole population. It is well-documented that, due to adverse selection, survival probabilities of insured individuals are generally different from those of the whole population (see, for example, Brouhns et al. 2002, and Denuit, 2008). Following Brouhns et al. (2002) and Denuit (2008), we will distinguish basis risk in case of group insureds, which is relevant in particular for pension funds, and basis risk in case of individual insureds, which is particularly relevant for insurance companies.

We use the Cox-type relational model to model mortality rates of the insured population. Specifically, the relationship between the gender-specific mortality rates of insured group h relative to the gender-specific mortality rates for the total (country-wide) population group g, is modeled as (see Brouhns et al. 2002, and Denuit 2008):

$$\log(\mu_{x,t}^{(h)}) = \alpha^{(h)} + \beta^{(h)} \cdot \log(\mu_{x,t}^{(g)}), \tag{20}$$

where $\alpha^{(h)}$ denotes the time- and age-independent difference in mortality rates between group g and h, and $\beta^{(h)}$ denotes the speed of the future mortality improvements of the group h relative to the general population with gender g. We use the estimated parameter reported in Denuit (2008), which are given in Table 2 for group insureds and individual insureds, and for both males and females.[11]

The negative sign of $\alpha^{(h)}$ indicates that the forces of mortality of group and individual insureds are lower than the general population. A larger negative value of $\alpha^{(h)}$ indicates that the difference in the forces of mortality between group h and the general population is larger. The value of $\beta^{(h)}$ smaller than one, in combination with a negative value of

[11] Notice that $\beta^{(h)} < 1$, which implies that the speed of the future mortality improvements in the insured population is smaller than the corresponding speed for the general population. This occurs because the adverse selection observed in the Belgian individual life market is so strong that the future improvements for the insured population are weaker than for the general population.

	$h=(m,group)$	$h=(f,group)$	$h=(m,individual)$	$h=(f,individual)$
$\alpha^{(h)}$	-0.71755	-0.577829	-1.54351	-1.024695
$\beta^{(h)}$	0.79180	0.843850	0.81849	0.906784

Table 2: Parameters estimates of the Cox relational model. Source: Denuit (2008).

$\alpha^{(h)}$, implies that the difference in the forces of mortality between group h and the general population are smaller at old ages than at young ages.

As before, we let $s_m = \gamma \cdot s$, and $s_f = (1-\gamma) \cdot s$, and we again consider the case where the reference population of the vanilla survivor swap is the general population of males and females, respectively, but we now let mortality rates of the insured persons be given by (20).[12]

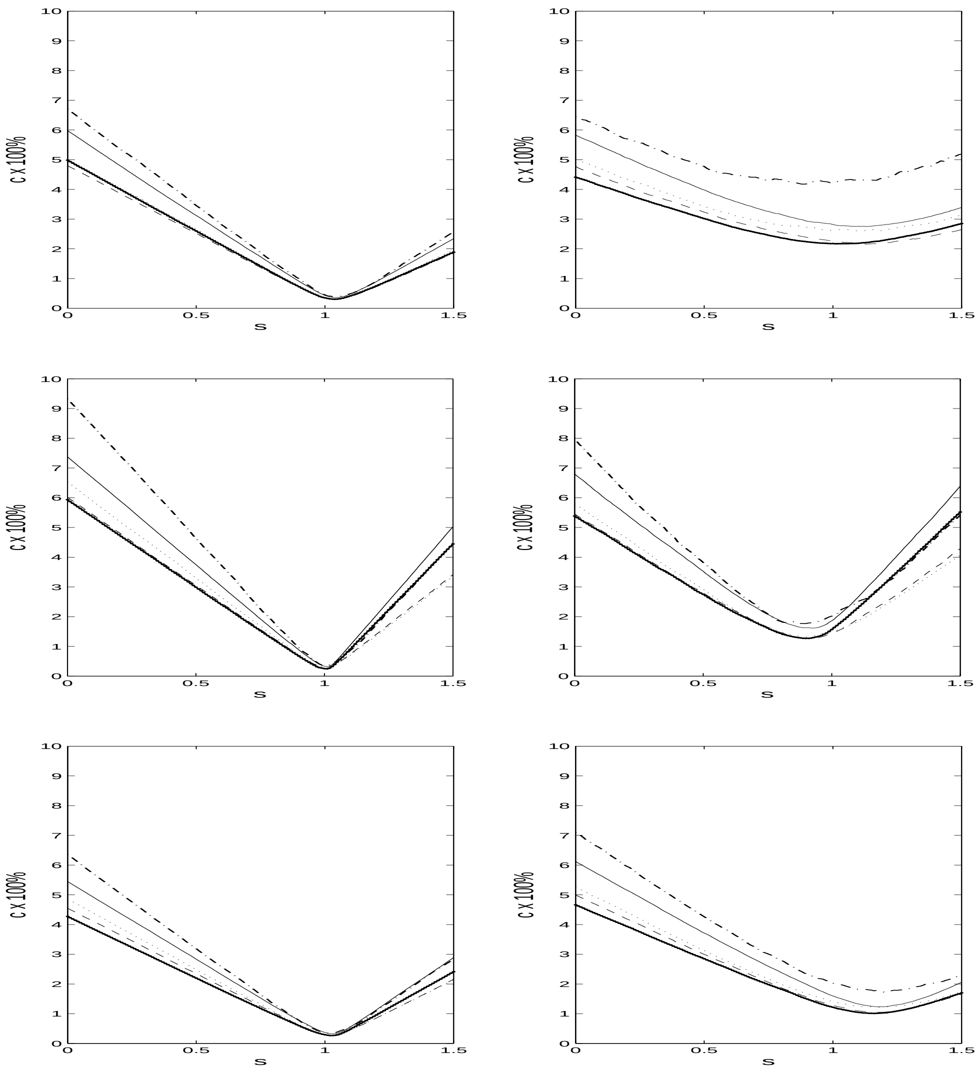
In Figures 5 and 6 we display the minimum required buffer as a function of s, for different asset and liability mixes, i.e., in portfolios of single life annuities (left panels), and in portfolios of single life and survivor annuities with $w = 0.5$ (right panels), for males (top panel), females (middle panel), and $\gamma = 0.5$ (bottom panel). In each case we consider four different investment strategies for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$. Figure 5 corresponds to group insureds, and Figure 6 corresponds to individual insureds. We assume that if an insured person belongs to group (individual) insureds, the same holds for the insured's partner.[13]

Comparing Figures 5 and 6 shows that the hedge effectiveness of survival swaps with basis risk is significantly smaller than without basis risk, especially for portfolios with both single life and survivor annuities.

[12]In our model, mortality probabilities of the general population and of the population of the insurer are perfectly correlated. The low hedge effectiveness of the survival swaps is caused by the fact that survival probabilities are non-linear transformations of the logarithm of the forces of mortality. The effect is stronger for portfolios with both single life and survivor annuities because the dependency between males and females.

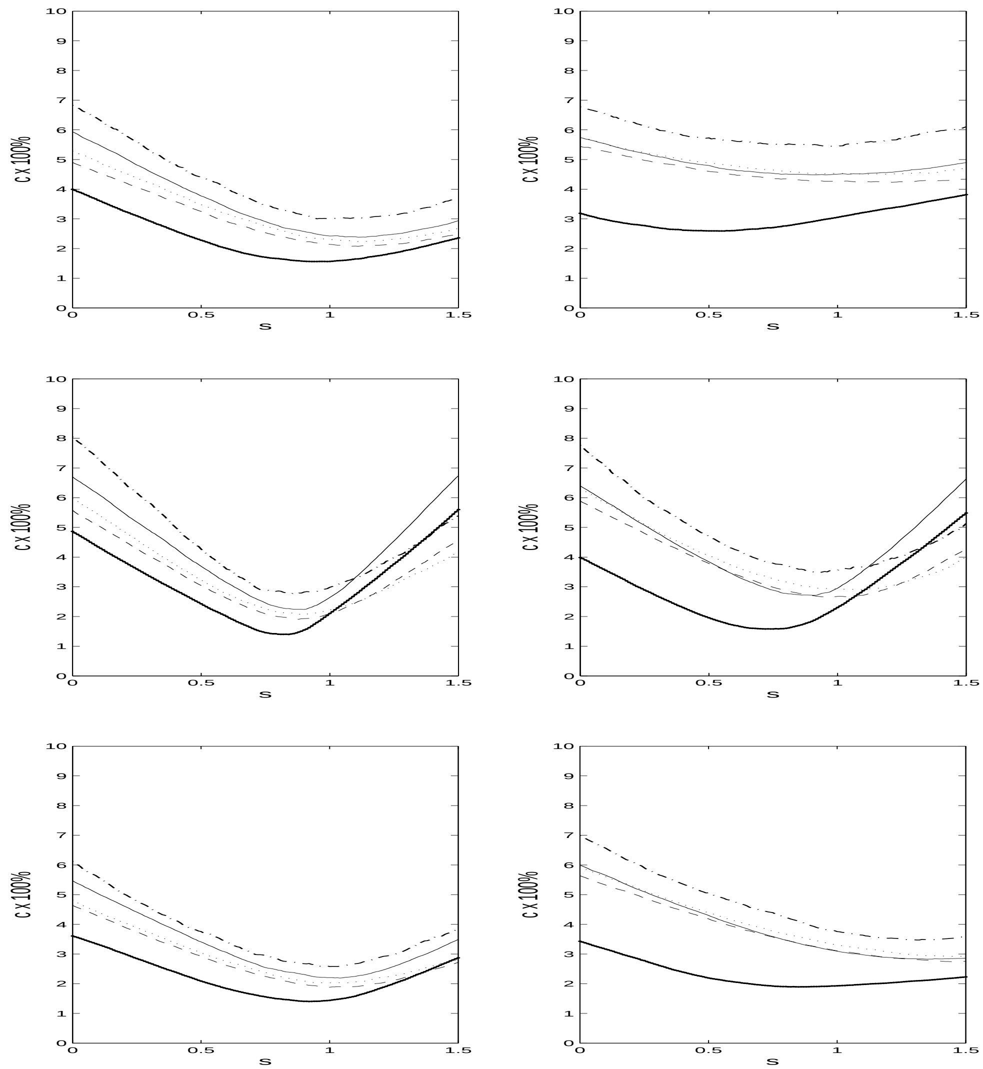
[13]Typically, the mortality probabilities of spouses are similar, due to, for instance, the living conditions.

Figure 5: Required buffer percentage for portfolios of annuities and vanilla survivor swaps with basis risk: group insureds



The figure displays the required buffer percentage as a function of s, for a fund with only single life annuities ($w = 0$) (left panels) and for a fund with single life and survivor annuities with $w = 0.5$ (right panels). The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves: 100% one-year zero-coupon bonds; dashed curves: 67% one-year zero-coupon bonds and 33% equity; dotted curves: 33% one-year zero-coupon bonds and 67% equity; dashed-dotted curves: 100% equity. The bold solid curves correspond to the liability-only approach.

Figure 6: Required buffer percentage for portfolios of annuities and vanilla survivor swaps with basis risk: individual insureds



The figure displays the required buffer percentage as a function of s, for a fund with only single life annuities ($w = 0$) (left panels) and for a fund with single life and survivor annuities with $w = 0.5$ (right panels). The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves: 100% one-year zero-coupon bonds; dashed curves: 67% one-year zero-coupon bonds and 33% equity; dotted curves: 33% one-year zero-coupon bonds and 67% equity; dashed-dotted curves: 100% equity. The bold solid curves correspond to the liability-only approach.

7 Conclusions

This paper quantifies the effect of longevity risk of portfolios of life insurance products, taking into account that longevity risk induces unhedgeable financial risk. We find that unhedgeable financial risk induces non-trivial interactions between asset mix and liability mix. These interactions affect the impact of longevity risk for any given type of liability, as well as the potential effects of combining different types of liabilities and/or investing in longevity-linked assets.

Our results suggest that analyzing the joint effect of liability mix and asset mix on the overall risk is important for two reasons. First, taking into account interactions between financial risk and longevity risk may lead to more accurate solvency measures. Separating investment risk and longevity risk, as is often proposed by regulators, unavoidably leads to inaccurate quantifications of the impact of longevity risk. Second, ignoring the impact of unhedgeable financial risk may lead to inaccurate quantification of the risk reduction that arises from combining different types of longevity-linked assets and liabilities. Specifically, insurers may be able to reduce their sensitivity to longevity risk by redistributing their risk. Our results indicate that the extent to which insurers may benefit from such mutual reinsurance depends not only on their liability portfolios, but also on their investment strategies. Finally, our results indicate that the hedge potential from investing in longevity-linked asset such as survivor swaps depends nontrivially on both the asset mix and the liability mix.

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A The distribution of the financial returns

In this section we briefly describe the quantification of the financial risk. Financial risk might arise due to investing in (default-free) zero-coupon bonds with different times to maturity or in an equity stock index. The bonds are described by the Vasicek-model, while the stock index is modeled by a Geometric Brownian Motion with time-varying drift. We allow for correlation between the bonds and the stock index.

In case of the Vasicek-model the instantaneous spot rate, r_t, evolves as an Ornstein-Uhlenbeck process with constant coefficients:

$$dr_t = (a - br_t)\, dt + \sigma dZ_t^1, \tag{21}$$

where a, b, and σ are model parameters, and Z_t^1 is a standard Brownian Motion. The stock index, S_t, follows a Geometric Brownian Motion with time-varying drift:

$$dS_t = \mu_t S_t dt + \sigma_S S_t dZ_t^2, \quad \mu_t = r_t + \lambda_S \sigma_S, \tag{22}$$

where λ_S and σ_S are model parameters, and Z_t^2 is a standard Brownian Motion. The correlation between the standard Brownian Motions Z_t^1 and Z_t^2 is equal to ρ.

Let $P_t^{(n)}$ be the price at time t of a zero-coupon bond with face value of one which matures at time $t + n$, and let $R_t^{(n)}$ be the corresponding yield to maturity $R_t^{(n)}$. Then we have:

$$R_t^{(n)} \equiv \frac{-\log\left(P_t^{(n)}\right)}{n} = \frac{A_t^{(n)}}{n} + \frac{B_t^{(n)}}{n} \cdot r_t, \tag{23}$$

with

$$
\begin{aligned}
A_t^{(n)} &= \frac{\sigma^2}{4b} \cdot \left(B_t^{(n)}\right)^2 - \left(B_t^{(n)} - n\right) \cdot \left(\frac{(a - \sigma\lambda) \cdot 2b - \sigma^2}{2b^2}\right), \\
B_t^{(n)} &= \frac{1 - \exp(-b \cdot n)}{b},
\end{aligned}
$$

with the additional parameter λ representing the price of risk.

To estimate the parameters of the Ornstein-Uhlenbeck process and the stock index process we discretize the stochastic differential equations (SDE) of equations (21) and (22). Let Δt be the time step, then we have, with $\alpha = a\Delta t$, $\beta = b\Delta t$, and $\sigma_{\Delta t} = \sigma\sqrt{\Delta t}$:

$$
\begin{aligned}
r_{t+\Delta t} - r_t &= \alpha - \beta r_t + \epsilon_{t+\Delta t}, \\
\frac{S_{t+\Delta t} - S_t}{S_t} &= (r_t + \lambda_S \sigma_S)\,\Delta t + \epsilon^S_{t+\Delta t}, \\
\begin{pmatrix} \epsilon_{t+\Delta t} \\ \epsilon^S_{t+\Delta t} \end{pmatrix} \mid \mathcal{F}_t &\sim \begin{pmatrix} \sigma_{\Delta t} & 0 \\ 0 & \sigma^S_{\Delta t} \end{pmatrix} \times N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right),
\end{aligned}
$$

where $\mathcal{F}_t$ denotes the information available at time t, and N stands for a normal distribution. For estimation purposes, we use five implied moment conditions:

$$
\begin{array}{lcllcl}
\mathbb{E}\left[\epsilon_{t+\Delta_t}\right] & = & 0, & \mathbb{E}\left[\epsilon^2_{t+\Delta_t}\right] & = & \sigma^2_{\Delta t}, \\
\mathbb{E}\left[\epsilon^S_{t+\Delta_t}\right] & = & 0, & \mathbb{E}\left[\left(\epsilon^S_{t+\Delta_t}\right)^2\right] & = & \left(\sigma^S_{\Delta t}\right)^2, \\
\mathbb{E}\left[\epsilon_{t+\Delta_t}\epsilon^S_{t+\Delta_t}\right] & = & \rho\sigma_{\Delta t}\sigma^S_{\Delta t}. & & &
\end{array}
\tag{24}
$$

In order to estimate the additional parameter λ we assume that the yield on a zero-coupon bond maturing in $n = 10$ years from time t is given by (23) plus a mean zero error term $\epsilon_t^{(n)}$:

$$
R_t^{(n)} = -\frac{D_t^{(n)}}{n} + \frac{B_t^{(n)} r_t}{n} + \epsilon_t^{(n)}, \ \mathbb{E}\left[\epsilon_t^{(n)}\right] = 0. \tag{25}
$$

We add to the moment restrictions in (24) and (25) as extra moment conditions

$$
\mathbb{E}\left[\epsilon_{t+\Delta t} r_t\right] = 0, \ \ \mathbb{E}\left[\epsilon^R_{t+\Delta t} r_t\right] = 0, \ \ \mathbb{E}\left[\left(\epsilon^2_{t+\Delta t} - \sigma^2_{\Delta t}\right) r_t\right] = 0. \tag{26}
$$

We use daily Dutch financial data obtained from *Datastream* from January 31, 1997 till January 1, 2007. We use three time series, namely the one month interest rate, the interest rate on a 10 years Dutch government bond, and the return on the Dutch stock index "AEX." When estimating the model parameters using the Generalized Method of Moments (GMM) (with optimal weighting matrix) based on the moment restrictions (24)–(26), we make use of the Newey-West covariance matrix estimator. We experimented with the lag length in this estimator. The reported estimates correspond to lag length equal to ... Table 3 displays the estimates and the standard deviation of the estimates of the model parameters.

Table 3: Parameter estimates of distribution of the financial returns

Parameter	a	b	σ	λ	λ_S	σ_S	ρ
Estimate	0.0045908	0.10399	0.0042971	-0.81134	0.40832	0.23663	-0.028284
St. dev.	0.0011086	0.026058	0.0005669	0.3307	0.17706	0.017793	0.008286

The table displays the estimates and the standard deviation of the estimates of the model parameters for the distribution of the returns of the assets in the financial market.

We include two sources of financial risk: process risk and parameter risk. First, using (22) and (23) and using the GMM-based estimates, there is *process risk* due to the fact that future values of r_t and S_t are risky. Next, these forecasts are based on estimates sensitive to estimation inaccuracy. The corresponding risk is referred to as *parameter risk*. Let θ be the vector of all parameters estimated by GMM. The GMM-estimator $\widehat{\theta}_{GMM}$ satisfies $\sqrt{T}\left(\widehat{\theta}_{GMM}-\theta\right) \xrightarrow{d} N\left(0, V_\theta\right)$. Let $\widehat{V}_\theta$ be a consistent estimator of V_θ. To quantify the financial risk, we simulate 15,000 scenarios as follows. First, we simulate a θ from the $N\left(\widehat{\theta}_{GMM}, \widehat{V}_\theta/T\right)$-distribution, to incorporate parameter risk, and then, given this θ, we simulate the relevant future values of r_t, $R_t^{(n)}$, and S_t, using (21)–(23), to incorporate process risk.

B The distribution of the mortality probabilities

In this section we describe the models used to quantify the systematic longevity risk affecting $p_{x,t}^{(g)}$. Let $\mu_{x,t}^{(g)}$ denote the force of mortality of a person with age x and gender g at time t. We assume that for any integer age x, any gender g, and any time t, it holds that $\mu_{x+u,t}^{(g)} = \mu_{x,t}^{(g)}$, for all $u \in [0,1)$. Then one can verify (see, for example, Pitacco, Denuit, Haberman, and Olivieri, 2009)

$$p_{x,t}^{(g)} = \exp\left(-\mu_{x,t}^{(g)}\right) = \exp\left(-m_{x,t}^{(g)}\right), \tag{27}$$

where $m_{x,t}^{(g)}$ is the central death rate. This rate is given by $m_{x,t}^{(g)} = D_{x,t}^{(g)}/E_{x,t}^{(g)}$, with $D_{x,t}^{(g)}$ the observed number of deaths in year t in the cohort with gender g and aged x at the beginning of year t, and with $E_{x,t}^{(g)}$ the corresponding number of person years, the so-called exposure. We use three variants of the Lee and Carter (1992)-model, a P-Spline model, based on Currie et al. (2004), and four variants of the Cairns, Blake, and Dowd (2006) (CBD)-model to quantify the systematic longevity risk. The three variants of the Lee-Carter model are described in Appendix B.1. In Appendix B.2 we describe the P-splines model. In Appendix B.3 we describe the four models for the CBD-model. In Appendix B.4 we then describe our approach of simulating scenarios to generate longevity risk, including model, parameter, and process risk.

B.1 Lee-Carter (1992) model

In this section we describe the three variants of the Lee-Carter model, namely the models proposed by Lee and Carter (1992), Brouhns, Denuit, and Vermunt (2002), and Cossette et al. (2007). The model by Lee and Carter (1992) is given by

$$\log\left(m_{x,t}^{(g)}\right) = a_x^{(g)} + b_x^{(g)} k_t^{(g)} + \epsilon_{x,t}^{(g)}, \tag{28}$$

where $k_t^{(g)}$ is an index of the level of mortality, $a_x^{(g)}$ is an age-specific constant describing the general pattern of mortality by age, $b_x^{(g)}$ is an age-specific constant describing the relative speed of the change in mortality by age, and where $\epsilon_{x,t}^{(g)}$ represents the measurement error, assumed to satisfy $\epsilon_{x,t}^{(g)} \mid \mathcal{K}_t \sim N\left(0, \sigma_{x,g}^2\right)$, conditional on $\mathcal{K}_t = \left\{k_\tau^{(g)} \mid g \in \{m, f\}, \tau = t, t-1, ...\right\}$. Moreover, we assume that the $\epsilon_{x,t}^{(g)}$ are independent for different x and g, conditional on $\mathcal{K}_t$.

To model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)'$ over time, we use an ARIMA(0,1,1) model (as best fitting ARIMA-model)

$$k_t^{(m)} = k_{t-1}^{(m)} + c^{(m)} + e_t^{(m)} + \theta^{(m)} e_{t-1}^{(m)}, \tag{29}$$

$$k_t^{(f)} = k_{t-1}^{(f)} + c^{(f)} + e_t^{(f)} + \theta^{(f)} e_{t-1}^{(f)}, \tag{30}$$

where $c^{(g)}$ is the gender g specific drift term which indicates the average annual change of $k_t^{(g)}$, $\theta^{(g)}$ is the gender specific moving average coefficient, and $e_t^{(g)}$ is the gender specific innovation such that

$$\begin{pmatrix} e_t^{(m)} \\ e_t^{(f)} \end{pmatrix} \mid \mathcal{K}_{t-1} \sim \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_f \end{pmatrix} \times N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{mf} \\ \rho_{mf} & 1 \end{pmatrix}\right),$$

where σ_g is the gender-specific standard deviation of the error term $e_t^{(g)}$, and where ρ_{mf} captures the correlation between $e_t^{(m)}$ and $e_t^{(f)}$.

In case of the model by Brouhns, Denuit, and Vermunt (2002), the age and gender specific numbers of deaths are modeled by a Poisson process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Poisson}\left(E_{x,t}^{(g)} e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}}\right), \tag{31}$$

with $\widetilde{\mathcal{K}}_t = \mathcal{K}_t \bigcup \left\{E_{x,\tau}^{(g)} \mid g \in \{m, f\}, \text{all } x, \tau = t, t-1, ...\right\}$. We assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$. The process for $\left(k_t^{(m)}, k_t^{(f)}\right)'$ is modeled as in case of the Lee and Carter (1992)-model, i.e., via equations (29)–(30).

As third model, we consider Cossette et al. (2007). These authors model the age specific numbers of deaths $D_{x,t}^{(g)}$ via the Binomial Gumbel process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Bin}\left(E_{x,t}^{(g)}, 1 - \exp\left(-e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}}\right)\right), \tag{32}$$

Table 4: Estimation results for the Lee-Carter models

Model	g	$c^{(g)}$	$\theta^{(g)}$	σ_g	ρ
Lee-Carter	m	-1.854	-0.131	1.612	0.881
	f	-1.576	-0.373	1.779	
Brouhns, Denuit, and Vermunt	m	-1.849	-0.096	1.376	0.897
	f	-1.519	-0.148	1.572	
Cossette et al.	m	-1.854	-0.097	1.386	0.916
	f	-1.529	-0.160	1.594	

Parameter estimates of equations (29)–(30)). Lee-Carter: Lee and Carter (1992)-model; Brouhns, Denuit, and Vermunt: Brouhns, Denuit, and Vermunt (2002)-model; Cossette et al.: Cossette et al. (2007)-model.

where we again assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$, and where we model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)'$ via equations (29)–(30).

The model-specific parameters are estimated imposing the required normalizations and using the estimation techniques as described in the corresponding papers. In order to avoid localized age induced anomalies in $\widehat{b}_x^{(g)}$ in the three models, we follow Renshaw and Haberman (2003). These authors proposed to smooth the age specific estimated parameters $\widehat{b}_x^{(g)}$ using cubic B-splines, with internal knots,

$$\zeta_0^{(g)} + \zeta_1^{(g)} x + \zeta_2^{(g)} x^2 + \zeta_3^{(g)} x^3 + \sum_{j=1}^{r} \zeta_{3+j}^{(g)} (x - x_j)_+^3, \tag{33}$$

where $(x-x_j)_+^3 = (x-x_j)^3$, in case $x - x_j > 0$, and zero otherwise. As internal knots we use $x_1 = 9.5$, $x_2 = 20.5$, $x_3 = 50.5$, $x_4 = 60.5$, and $x_r = x_5 = 80.5$. The cubic B-splines are fitted to the (model specific) estimated $\widehat{b}_x^{(g)}$ using the method of least squares.

Age, gender, and time specific numbers of death and exposed to death are obtained from the Human Mortality Database.[14] In our case $x \in \{0, 1, 2, ..., 99, 100^+\}$, with 100^+ the age group of people aged 100 years or more. We use the time period 1977–2006, so that $T = 2006$. This time period minimizes the statistic proposed by Booth et al. (2002) to test the hypothesis that the age components in the original Lee-Carter model are invariant over time. The parameter estimates relevant for the quantification of the systematic longevity risk are plotted in Figure 7 (the $\widehat{b}_x^{(g)}$) and Table I (the parameter estimates of equations (29)–(30)).

To forecast the future mortality probabilities, we use (27), combined with (28), (31), or (32) (depending on the model), together with (29)–(30) and (33). Let $\widehat{q}_{x,T+s}^{(g)} = 1 - \widehat{p}_{x,T+s}^{(g)}$ be the s-periods ahead model-specific forecasted one-year death probability (starting from the end of the sample $T = 2006$). To avoid a jump-off bias in the forecasts,

[14] See www.mortality.org.

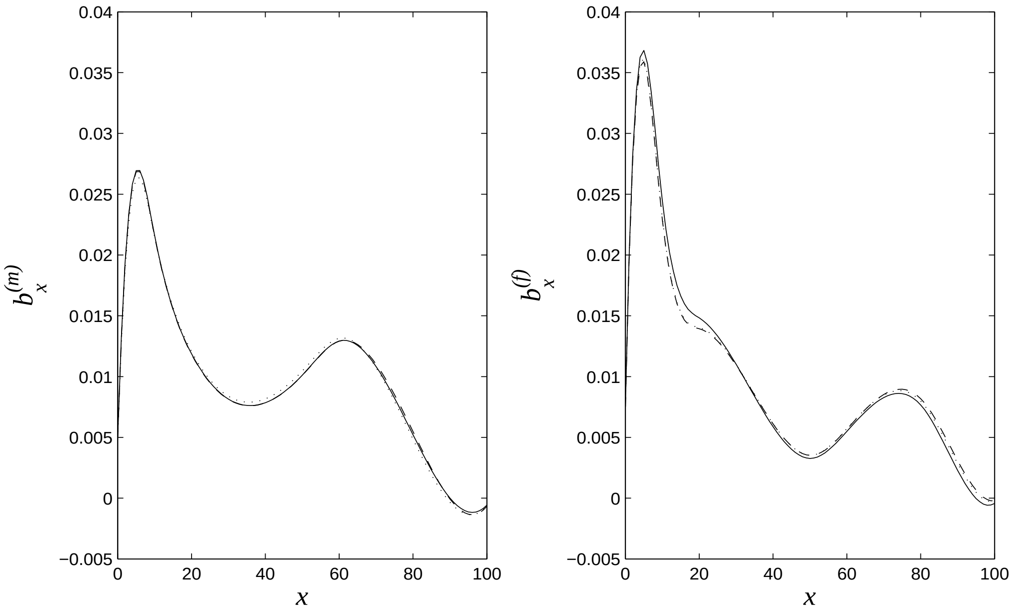

Figure 7: Estimated $b_x^{(g)}$ after smoothing using cubic B-splines. Left panel: $g = m$; right panel: $g = f$. The solid curve corresponds to the Lee and Carter (1992)-model; the dashed curve corresponds to the Brouhns, Denuit, and Vermunt (2002)-model, and the dotted curve corresponds to the Cossette et al. (2007)-model.

we correct this forecast using as correction factor $q_{x,T}^{(g)}/\widehat{q}_{x,T}^{(g)}$, with $q_{x,T}^{(g)}$ the observed one-year death probability in year T and $\widehat{q}_{x,T}^{(g)}$ the corresponding model-specific one-year death probability.

B.2 P-Splines

In this section we describe the P-spline model proposed by Currie, Durbin, and Eilers (2004). Let $B_y = B_y(x_y)$, be a $n_y \times c_y$ regression matrix of B-splines based on explanatory variable x_y and let $B_a = B_a(x_a)$, be a $n_a \times c_a$ regression matrix of B-splines based on explanatory variable x_a. The regression matrix for our model is the Kronecker product:

$$B = B_y \otimes B_a.$$

For the general population we assume:

$$\underline{D}^{(m)} + \underline{D}^{(f)} \mid \underline{E}^{(m)} + \underline{E}^{(f)} \sim \text{Poisson}\left(\left(\underline{E}^{(m)} + \underline{E}^{(f)}\right) \exp\left(B\underline{\alpha}^{(p)}\right)\right), \tag{34}$$

where the data is arranged in column order, that is $\underline{D}^{(g)} = \text{vec}\left(D^{(g)}\right)$ and $\underline{E}^{(g)} = \text{vec}\left(E^{(g)}\right)$, and the log of a vector is the log applied componentwise. The general trend

Table 5: Parameter settings and Output P-spline model

	General	Males-general	Females-general
$bdeg_a$	3	3	3
$pord_a$	2	2	2
n_a	91	91	91
c_a	21	21	21
λ_a	15	1400	820
$bdeg_a$	3	3	3
$pord_y$	2	2	2
n_y	30	30	30
c_y	8	8	8
λ_y	72	3000	1800

This table displays the parameter settings and output of the P-spline model.

in the force of mortality of the whole population is given by $B\underline{\alpha}^{(p)}$. For the difference in the forces of mortality between the general population and the gender specific forces of mortality we regress for both $g = m$ and $g = f$:

$$\underline{D}^{(g)} \mid \underline{E}^{(g)} \sim \text{Poisson}\left(\underline{E}^{(g)} \exp\left(B\underline{\widehat{\alpha}}^{(p)} + B\underline{\alpha}^{(g)}\right)\right), \tag{35}$$

where $B\underline{\alpha}^{(p)}$ is estimated in the previous step and, thus, assumed to be known in the second step.

To avoid under-smoothing, we use a penalty on α of the form $\alpha' P\alpha$, where the penalty matrix P is given by:

$$P = \lambda_a I_{c_y} \otimes D_a^{'} D_a + \lambda_y D_y^{'} D_y \otimes I_{c_a},$$

with λ_a and λ_y smoothing parameters, I_{c_y} an identity matrix of size c_y, D_a a so-called difference matrix of dimension $(c_a - p_a) \times c_a$ (that takes the column-wise difference of another matrix when post-multiplied), where p_a is the order of the penalty on age, and with I_{c_a} and D_y defined similarly. Given the smoothing parameters λ_a and λ_y, the parameter vector α is estimated by maximizing the log-likelihood based on (34) or (35) (with $B\underline{\widehat{\alpha}}^{(p)}$ given), corrected for the penalty $\frac{1}{2}\alpha' P\alpha$. The smoothing parameters λ_a and λ_y are set such that they optimize the Bayesian Information Criterion (BIC).

Currie, Durbin, and Eilers (2004) provide an easy way not only to estimate α, but also to calculate forecasts given α. Moreover, these authors provide an approximate normal distribution by which the sampling inaccuracy in the estimate $\widehat{\alpha}$ can be quantified.

The application of the P-spline method requires a large number of settings. Table 5 presents the settings that we used. As data we used the Dutch mortality data from 1977 till 2006 for the ages 20 till 110.

B.3 CBD models

In this section we describe the third class of models, the CBD-models, first introduced in Cairns, Blake, and Dowd (2006). Later several extensions have been proposed, see for example, Cairns et al. (2009). The CBD models fit the one-year mortality probabilities $q_{x,t}^{(g)} = 1 - p_{x,t}^{(g)}$. The general specification of the CBD-model is given by

$$\log\left(\frac{q_{x,t}^{(g)}}{1-q_{x,t}^{(g)}}\right) = \beta_{1,x}^{(g)}\kappa_{1,t}^{(g)} + \beta_{2,x}^{(g)}\kappa_{2,t}^{(g)} + \beta_{3,x}^{(g)}\kappa_{3,t}^{(g)} + \beta_{4,x}^{(g)}\gamma_{t-x}^{(g)}, \tag{36}$$

where $\beta_{j,x}^{(g)}$, $j = 1, \cdots, 4$, are possibly age dependent constants, and $\kappa_{j,t}^{(g)}$, $j = 1, 2, 3$, represent time effects, $\gamma_{t-x}^{(g)}$ is a cohort effect, and $\epsilon_{x,t}^{(g)}$ is a residual. We consider the four following possibilities. We define the set C as the set of all cohort years that have been included in the analysis, i.e., $C = \{c = t - x \mid t \in \mathcal{T}, x \in \mathcal{X}\}$, where $\mathcal{T}$ is the sample period and $\mathcal{X}$ is the set of ages considered.

1) $\beta_{1,x}^{(g)} = 1$, $\beta_{2,x}^{(g)} = x - \overline{x}$, $\beta_{3,x}^{(g)} = \beta_{4,x}^{(g)} = 0$ (where $\overline{x}$ is the mean of the ages in $\mathcal{X}$).

2) As 1) but with $\beta_{4,x}^{(g)} = 1$, together with the identification constraints $\sum_{c\in C}\gamma_c^{(g)} = \sum_{c\in C} c \cdot \gamma_c^{(g)} = 0$.

3) As 2) but with $\beta_{3,x}^{(g)} = (x - \overline{x})^2 - \sigma_x^2$ (where σ_x^2 is the variance of the ages in $\mathcal{X}$), together with the extra identification constraint $\sum_{c\in C} c^2 \cdot \gamma_c^{(g)} = 0$.

4) As 2) but with $\beta_{4,x}^{(g)} = C^{(g)} - x$, for some constant parameter $C^{(g)}$, together with the (single) identification constraint $\sum_{c\in C}\gamma_c^{(g)} = 0$.

Version 1) is the original CBD-model, proposed in Cairns, Blake, and Dowd (2006). Let $\kappa_t = \left(\kappa_{1,t}^{(m)}, \kappa_{2,t}^{(m)}, \kappa_{3,t}^{(m)}, \kappa_{1,t}^{(f)}, \kappa_{2,t}^{(f)}, \kappa_{3,t}^{(f)}\right)'$, and $\mathcal{K}_t = \{\kappa_\tau \mid \tau = t, t-1, \cdots\}$. Similar to the model by Brouhns, Denuit, and Vermunt (2002), the age and gender specific numbers of deaths are modeled by a Poisson process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Poisson}\left(m_{x,t}^{(g)} E_{x,t}^{(g)}\right),$$

with $\widetilde{\mathcal{K}}_t = \mathcal{K}_t \bigcup \left\{E_{x,\tau}^{(g)} \mid g \in \{m, f\}, \text{all } x, \tau = t, t-1, ...\right\}$, together with the assumption that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$. Here, $m_{x,t}^{(g)}$ is linked to $q_{x,t}^{(g)}$ via $m_{x,t}^{(g)} = -\log\left(1 - \widehat{q}_{x,t}^{(g)}\right)$, cf. (27). The parameters κ_t, for $t \in \mathcal{T}$, γ_c, for $c \in C$, and $C^{(g)}$ are estimated by maximizing the corresponding log likelihood, where we use for $\mathcal{T}$ the sample period from 1977 until 2006 and for the set $\mathcal{X}$ of ages the ages 60 until 100+.

Table 6: Parameter estimates of the CBD-models

	$\mu_1^{(m)} \cdot 10^2$	$\mu_2^{(m)} \cdot 10^4$	$\mu_3^{(m)} \cdot 10^5$	$\mu_1^{(f)} \cdot 10^2$	$\mu_2^{(f)} \cdot 10^4$	$\mu_3^{(f)} \cdot 10^5$
CBD 1	-1.3723	8.3578		-1.1211	1.3925	
CBD 2	-1.3203	3.9099		-1.0141	17.7359	
CBD 3	-1.3708	8.0533	2.1365	-0.87047	1.7736	-5.8667
CBD 4	-3.9336	-1.7694		6.7977	36.4297	
	LogL		# par		BIC	
CBD 1	-12042		120		24905	
CBD 2	-9344		236		20302	
CBD 3	-9220		294		20449	
CBD 4	-9431		240		20503	

The table displays the estimation of the parameter μ and the log likelihood, number of parameter, and the Bayesian Information Criterion (BIC) for the different CBD-models. For model CBD 4 we have used $C^{(m)} = 74$ and $C^{(f)} = 75$.

In terms of κ_t, we assume, cf. (29)–(30),

$$\kappa_t = \kappa_{t-1} + \mu + e_t, \quad e_t \mid \mathcal{K}_{t-1} \sim N(0, V), \tag{37}$$

where μ and V represent the mean vector and covariance matrix of $D_t = \kappa_t - \kappa_{t-1}$. Following Cairns, Blake, and Dowd (2006) we assume as non-informative prior distribution for (μ, V) the Jeffreys prior:

$$p(\mu, V) \propto |V|^{-3/2},$$

where $|V|$ is the determinant of the covariance matrix V. The posterior distribution for $(\mu, V|D)$, with $D = (D_1, \cdots, D_T)$, then satisfies

$$V^{-1}|D \sim \text{Wishart}\left(T-1, T^{-1}\widehat{V}^{-1}\right),$$
$$\mu|V, D \sim MVN\left(\widehat{\mu}, T^{-1}V\right),$$
$$\text{where } \widehat{\mu} = T^{-1}\sum_{t=1}^{n} D_t,$$
$$\text{and } \widehat{V} = T^{-1}\sum_{t=1}^{T}(D_t - \widehat{\mu})(D_t - \widehat{\mu})'.$$

Table 6 displays the estimates of μ for the different models.

B.4 Quantifying Longevity Risk

We include three sources of systematic longevity risk: process risk, parameter risk, and model risk. First, given a specific model and given the corresponding model specific estimates, there is *process risk* due to fact that future values of $\widehat{q}_t^{(g)}$ are still risky. Next, given a specific model, the forecasts of $\widehat{q}_t^{(g)}$ are based on model specific estimates, sensitive to estimation inaccuracy. The corresponding risk is referred to as *parameter risk*. Finally, different models might be used to calculate the forecasts, resulting. Assuming that some prior distribution is used to do the forecast calculations, there is in *model risk*.

To incorporate model risk, we generate 5000 scenarios from each class of models: 5000 scenarios from the Lee-Carter (1992)-type models (1666 scenarios from the Lee-Carter (1992) model, 1666 scenarios from the Brouhns, Denuit, and Vermunt (2002) model, and 1667 scenarios from the Cossete et al. (2007) model); 5000 scenarios from Cairns-Blake-Dowd (2006) models (1250 scenarios from each of the four variants), and 5000 scenarios from the P-Splines model.

To incorporate parameter risk, we simulate in each of the scenarios parameters in a model-specific way. For example, in case of the Lee and Carter (1992) model we simulate $\alpha_x^{(g)}$, $\beta_x^{(g)}$, $\sigma_{x,g}^2$, $c^{(g)}$, $\theta^{(g)}$, σ_g, and ρ_{mf}, using a bootstrap procedure, following Koissi, Shapiro, and Högnäs (2006). A similar approach is used in case of the Brouhns, Denuit, and Vermunt (2002) model and Cossete et al. (2007) model. In case of the P-Splines model we simulate α-s, using the approximate normal distribution of the estimated $\widehat{\alpha}$. In case of the CBD-models we simulate μ and V from the corresponding posterior distribution.

To incorporate process risk, we simulate in case of the Lee-Carter (1992) model, given the simulated parameter values, future values of $k_t^{(g)}$ (by simulating future values of $e_t^{(g)}$) and future values of $\varepsilon_{x,t}^{(g)}$. This results in scenario-specific future values of $q_{x,t}^{(g)}$. In case of the Brouhns, Denuit, and Vermunt (2002) model and Cossete et al. (2007) model we proceed in a similar way. However, in these models we ignore the potential process risk in the error terms $\varepsilon_{x,t}^{(g)}$, which are set equal to zero (in fact, we did not present these error terms in these cases). In case of the P-Splines model the simulated α-s also incorporate process risk. In case of the CBD-models we simulate, given the simulated μ and V, future values of κ_t (by simulating future values of e_t). Similar to the Brouhns, Denuit, and Vermunt (2002) and Cossete et al. (2007) models, we ignore both in the P-spline model and the CBD-models the potential process risk in the error terms $\varepsilon_{x,t}^{(g)}$ (these error terms are also not presented in case of these models).