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Publication date:
2011

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Grundel, S., Borm, P. E. M., & Hamers, H. J. M. (2011). *A Compromise Stable Extension of Bankruptcy Games: Multipurpose Resource Allocation*. (CentER Discussion Paper; Vol. 2011-029). Tilburg University.

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No. 2011-029

**A COMPROMISE STABLE EXTENSION OF BANKRUPTCY
GAMES: MULTIPURPOSE RESOURCE ALLOCATION**

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24 February 2011

ISSN 0924-7815

A Compromise Stable Extension of Bankruptcy Games: Multipurpose Resource Allocation

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February 24, 2011

Abstract

This paper considers situations characterized by a common-pool resource, which needs to be divided among agents. Each of the agents has some claim on this pool and an individual reward function for assigned resources. This paper analyzes not only the problem of maximizing the total joint reward, but also the allocation of these rewards among the agents. Analyzing these situations a new class of transferable utility games is introduced, called *multipurpose resource games*. These games are based on the bankruptcy model, as introduced by O'Neill (1982). It is shown that every multipurpose resource game is compromise stable. Moreover, an explicit expression for the nucleolus of these games is provided.

Keywords: bankruptcy games, compromise stability, nucleolus

JEL classification code: C71

1 Introduction

The model of bankruptcy situations was first analyzed from a game theoretic perspective by O'Neill (1982). In a *bankruptcy situation* a certain amount of money, the estate, has to be divided among a group of claimants. Each claimant has a justified claim on the estate such that the sum of these claims exceeds the available estate. The example originally given by O'Neill is of the division of an estate amongst several heirs when the estate cannot meet all the deceased's commitments. Another example is of a firm going bankrupt, whose remaining assets do not cover the total demand of all creditors.

Many rules have been proposed to fairly allocate the estate in bankruptcy situations. Some of these rules are based on the associated cooperative *bankruptcy game* where the worth of a coalition is equal to what is left of the estate if all other claimants would receive their demands. Aumann and Maschler (1985) proposed and characterized a rule that coincides with the nucleolus of this corresponding bankruptcy game. For an overview on bankruptcy rules we refer to Thomson (2003).

The bankruptcy model is a general framework for various kinds of allocation problems and is applied to many cases such as cost-sharing problems (Moulin, 1991), taxation problems (Young, 1988), and apportionment of indivisible good(s) problems (Young, 1995). In the past, several extensions are proposed for the basic bankruptcy model. One of them is Calleja et al. (2005)

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where multi-issue allocation problems are analyzed. A second is Kaminski (2000), where claims are multidimensional rather than one-dimensional.

In this paper we present another extension for the traditional bankruptcy problem. In a *multipurpose resource allocation* (MPRA) *situation* there is a limited supplied resource (e.g. water). Agents are characterized by a justified claim on the resource and a reward function which describes the (monetary) reward for assigned resources. A specific resource assignment leads to some total joint reward obtained by the agents. The aim is to find a *fair* allocation of the maximum joint reward. In particular, rewards may be redistributed among the agents such that agents are compensated who cede their resources to others.

The general framework we propose in this paper is applicable to the field of water resource management. Water resource management often involves a multitude of different agents with different interests who put their claim on a common-pool resource. The Tennessee Valley Authority case (Ransmeier, 1942) is one of the earliest cases that offered game theorists an opportunity to examine a practical problem of cost allocation in a water resources development project. Straffin and Heaney (1981) outlined some basic cooperative game theoretic principles embedded in the analysis of this case and translated the main cost allocation methods into ‘game theory language’. In Carraro et al. (2005), Parrachino et al. (2006), and Zara et al. (2006) an elaborated review is provided of game theoretic water conflict resolution studies. For a recent overview of the literature about game theory and water resources we refer to Madani (2010).

A game theoretic analysis of MPRA-situations falls within the framework of operations research games. These games are concerned with the combinatorial optimization problem of finding a joint optimal structure like a network or processing order. After the optimal structure is determined, cooperative game theory is applied subsequently to analyze the allocation of rewards. A survey of operations research games is provided by Borm et al. (2001).

Also for MPRA-situations we first analyze a joint optimization problem, the maximization of total joint reward via an optimal assignment of resources. Secondly, the maximum total joint reward is allocated to the agents. This is done by analyzing the associated cooperative *multipurpose resource* (MR) *game*. For this game the value of a particular coalition reflects the maximum total joint reward that can be derived from the amount of resources not claimed by agents outside the coalition. It is shown that MR-games extend bankruptcy games and that the property of compromise stability can be extended to MR-games, although the property of convexity is not maintained in general. Moreover, an explicit expression for the nucleolus (Schmeidler, 1969) for MR-games is provided.

This paper is organized as follows. In section 2 the formal model of multipurpose resource allocation situations is described and the optimal assignments of resources are characterized. In section 3, we introduce and analyze corresponding multipurpose resource games. Section 4 is devoted to compromise stability and the computation of the nucleolus. Technical proofs are relegated to the Appendix.

2 Multipurpose Resource Allocation Situations

This section formally introduces *multipurpose resource allocation situations*. After introducing the model, optimal assignments of resources, in which the total joint reward is maximized, are characterized .

A *bankruptcy situation* is a triple (N, E, d) , where N represents a finite set of claimants, $E \geq 0$ is the estate (e.g money) which has to be divided among the claimants, and $d \in \mathbb{R}_{++}^N$ is a vector of claims, where for $i \in N$, d_i represents agent i 's claim on the estate. To justify the term ‘bankruptcy’ it is assumed that $\sum_{j \in N} d_j \geq E$.

An MPRA-situation extends a bankruptcy situation (N, E, d) , with an estate E of some resource (e.g. water) and a claim vector d on this resource, by adding a reward vector $\alpha \in \mathbb{R}_{++}^N$ in the following way. For every agent $i \in N$ there exists a *reward function* $r_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ describing the monetary gain of resources for agent i . For every $z \in \mathbb{R}_+$, $r_i(z)$ denotes the monetary reward for agent i if he is assigned z units of resource. In this paper the focus is on linear reward functions. Hence, for every $i \in N$ there exists an $\alpha_i \in \mathbb{R}_{++}$ such that $r_i(z) = \alpha_i z$. An MPRA-situation will be summarized by (N, E, d, α) . The class of all MPRA-situations with set of agents N is denoted by $MPRA^N$.

An outcome for an MPRA-situation consists of two elements: an *assignment* of resources and an *allocation* of the associated monetary reward. Throughout this paper ‘assignment’ refers to the distribution of resources (e.g. water) and ‘allocation’ to the distribution of rewards (e.g. money). Formally, a solution $f : MPRA^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N$ is defined by

$$f(N, E, d, \alpha) = (x, y(x))$$

$$s.t. \sum_{j \in N} x_j = E \tag{1}$$

$$0 \leq x_i \leq d_i \quad \text{for all } i \in N \tag{2}$$

$$\sum_{j \in N} y_j(x) = \sum_{j \in N} r_j(x_j) \tag{3}$$

$$y_i(x) \geq 0 \quad \text{for all } i \in N \tag{4}$$

for all $(N, E, d, \alpha) \in MPRA^N$. Constraint (1) tells us that the sum of assigned resources is equal to the estate. Constraint (2) ensures that the assigned resources do not exceed the demand and are non-negative for all agents in N . Let $F(N, E, d, \alpha)$ be the set of *feasible* assignments x determined by conditions (1) and (2), i.e.

$$F(N, E, d, \alpha) = \left\{ x \in \mathbb{R}^N \mid \sum_{j \in N} x_j = E; 0 \leq x_i \leq d_i \text{ for all } i \in N \right\}.$$

Constraint (3) ensures that the total reallocated sum of rewards is equal to the total joint reward obtained from the underlying assignment x . One specific type of solutions are the *direct solutions*. A solution f is called direct, if for all $(N, E, d, \alpha) \in MPRA^N$, we have that $f(N, E, d, \alpha) = (x, r(x))$ for some $x \in F(N, E, d, \alpha)$, where $r(x) = (\alpha_i x_i)_{i \in N}$ is the direct reward vector with respect to x .

Example 2.1. Consider an MPRA-situation $(N, E, d, \alpha) \in MPRA^N$ with $N = \{1, 2, 3\}$ and estate $E = 3$. Assume that agent 1 claim 2 units of resource, the others both claim 1. Therefore, the demand vector equals $d = (2, 1, 1)$. Note that the sum of these demands exceeds the available estate such that not all agents can obtain their full demand. With reward vector $\alpha = (1, 1, 2)$ e.g. the reward function of agent 1 is given by $r_1(z) = z$ for z such that $0 \leq z \leq d_1$. Consider the feasible assignment $x = (2, 1, 0)$ with total joint reward 3. Note that $r(x) = (2, 1, 0)$ while e.g. also $y(x) = (1, 1, 1)$ satisfies conditions (3) and (4). \diamond

The set of feasible assignments is large and there are many possible (re)allocations of the corresponding rewards. Throughout this article, assignments of resources are considered which maximize the total joint reward. The remainder of this section is dedicated to finding these *optimal assignments*.

Let $(N, E, d, \alpha) \in MPRA^N$. The maximum total joint reward $v(N, E, d, \alpha)$ is determined by

$$v(N, E, d, \alpha) = \max \left\{ \sum_{j \in N} r_j(x_j) \mid x \in F(N, E, d, \alpha) \right\}.$$

The set $X(N, E, d, \alpha)$ of optimal assignments is given by

$$X(N, E, d, \alpha) = \left\{ x \in F(N, E, d, \alpha) \left| \sum_{j \in N} r_j(x_j) = v(N, E, d, \alpha) \right. \right\}.$$

Before characterizing the set $X(N, E, d, \alpha)$ of optimal assignments we introduce some additional notations. Let $m = |\{\alpha_i | i \in N\}|$ be the number of different reward parameters. Recursively, a partition of N is introduced by defining sets N_1, N_2, \dots, N_m by

$$N_1 = \left\{ i \in N \left| \alpha_i = \max \{ \alpha_j \mid j \in N \} \right. \right\}$$

and

$$N_l = \left\{ i \in N \left| \alpha_i = \max \{ \alpha_j \mid j \in N \setminus \bigcup_{t=1}^{l-1} N_t \} \right. \right\}$$

for all $l \in \{2, \dots, m\}$. For $l \in \{1, \dots, m\}$ let β_l be the uniquely defined reward parameter for every agent in N_l . Clearly, $\beta_1 > \beta_2 > \dots > \beta_m$. Denote the total demand of agents in N_l as $d(N_l)$, i.e. $d(N_l) = \sum_{j \in N_l} d_j$. Define the *pivot index* $k \in \{1, \dots, m\}$ such that

$$\sum_{l=1}^{k-1} d(N_l) \leq E < \sum_{l=1}^k d(N_l). \quad (5)$$

If $E = \sum_{j \in N} d_j$, then we assume $k = m$. With pivot index k define three disjoint sets P_1, P_2 and P_3 of agents in the following way

$$\begin{aligned} P_1 &= \bigcup_{l=1}^{k-1} N_l; \\ P_2 &= N_k; \\ P_3 &= \bigcup_{l=k+1}^m N_l. \end{aligned}$$

Note that P_1 and P_3 may be empty, but P_2 is always non-empty.

Example 2.2. Reconsider the MPRA-situation of Example 2.1. Clearly, $N_1 = \{3\}$ with $\beta_1 = \alpha_3 = 2$ and $N_2 = \{1, 2\}$ with $\beta_2 = \alpha_1 = \alpha_2 = 1$. Observe that

$$d(N_1) = 1 < E = 3 < d(N_1) + d(N_2) = 4$$

Hence, the pivot index is $k = 2$. Consequently, $P_1 = N_1 = \{3\}$, $P_2 = N_2 = \{1, 2\}$, and $P_3 = \emptyset$. \diamond

Optimal assignments, which maximize the total joint reward, are characterized in the following theorem.

Theorem 2.1. *Let $(N, E, d, \alpha) \in \text{MPRA}^N$ and $x \in F(N, E, d, \alpha)$. Then $x \in X(N, E, d, \alpha)$ if and only if $x_i = d_i$ for all $i \in P_1$ and $x_i = 0$ for all $i \in P_3$.*

Proof. Let $x \in X(N, E, d, \alpha)$. First suppose there exists an agent $i \in P_1$ such that $x_i < d_i$. This implies that

$$\sum_{j \in P_2 \cup P_3} x_j = E - \sum_{j \in P_1} x_j > E - \sum_{j \in P_1} d_j = E - \sum_{l=1}^{k-1} d(N_l) \geq 0.$$

We may conclude that there is at least one agent $j \in P_2 \cup P_3$ for which $x_j > 0$. Since $\alpha_j < \alpha_i$, the total joint reward would strictly increase if agent j transfers $\min\{d_i - x_i, x_j\}$ to agent i . This establishes a contradiction.

Secondly, suppose there exists an agent $i \in P_3$ such that $x_i > 0$. This implies that

$$E - \sum_{j \in P_1 \cup P_2} x_j = \sum_{j \in P_3} x_j > 0 > E - \sum_{l=1}^k d(N_l) = E - \sum_{j \in P_1 \cup P_2} d_j.$$

Consequently, there is at least one agent $j \in P_1 \cup P_2$ whose demand is not fully satisfied, i.e. $x_j < d_j$. Since $\alpha_j > \alpha_i$, the total joint reward would increase if agent i transfers $\min\{d_j - x_j, x_i\}$ to agent j . This establishes a contradiction and proves the only if part.

The fact that all assignments $x \in F(N, E, d, \alpha)$ such that $x_i = d_i$ for all $i \in P_1$ and $x_i = 0$ for all $i \in P_3$ lead to the same total joint reward finishes the proof. \square

Example 2.3. The set of optimal assignments of the MPRA-situation of Example 2.1 can be written as follows:

$$\text{Conv}\{(2, 0, 1), (1, 1, 1)\}.$$

\diamond

In general there is not a unique optimal assignment. The MPRA-situations for which there is exactly one optimal assignment are characterized in the following corollary.

Corollary 2.2. *Let $(N, E, d, \alpha) \in MPRA^N$. Then $|X(N, E, d, \alpha)| = 1$ if and only if $E = \sum_{j \in P_1} d_j$ or $|P_2| = 1$.*

3 Multipurpose Resource Games

In this section we introduce the class of *multipurpose resource (MR) games*. A *transferable utility (TU) game* is an ordered pair (N, v) where N is the finite set of agents, and v the characteristic function on 2^N , the set of all subsets of N . The function v assigns to every coalition $S \in 2^N$ a real number $v(S)$ with $v(\emptyset) = 0$. Here, $v(S)$ is called the worth or value of the coalition S . The set of all TU-games with set of agents N is denoted by TU^N . Where no confusion arises, we write v rather than (N, v) .

Consider a bankruptcy situation (N, E, d) . For the associated bankruptcy (BR) game $v_{E,d}$ the value of a coalition S is determined by the amount of E that is not claimed by agent in $N \setminus S$. Hence,

$$v_{E,d}(S) = \max\left\{0, E - \sum_{j \in N \setminus S} d_j\right\}$$

for all $S \in 2^N$.

Now consider an MPRA-situation $M = (N, E, d, \alpha)$. We assume that a coalition S can only use the amount of resources $D(S)$ not demanded by the agents in $N \setminus S$. Let $(S, D(S), d|_S, \alpha|_S) \in MPRA^S$ describe the associated multipurpose resource allocation situation for S , where

$$D(S) = \max\left\{0, E - \sum_{j \in N \setminus S} d_j\right\}.$$

Note that $D(N) = E$. The class of MR-games extends the class of BR-games, i.e. every BR-game can be written as an MR-game in which $\alpha_i = \alpha_j$ for all $i, j \in N$. In the MR-game v^M , associated to an MPRA-situation $M = (N, E, d, \alpha)$, the worth of coalition S equals

$$v^M(S) = v(S, D(S), d|_S, \alpha|_S).$$

Let $S_l = S \cap N_l$ for all $l \in \{1, \dots, m\}$. We extend the definition of the pivot index, as provided in (5) for the grand coalition, to every possible coalition S as follows. The pivot index $k(S)$ is such that

$$\sum_{l=1}^{k(S)-1} d(S_l) \leq D(S) < \sum_{l=1}^{k(S)} d(S_l). \quad (6)$$

If $D(S) = 0$, then we assume that $k(S) = 1$ and if $D(S) = \sum_{j \in S} d_j$, then we assume $k(S) = m$.

Let $x^S \in X(S, D(S), d|_S, \alpha|_S)$ be an optimal assignment of resources to agents in S . Theorem 2.1 tells us that $x_i^S = d_i$ for all $i \in S_1 \cup \dots \cup S_{k(S)-1}$ and $x_i^S = 0$ for all $i \in S_{k(S)+1} \cup \dots \cup S_m$. Clearly, $v^M(S)$ equals the total direct reward of agents in S associated to x^S . This allows us to construct an explicit formula for $v^M(S)$.

Theorem 3.1. *Let $M = (N, E, d, \alpha) \in MPRA^N$ and let v^M be the associated MR-game. Then*

$$v^M(S) = \sum_{l=1}^{k(S)-1} \beta_l d(S_l) + \beta_{k(S)} \left(D(S) - \sum_{l=1}^{k(S)-1} d(S_l) \right)$$

for all $S \in 2^N \setminus \{\emptyset\}$.

Example 3.1. Consider the MPRA-situation of Example 2.1 where $E = 3, d = (2, 1, 1)$ and $\alpha = (1, 1, 2)$. The corresponding values of $D(S)$, $k(S)$, and $v^M(S)$ are given in the table below.

S	1	2	3	1,2	1,3	2,3	N
$D(S)$	1	0	0	2	2	1	3
$k(S)$	2	1	1	2	2	2	2
$v^M(S)$	1	0	0	2	3	2	4

We illustrate the underlying computations for coalition $S = \{1, 2\}$. Note that $N_1 = \{3\}$, $S_1 = \emptyset$ and $N_2 = S_2 = \{1, 2\}$. The amount of available resources equals

$$D(S) = \max\{0, 3 - 1\} = 2.$$

This yields,

$$\begin{array}{rcl} d(S_1) & < D(S) < & d(S_1) + d(S_2) \\ 0 & < 2 < & 0 + 3 \end{array}$$

and consequently, $k(S) = 2$. Therefore,

$$\begin{aligned} v^M(12) &= \beta_1 d(S_1) + \beta_2 (D(S) - d(S_1)) \\ &= 2 \cdot 0 + 1(2 - 0) = 2. \end{aligned}$$

◇

From Example 3.1 we immediately see that MR-games are not *convex* in general since the condition

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

for all $S \subset T \subset N \setminus \{i\}$ is violated for $S = \{3\}, T = \{1, 3\}$ and $i = 2$.

Lemma 3.2 describes a relation between pivot indices of two coalitions.

Lemma 3.2. *Let $(N, E, d, \alpha) \in MPRA^N$. Let $S, T \in 2^N$ be such that $S \subset T$. Then*

$$k(S) \leq k(T).$$

Lemma 3.3 provides a monotonicity result for optimal assignments.

Lemma 3.3. *Let $(N, E, d, \alpha) \in MPRA^N$. Let $S, T \in 2^N$ be such that $S \subset T$. Let $x^T \in X(T, D(T), d|_T, \alpha|_T)$ and $x^S \in X(S, D(S), d|_S, \alpha|_S)$. Then*

$$\sum_{j \in S_l} x_j^S \leq \sum_{j \in S_l} x_j^T$$

for all $l \in \{1, \dots, m\}$.

The proofs of Lemma 3.2 and Lemma 3.3 can be found in the Appendix.

A game $v \in TU^N$ is called *balanced* if the core $C(v)$ of the game is non-empty. The core is defined by

$$C(v) = \left\{ y \in \mathbb{R}^N \mid \sum_{j \in N} y_j = v(N), \sum_{j \in S} y_j \geq v(S) \text{ for all } S \in 2^N \right\}.$$

For a game $v \in TU^N$ and $T \subset N, T \neq \emptyset$ the *subgame* $v|_T \in TU^T$ is defined by

$$v|_T(S) = v(S)$$

for all $S \in 2^T$. Note that $v|_N = v$. A game $v \in TU^N$ is called *totally balanced* if every subgame is balanced, i.e. if every subgame has a non-empty core.

Theorem 3.4. *Every MR-game is totally balanced.*

Proof. Let $M = (N, E, d, \alpha) \in MPRA^N$ with corresponding MR-game v^M . Let $T \subset N$ and let the assignment $x^T \in X(T, D(T), d|_T, \alpha|_T)$ be such that $v^M(T) = \sum_{j \in T} \alpha_j x_j^T$. Let $y(x^T) = (\alpha_i x_i^T)_{i \in T}$ be the direct allocation. To prove that v^M is totally balanced, it is sufficient to prove that $y(x^T) \in C(v^M|_T)$. First note that $\sum_{j \in T} y_j(x^T) = v^M(T)$ by definition. Secondly, consider $S \subset T$ and take $x^S \in X(S, D(S), d|_S, \alpha|_S)$. Then

$$\begin{aligned} \sum_{j \in S} y_j(x^T) &= \sum_{j \in S} \alpha_j x_j^T \\ &= \sum_{l=1}^{k(T)-1} \beta_l d(S_l) + \beta_{k(T)} \sum_{j \in S_{k(T)}} x_j^T \\ &\geq \sum_{l=1}^{k(S)-1} \beta_l d(S_l) + \beta_{k(S)} \sum_{j \in S_{k(S)}} x_j^S \\ &= \sum_{l=1}^{k(S)-1} \beta_l d(S_l) + \beta_{k(S)} \left(D(S) - \sum_{l=1}^{k(S)-1} d(S_l) \right) \\ &= v^M(S). \end{aligned}$$

The second and third equality follows from Theorem 2.1. From Lemma 3.2 we see that $k(S) \leq k(T)$. If $k(S) < k(T)$, then the inequality is readily verified. If $k(S) = k(T)$, then we use Lemma 3.3 to see that this inequality holds. The last equality follows from Theorem 3.1. \square

4 Compromise Stability and the Nucleolus

In this section we prove that MR-games are compromise stable and we derive an explicit expression for the nucleolus of MR-games.

Let $v \in TU^N$. The *core cover* $CC(v)$, as introduced by Tijs and Lipperts (1982), is given by

$$CC(v) = \left\{ y \in \mathbb{R}^N \left| \sum_{j \in N} y_j = v(N), m(v) \leq y \leq M(v) \right. \right\}$$

where the *utopia demand* $M_i(v)$ of agent $i \in N$ is defined by

$$M_i(v) = v(N) - v(N \setminus \{i\})$$

and the *minimum right* $m_i(v)$ of agent $i \in N$ equals

$$m_i(v) = \max_{S: S \ni i} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right\}.$$

Note that $m_i(v) \geq v(\{i\})$ for all $i \in N$. Moreover, for any TU-game v it holds that $C(v) \subset CC(v)$.

Example 4.1. Let $M = (N, E, d, \alpha) \in MPRA^N$, with $N = \{1, 2, 3, 4\}$, $E = 5$, $d = (2, 1, 3, 1)$ and $\alpha = (4, 3, 2, 1)$. Then the corresponding MR-game v^M is given by

S	1	2	3	4	1,2	1,3	1,4	2,3	2,4	3,4	1,2,3	1,2,4	1,3,4	2,3,4	N
$v^M(S)$	0	0	2	0	4	10	4	5	0	4	13	8	12	7	15

It is readily verified that $M(v^M) = (8, 3, 7, 2)$ and $m(v^M) = (3, 0, 2, 0)$ while

$$CC(v^M) = \text{Conv}\{(8, 3, 4, 0), (8, 3, 2, 2), (8, 0, 7, 0), (8, 0, 5, 2), (5, 3, 7, 2), (3, 3, 7, 2), (6, 0, 7, 2)\}.$$

Moreover, since each of the seven extreme points of the core cover belong to the core, we may conclude that $C(v^M) = CC(v^M)$ \diamond

Quant et al. (2005) define a game $v \in TU^N$ to be *compromise stable* if

$$C(v) = CC(v)$$

and $CC(v) \neq \emptyset$. Moreover, it was proved that each BR-game is both convex and compromise stable while reversely, each convex and compromise stable game is strategically equivalent to a BR-game.

In what follows N_{-i} is a shorthand notation for $N \setminus \{i\}$.

Lemma 4.1. Let $M = (N, E, d, \alpha) \in MPRA^N$, let v^M be the corresponding MR-game, and let $i \in N$.

1. If $d_i \geq E$, then $M_i(v^M) = v^M(N)$.

2. If $d_i < E$, then

a. $M_i(v^M) = \alpha_i d_i$ if $i \in P_1$;

b. $M_i(v^M) = \beta_{k(N)} d_i$ if $i \in P_2 \cup P_3$ and $k(N_{-i}) = k(N)$;

c. $M_i(v^M) = \beta_{k(N_{-i})} \left(\sum_{l=1}^{k(N_{-i})} d(N_l) - E + d_i \right) + \sum_{l=k(N_{-i})+1}^{k(N)-1} \beta_l d(N_l) + \beta_{k(N)} \left(E - \sum_{l=1}^{k(N)-1} d(N_l) \right)$
if $i \in P_2 \cup P_3$ and $k(N_{-i}) < k(N)$.

Proof. 1: Let $d_i \geq E$. Then $D(N_{-i}) = 0$ and $v^M(N_{-i}) = 0$. Therefore,

$$M_i(v^M) = v^M(N) - v^M(N_{-i}) = v^M(N).$$

2: Let $d_i < E$. Set $k = k(N)$,

2a: Let $i \in P_1$. We start by showing that $k(N_{-i}) = k$. By definition of k we have

$$\sum_{l=1}^{k-1} d(N_l) - d_i \leq E - d_i < \sum_{l=1}^k d(N_l) - d_i.$$

Since $i \in N_1 \cup \dots \cup N_{k-1}$, this implies

$$\sum_{l=1}^{k-1} d((N_{-i})_l) \leq D(N_{-i}) < \sum_{l=1}^k d((N_{-i})_l).$$

Consequently, $k(N_{-i}) = k$. Hence,

$$\begin{aligned} v^M(N_{-i}) &= \sum_{l=1}^{k-1} \beta_l d((N_{-i})_l) + \beta_k \left(D(N_{-i}) - \sum_{l=1}^{k-1} d((N_{-i})_l) \right) \\ &= \sum_{l=1}^{k-1} \beta_l d(N_l) - \alpha_i d_i + \beta_k \left(E - d_i - \sum_{l=1}^{k-1} d(N_l) + d_i \right) \\ &= v^M(N) - \alpha_i d_i. \end{aligned}$$

and

$$M_i(v^M) = v^M(N) - v^M(N_{-i}) = \alpha_i d_i.$$

2b/2c: Let $i \in P_2 \cup P_3$. By Theorem 3.1 it holds that

$$\begin{aligned} v^M(N_{-i}) &= \sum_{l=1}^{k(N_{-i})-1} \beta_l d((N_{-i})_l) + \beta_{k(N_{-i})} \left(D(N_{-i}) - \sum_{l=1}^{k(N_{-i})-1} d((N_{-i})_l) \right) \\ &= \sum_{l=1}^{k(N_{-i})-1} \beta_l d(N_l) + \beta_{k(N_{-i})} \left(E - d_i - \sum_{l=1}^{k(N_{-i})-1} d(N_l) \right) \end{aligned}$$

while

$$v^M(N) = \sum_{l=1}^{k-1} \beta_l d(N_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right).$$

Then

$$\begin{aligned} M_i(v^M) &= v^M(N) - v^M(N_{-i}) \\ &= \sum_{l=k(N_{-i})}^{k-1} \beta_l d(N_l) - \beta_{k(N_{-i})} \left(E - d_i - \sum_{l=1}^{k(N_{-i})-1} d(N_l) \right) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right). \end{aligned}$$

If $k(N_{-i}) = k$, then this simplifies into

$$M_i(v^M) = \beta_k d_i.$$

If $k(N_{-i}) < k$, then

$$M_i(v^M) = \beta_{k(N_{-i})} \left(\sum_{l=1}^{k(N_{-i})} d(N_l) - E + d_i \right) + \sum_{l=k(N_{-i})+1}^{k-1} \beta_l d(N_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right).$$

This finishes 2b/2c. □

Quant et al. (2005) prove that a game $v \in TU^N$ is compromise stable if and only if

$$v(S) \leq \max \left\{ \sum_{j \in S} m_j(v), v(N) - \sum_{j \in N \setminus S} M_j(v) \right\}$$

for all $S \in 2^N \setminus \{\emptyset\}$. Observe that if $m_i(v) \geq 0$ for all $i \in N$, then it suffices to prove

$$v(S) \leq \max \left\{ 0, v(N) - \sum_{j \in N \setminus S} M_j(v) \right\} \quad (7)$$

to establish that v is compromise stable.

Clearly, $m(v^M) \geq 0$ for an MR-game v^M . Using this observation the appendix shows that every MR-game is compromise stable.

Theorem 4.2. *Let $M = (N, E, d, \alpha) \in MPRA^N$ with corresponding MR-game v^M . Then*

$$v^M(S) \leq \max \left\{ 0, v^M(N) - \sum_{j \in N \setminus S} M_j(v^M) \right\}$$

for all $S \in 2^N \setminus \{\emptyset\}$. Consequently, v^M is compromise stable.

Next we derive an explicit expression for the nucleolus of an MR-game. Recall (cf. Aumann and Maschler (1985)) that the nucleolus $n(v_{E,d})$ of bankruptcy game $v_{E,d} \in TU^N$ can be computed as follows:

$$n(v_{E,d}) = AM(E, d) = \begin{cases} CEA(E, \frac{1}{2}d) & \text{if } \sum_{j \in N} d_j \geq 2E; \\ d - CEA\left(\sum_{j \in N} d_j - E, \frac{1}{2}d\right) & \text{if } \sum_{j \in N} d_j < 2E; \end{cases}$$

where

$$CEA(E, d) = (\min\{\lambda, d_i\})_{i \in N}$$

with λ such that $\sum_{j \in N} \min\{\lambda, d_j\} = E$.

Although v^M is not necessarily convex and therefore not strategically equivalent to a BR-game, the nucleolus of v^M corresponds to the AM-rule of a related BR-problem.

Theorem 4.3. *Let $M = (N, E, d, \alpha) \in MPRA^N$ and let v^M be the corresponding MR-game. Then*

$$n(v^M) = AM\left(v^M(N), M(v^M)\right).$$

Proof. Set $\hat{E} = v^M(N)$ and $\hat{d} = M(v^M)$. Observe that

$$\sum_{j \in N} \hat{d}_j = \sum_{j \in N} M_j(v^M) \geq v^M(N) = \hat{E}$$

because $C(v^M) \neq \emptyset$ and the core is a subset of the core cover for any game. Hence (\hat{E}, \hat{d}) is a BR-problem and, consequently

$$n(v_{\hat{E}, \hat{d}}) = AM(\hat{E}, \hat{d}).$$

Next we show that $C(v^M) = C(v_{\hat{E}, \hat{d}})$. Observe that $v^M(N) = v_{\hat{E}, \hat{d}}(N)$. Since Theorem 4.2 implies that

$$v^M(S) \leq v_{\hat{E}, \hat{d}}(S)$$

for all $S \in 2^N \setminus \{\emptyset\}$ it is clear that $C(v_{\hat{E}, \hat{d}}) \subset C(v^M)$. To prove also the reverse inclusion, let $y \in C(v^M)$ and $S \subset N$. We will prove that $\sum_{j \in S} y_j \geq v_{\hat{E}, \hat{d}}(S)$. Since

$$\sum_{j \in S} y_j \geq \sum_{j \in S} v^M(\{j\}) \geq 0$$

and

$$\begin{aligned} \sum_{j \in S} y_j &= v^M(N) - \sum_{j \in N \setminus S} y_j \\ &\geq v^M(N) - \sum_{j \in N \setminus S} M_j(v^M). \end{aligned}$$

we find that

$$\sum_{j \in S} y_j \geq v_{\hat{E}, \hat{d}}(S).$$

Potters and Tijs (1994) prove that for any two games $v, w \in TU^N$ with $C(v) = C(w)$ and v convex it holds that $n(v) = n(w)$. From this we may conclude that

$$n(v^M) = n(v_{\hat{E}, \hat{d}})$$

and hence

$$n(v^M) = AM(\hat{E}, \hat{d}) = AM(v^M(N), M(v^M)).$$

□

Appendix

In this Appendix we present the proofs of Lemma 3.2, Lemma 3.3, and Theorem 4.2.

Proof of Lemma 3.2. Let $S \subset T \subset N$. We will prove that

$$k(S) \leq k(T).$$

First assume that $D(T) = 0$. Then $k(T) = 1$ and

$$E \leq \sum_{j \in N \setminus T} d_j.$$

Hence,

$$E \leq \sum_{j \in N \setminus T} d_j + \sum_{j \in T \setminus S} d_j = \sum_{j \in N \setminus S} d_j.$$

This implies $D(S) = 0$ and $k(S) = 1$.

Next assume that $D(T) > 0$. First note that

$$\begin{aligned} D(S) &= \max \left\{ 0, E - \sum_{j \in N \setminus S} d_j \right\} \\ &= \max \left\{ 0, E - \sum_{j \in N \setminus T} d_j - \sum_{j \in T \setminus S} d_j \right\} \\ &= \max \left\{ 0, D(T) - \sum_{j \in T \setminus S} d_j \right\}. \end{aligned}$$

If $D(S) = 0$, then $k(S) = 1 \leq k(T)$ by definition. Therefore, we may assume that $D(S) > 0$ and $D(S) = D(T) - \sum_{j \in T \setminus S} d_j$. It follows that

$$\begin{aligned} \sum_{l=1}^{k(S)-1} d(T_l) &= \sum_{l=1}^{k(S)-1} d(S_l) + \sum_{l=1}^{k(S)-1} d(T_l \setminus S_l) \\ &\leq D(S) + \sum_{l=1}^{k(S)-1} d(T_l \setminus S_l) \\ &\leq D(S) + \sum_{j \in T \setminus S} d_j \\ &= D(T) \\ &< \sum_{l=1}^{k(T)} d(T_l) \end{aligned}$$

where the first and last inequality follow from (6). This implies that $k(S) - 1 < k(T)$ and consequently $k(S) \leq k(T)$. \square

Proof of Lemma 3.3. Let $l \in \{1, \dots, m\}$. We will prove that

$$\sum_{j \in S_l} x_j^S \leq \sum_{j \in S_l} x_j^T.$$

If $D(S) = 0$, then it holds that

$$\sum_{j \in S_l} x_j^S = 0 \leq \sum_{j \in S_l} x_j^T.$$

Assume $D(S) > 0$. As before, this implies that $D(S) = D(T) - \sum_{j \in T \setminus S} d_j$. If $l > k(S)$, then $x_j^S = 0$ for all $j \in S_l$ and

$$\sum_{j \in S_l} x_j^S = 0 \leq \sum_{j \in S_l} x_j^T.$$

If $l < k(T)$, then it holds that $x_j^T = d_j$ for all $j \in T_l$. Hence, since $S_l \subset T_l$,

$$\sum_{j \in S_l} x_j^S \leq \sum_{j \in S_l} d_j = \sum_{j \in S_l} x_j^T.$$

Since $S \subset T$, it follows from Lemma 3.2 that $k(S) \leq k(T)$. Therefore, the only case that remains to be considered is $l = k(S) = k(T)$. In that case

$$\begin{aligned}
\sum_{j \in S_l} x_j^S &= D(S) - \sum_{t=1}^{l-1} d(S_t) \\
&\leq D(S) - \sum_{t=1}^{l-1} d(S_t) + d(T_l \setminus S_l) - \sum_{j \in T_l \setminus S_l} x_j^T + \sum_{t=l+1}^m d(T_t \setminus S_t) \\
&= D(S) - \sum_{t=1}^{l-1} d(S_t) - \sum_{t=1}^{l-1} d(T_t \setminus S_t) + \sum_{j \in T \setminus S} d_j - \sum_{j \in T_l \setminus S_l} x_j^T \\
&= D(S) + \sum_{j \in T \setminus S} d_j - \sum_{t=1}^{l-1} d(T_t) - \sum_{j \in T_l \setminus S_l} x_j^T \\
&= D(T) - \sum_{t=1}^{l-1} d(T_t) - \sum_{j \in T_l \setminus S_l} x_j^T \\
&= \sum_{j \in T_l} x_j^T - \sum_{j \in T_l \setminus S_l} x_j^T \\
&= \sum_{j \in S_l} x_j^T
\end{aligned}$$

where the first and fifth equality hold by Theorem 2.1. □

Proof of Theorem 4.2. Let $S \in 2^N \setminus \{\emptyset\}$. We will prove that

$$v^M(S) \leq \max \left\{ 0, v^M(N) - \sum_{j \in N \setminus S} M_j(v^M) \right\}. \quad (8)$$

For $D(S) = 0$ it holds that $v^M(S) = 0$ and inequality (8) is easily verified.

Let $D(S) > 0$. This implies that $v^M(S) > 0$ and $d_i < E$ for all $i \in N \setminus S$. It remains to prove that

$$v^M(N) - \sum_{j \in N \setminus S} M_j(v^M) \geq v^M(S) \quad (9)$$

Set $k = k(N)$. First let $\{N_l \setminus S_l | l \in \{k, \dots, m\}\} = \emptyset$. This tells us that

$$D(S) = E - \sum_{j \in N \setminus S} d_j = E - \sum_{l=1}^{k-1} d(N_l \setminus S_l),$$

which implies that $k(S) = k$. Hence,

$$\begin{aligned}
v^M(N) - \sum_{j \in N \setminus S} M_j(v^M) &= v^M(N) - \sum_{j \in N_l \setminus S_l : l < k} M_j(v^M) \\
&= \sum_{l=1}^{k-1} \beta_l d(N_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right) - \sum_{l=1}^{k-1} \beta_l d(N_l \setminus S_l) \\
&= \sum_{l=1}^{k-1} \beta_l d(S_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=1}^{k-1} \beta_l d(S_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l \setminus S_l) - \sum_{l=1}^{k-1} d(S_l) \right) \\
&= \sum_{l=1}^{k-1} \beta_l d(S_l) + \beta_k \left(D(S) - \sum_{l=1}^{k-1} d(S_l) \right) \\
&= v^M(S).
\end{aligned}$$

Here, the second and last equality follows from Theorem 3.1 and Lemma 4.1(2a).

Secondly, let $\{N_l \setminus S_l | l \in \{k, \dots, m\}\} \neq \emptyset$ and assume that $k(N_{-i}) = k$ for all $i \in \bigcup_{l=k}^m (N_l \setminus S_l)$.

We will prove inequality (9) by showing

$$v^M(N) - \sum_{j \in N_i \setminus S_i : l < k} M_j(v^M) - v^M(S) \geq \sum_{j \in N_i \setminus S_i : l \geq k} M_j(v^M).$$

For this,

$$\begin{aligned}
&v^M(N) - \sum_{j \in N_i \setminus S_i : l < k} M_j(v^M) - v^M(S) \\
&= \sum_{l=1}^{k-1} \beta_l d(N_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right) - \sum_{l=1}^{k-1} \beta_l d(N_l \setminus S_l) \\
&\quad - \sum_{l=1}^{k(S)-1} \beta_l d(S_l) - \beta_{k(S)} \left(E - \sum_{j \in N \setminus S} d_j - \sum_{l=1}^{k(S)-1} d(S_l) \right) \\
&= \beta_{k(S)} \left(\sum_{l=1}^{k(S)} d(S_l) - E + \sum_{j \in N \setminus S} d_j \right) + \sum_{l=k(S)+1}^{k-1} \beta_l d(S_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right) \\
&\geq \beta_k \left(\sum_{l=1}^{k(S)} d(S_l) - E + \sum_{j \in N \setminus S} d_j + \sum_{l=k(S)+1}^{k-1} d(S_l) + E - \sum_{l=1}^{k-1} d(N_l) \right) \\
&= \beta_k \left(\sum_{l=k}^m d(N_l \setminus S_l) \right) \\
&= \beta_k \left(\sum_{j \in N_i \setminus S_i : l \geq k} d_j \right) \\
&= \sum_{j \in N_i \setminus S_i : l \geq k} M_j(v^M).
\end{aligned}$$

Here, the first equality follows from Theorem 3.1 and Lemma 4.1(2a), the inequality because $\beta_k < \beta_l$ for all $l \in \{k(S), \dots, k-1\}$. The last equality follows from the fact that $M_i(v^M) = \beta_k d_i$ for all $i \in \bigcup_{l=k}^m (N_l \setminus S_l)$.

Thirdly, let $\{N_l \setminus S_l | l \in \{k, \dots, m\}\} \neq \emptyset$ and let agent $i^* \in \bigcup_{l=k}^m (N_l \setminus S_l)$ be such that $k(N_{-i^*}) < k$. Without loss of generality, assume that $d_{i^*} \geq d_j$ for all $j \in \bigcup_{l=k}^m (N_l \setminus S_l)$. We will

prove (9) by showing

$$v^M(N) - \sum_{j \in N_i \setminus S_i: l < k} M_j(v^M) - v^M(S) - M_{i^*}(v^M) \geq \sum_{j \in N_i \setminus S_i: l \geq k, j \neq i^*} M_j(v^M).$$

For this, observe that

$$\begin{aligned}
& v^M(N) - \sum_{j \in N_i \setminus S_i: l < k} M_j(v^M) - v^M(S) - M_{i^*}(v^M) \\
&= \sum_{l=1}^{k-1} \beta_l d(N_l) + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right) - \sum_{l=1}^{k-1} \beta_l d(N_l \setminus S_l) \\
&\quad - \sum_{l=1}^{k(S)-1} \beta_l d(S_l) - \beta_{k(S)} \left(E - \sum_{j \in N \setminus S} d_j - \sum_{l=1}^{k(S)-1} d(S_l) \right) \\
&\quad - \beta_{k(N_{-i^*})} \left(\sum_{l=1}^{k(N_{-i^*})} d(N_l) - E + d_{i^*} \right) - \sum_{l=k(N_{-i^*})+1}^{k-1} \beta_l d(N_l) - \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right) \\
&= \beta_{k(S)} \left(\sum_{l=1}^{k(S)} d(S_l) - E + \sum_{j \in N \setminus S} d_j \right) + \sum_{l=k(S)+1}^{k-1} \beta_l d(S_l) \\
&\quad - \beta_{k(N_{-i^*})} \left(\sum_{l=1}^{k(N_{-i^*})} d(N_l) - E + d_{i^*} \right) - \sum_{l=k(N_{-i^*})+1}^{k-1} \beta_l d(N_l) \\
&= \beta_{k(S)} \left(\sum_{l=1}^{k(S)} d(S_l) - E + \sum_{j \in N \setminus S} d_j \right) + \sum_{l=k(S)+1}^{k(N_{-i^*})-1} \beta_l d(S_l) \\
&\quad + \beta_{k(N_{-i^*})} \left(d(S_{k(N_{-i^*})}) - \sum_{l=1}^{k(N_{-i^*})} d(N_l) + E - d_{i^*} \right) - \sum_{l=k(N_{-i^*})+1}^{k-1} \beta_l d(N_l \setminus S_l) \\
&\geq \beta_{k(N_{-i^*})} \left(\sum_{l=1}^{k(S)} d(S_l) - E + \sum_{j \in N \setminus S} d_j + \sum_{l=k(S)+1}^{k(N_{-i^*})-1} d(S_l) \right. \\
&\quad \left. + d(S_{k(N_{-i^*})}) - \sum_{l=1}^{k(N_{-i^*})} d(N_l) + E - d_{i^*} - \sum_{l=k(N_{-i^*})+1}^{k-1} d(N_l \setminus S_l) \right) \\
&= \beta_{k(N_{-i^*})} \left(\sum_{j \in N_i \setminus S_i: l \geq k} d_j - d_{i^*} \right) \\
&= \beta_{k(N_{-i^*})} \left(\sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N_{-j})=k}} d_j + \sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N_{-j}) < k; j \neq i^*}} d_j \right) \\
&\geq \sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N_{-j})=k}} \beta_k d_j + \sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N_{-j}) < k; j \neq i^*}} \beta_{k(N_{-j})} d_j
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N-j)=k}} M_j(v^M) + \sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N-j) < k; j \neq i^*}} \beta_{k(N-j)} \left(\sum_{l=1}^{k(N-j)} d(N_l) - E + d_j + \sum_{l=k(N-j)+1}^{k-1} d(N_l) \right. \\
&\quad \left. + E - \sum_{l=1}^{k-1} d(N_l) \right) \\
&\geq \sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N-j)=k}} M_j(v^M) + \sum_{\substack{j \in N_i \setminus S_i: l \geq k; \\ k(N-j) < k; j \neq i^*}} \left(\beta_{k(N-j)} \left(\sum_{l=1}^{k(N-j)} d(N_l) - E + d_j \right) + \sum_{l=k(N-j)+1}^{k-1} \beta_l d(N_l) \right. \\
&\quad \left. + \beta_k \left(E - \sum_{l=1}^{k-1} d(N_l) \right) \right) \\
&= \sum_{j \in N_i \setminus S_i: l \geq k; j \neq i^*} M_j(v^M).
\end{aligned}$$

The first equality follow from Theorem 3.1, Lemma 4.1(2a,2b). The third equality holds by using Lemma 3.2 such that $k(N_{-i^*}) \geq k(S)$ for $i^* \in N \setminus S$. The first and second inequality are due to the fact that $\beta_{k(N_{-i^*})} < \beta_l$ for all $l \in \{k(S), \dots, k(N_{-i^*}) - 1\}$, $\beta_{k(N_{-i^*})} > \beta_l$ for all $l \in \{k(N_{-i^*}) + 1, \dots, k - 1\}$, and $\beta_{k(N_{-i^*})} \geq \beta_{k(N-j)}$ for all $j \in \bigcup_{l=k}^m (N_l \setminus S_l)$ \square

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