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Necessary and Sufficient Conditions for Feedback Nash Equilibria for the Affine-Quadratic Differential Game

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Abstract

In this note we consider the non-cooperative linear feedback Nash quadratic differential game with an infinite planning horizon. The performance function is assumed to be indefinite and the underlying system affine. We derive both necessary and sufficient conditions under which this game has a Nash equilibrium.

Keywords: linear-quadratic games, linear feedback Nash equilibrium, affine systems, solvability conditions, Riccati equations.

Jel-codes: C61, C72, C73.

1 Introduction

In the last decades, there is an increased interest in studying diverse problems in economics and optimal control theory using dynamic games. In particular in environmental economics and macroeconomic policy coordination, dynamic games are a natural framework to model policy coordination problems (see e.g. the books and references in Dockner et al. [5], Jørgensen et al. [13], Plasmans et al. [19] and Grass et al. [10]). Moreover, in optimal control theory it is well-known that, e.g., the issue to obtain robust control strategies can be approached as a dynamic game problem (see e.g. [1], [15] and [3]).

In this note we consider the linear quadratic differential game under a feedback information structure. The reason to consider this information structure is that the corresponding linear feedback Nash equilibria (FBNE) have the nice property of strong time consistency. A property which, e.g., does not hold under an open-loop information structure.

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This problem has been considered by many authors and dates back to the seminal work of Starr and Ho in [20].

For the fixed finite planning horizon, one can show that there exists at most one FBNE (see e.g. [16], [17]). The question whether a solution exists depends on the solvability of a related set of coupled Riccati differential equations. Global existence and convergence properties of solutions of these differential equations has, e.g., been studied in [18], [9] and [21]. Further, the problem of calculating the solutions of these differential equations was considered in, e.g., [4] and [12]. In [2] the more general affine-quadratic differential game was considered, and conditions were derived under which the game admits a FBN solution, affine in the current state of the system. In both [2] and [6] one can find additional references and generalizations of the above results. In particular one can find here results for an infinite planning horizon and indefnite cost functions (that is the case that the state weighting matrices Q_i (see below) are indefinite). Some more recent generalizations are [7], where the game problem is solved assuming that the players use static output feedback control, and [8], where the problem is considered for descriptor systems.

All of the above results are, for an infinite planning horizon, formulated for a performance criterion that is a pure quadratic form of the state and control variables. In this note we generalize this result for performance criteria that also include "cross-terms", i.e. products of the state and control variables. Performance criteria of this type often naturally appear in economic policy making. Moreover, we assume that the linear system describing the dynamics is affected by a deterministic variable.

The outline of this note is as follows. Section two introduces the problem and contains some preliminary results. The main results of this paper are stated in Section three, whereas Section four contains some concluding remarks. The proof of the main theorem is included in the Appendix.

2 Preliminaries

In this paper we assume that player $i \in \bar{N}$ (see the end of this paper for the introduced notation) likes to minimize w.r.t. u_i: $\lim_{t_f \to \infty} J_i(t_f, x_0, u_1, \cdots, u_N)$

$$\text{where } J_i(t_f, x_0, u_1, \cdots, u_N) := \int_0^{t_f} [x^T(t),\ u_1^T(t), \cdots,\ u_N^T(t)] M_i \begin{bmatrix} x(t) \\ u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix} dt, \tag{1}$$

$$M_i = \begin{bmatrix} Q_i & V_{i11} & \cdots & \cdots & V_{i1N} \\ V_{i11}^T & R_{i1} & V_{i22} & \cdots & V_{i2N} \\ & & \ddots & & \\ V_{i1N}^T & V_{i2N}^T & \cdots & \cdots & R_{iN} \end{bmatrix},\ M_i = M_i^T,\ R_{ii} > 0,\ i \in \bar{N},$$

and $x(t)$ satisfies the linear differential equation

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t) + c(t),\ x(0) = x_0. \tag{2}$$

The variable $c \in L_2(0, \infty)$ here is some given trajectory. Notice that we make no definiteness assumptions w.r.t. matrix Q_i.

We assume that the matrix pairs $(A, B_i),\ i \in \bar{N}$, are stabilizable. So, in principle, each player is capable to stabilize the system on his own.

The linear feedback information structure of the game means that both players know the current state of the system and that the set of admissible control actions are affine functions of the current state of the system[1]

$$\mathcal{U}_s = \left\{ u = [u_1^T \ \cdots u_N^T]^T \ | u_i(t) = F_i x(t) + g_i(t), \text{ where } g_i \in L_2(0,\infty) \text{ and } \sigma(A+BF) \subset \mathbb{C}^- \right\}.$$

Notice that the assumption that the players use simultaneously stabilizing controls introduces the cooperative meta-objective of both players to stabilize the system (see e.g. [6] for a discussion).

Then, $u^* := (u_1^*, \cdots, u_N^*) \in \mathcal{U}_s$ is called a feedback Nash equilibrium if the usual inequalities apply, i.e., no player can improve his performance by a unilateral deviation from this set of equilibrium actions. Introducing the notation $u^*_{-i}(\alpha) := u^*$ where u_i^* has been replaced by the arbitrary input function α the formal definition reads as follows

Definition 2.1 $((F_1^*, g_1^*), \cdots, (F_N^*, g_N^*))$ or $u^* \in \mathcal{U}_s$ is called a *feedback Nash equilibrium* if for $i \in \bar{N}$, $J_i(x_0, u^*) \le J_i(x_0, u^*_{-i}(\alpha))$ for every x_0 and input α such that $u^*_{-i}(\alpha) \in \mathcal{U}_s$. □

3 Main results

In the Appendix the following theorem is proved.

Theorem 3.1 *The affine differential game (1,2) has a feedback Nash equilibrium* $((F_1, g_1), \cdots, (F_N, g_N))$ *for every initial state if and only if*

$$F = -G^{-1}(Z + \tilde{B}^T K) \tag{3}$$

and

$$g = -G^{-1}\tilde{B}^T m(t). \tag{4}$$

Here $K_i,\ i \in \bar{N}$, are symmetric solutions of the coupled algebraic Riccati equations

$$A_{cl}^T K_i + K_i A_{cl} + K_i S_i K_i + [I \ F_{-i}^T]\bar{M}_i \begin{bmatrix} I \\ F_{-i} \end{bmatrix} = 0,\ i \in \bar{N}, \tag{5}$$

that have the property that $\sigma(A_{cl}) \subset \mathbb{C}^-$, where $A_{cl} := A + BF$. Further, $m(t)$ is the unique solution of the integral equation

$$m(t) = \int_t^\infty \text{diag}(e^{-A_{cl}^T(t-s)}) \left\{ -\text{col}\left(\left(K_i B_{-i} + [I \ F^T] M_{i,/1} \right) I_{N,-i} \right) G^{-1} \tilde{B}^T m(s) + Kc(s) \right\} ds. \tag{6}$$

□

[1] $\sigma(H)$ denotes the spectrum of matrix H; $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) < 0\}$; $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) \ge 0\}$.

Remark 3.2
Introducing $H := \text{diag}(-A_{cl}^T) + \text{col}\left(\left(K_iB_{-i} + [I\ F^T]M_{i,/1}\right) I_{N,-i}\right) G^{-1}\tilde{B}^T$ we obtain by differentiation of (6) that $m(t)$ solves the next differential equation

$$\dot{m}(t) = Hm(t) - Kc(t). \tag{7}$$

□

Corollary 3.3 *Assume that $\sigma(H) \subset \mathbb{C}^+$, where H is as defined in Remark 3.2 above. Then* (6) *has the unique solution*

$$m(t) = \int_t^\infty e^{H(t-s)}Kc(s)ds. \tag{8}$$

Proof. Clearly, due to our assumption on $\sigma(H)$, $m(t)$ in (8) is well-defined. By straightforward differentiation it follows that $m(t)$ satisfies (7). Next, consider the right-hand side of (6). Using (7) we have

$$\begin{aligned}
&\int_t^\infty \text{diag}(e^{-A_{cl}^T(t-s)}) \left\{-\text{col}\left(\left(K_iB_{-i} + [I\ F^T]M_{i,/1}\right) I_{N,-i}\right) G^{-1}\tilde{B}^T m(s) + Kc(s)\right\} ds \\
&= \text{diag}(e^{-A_{cl}^T t}) \int_t^\infty \text{diag}(e^{A_{cl}^T(s)}) \left\{\left(\text{diag}(-A_{cl}^T) - H\right) m(s) + Kc(s)\right\} ds \\
&= \text{diag}(e^{-A_{cl}^T t}) \int_t^\infty \text{diag}(e^{A_{cl}^T(s)}) \left\{\text{diag}(-A_{cl}^T)m(s) - \dot{m}(s)\right\} ds \\
&= \text{diag}(e^{-A_{cl}^T t}) \int_t^\infty -\frac{d}{ds}\left\{\text{diag}(e^{A_{cl}^T(s)})m(s)\right\} ds \\
&= m(t).
\end{aligned}$$

So, $m(t)$ satisfies (6). Since (6) has a unique solution this concludes the proof. □

Corollary 3.4 In case $c(.) = 0$ it is clear that $m_i = 0$ and it follows that $J_i = x_0^T K_i x_0$. □

Notice that in case the set of algebraic Riccati equations (5) has more than one set of stabilizing solutions, there exists more than one FBNE equilibrium. This may happen even if all Q_i matrices are positive definite (see e.g. [6][Theorem 8.10]).

Remark 3.5 Consider the two-player zero-sum game, i.e. $J_1 = -J_2$, where for simplicity of notation we denote $M_1 =: \begin{bmatrix} Q_1 & V_1 & W_1 \\ V_1^T & R_{11} & N_1 \\ W_1^T & N_1^T & R_{12} \end{bmatrix}$. By addition of the two equations we get from (5) (followed by some elementary rewriting) that K_i satisfy the equation

$$A_{cl}^T(K_1 + K_2) + (K_1 + K_2)A_{cl} = 0.$$

Since A_{cl} is a stable matrix it follows from this linear matrix equation that necessarily $K_1 + K_2 = 0$. So we have that $K_2 = -K_1$. Substitution of this into (5) shows that these equations have a stabilizing solution if and only if the equation

$$A^TK_1 + K_1A + Q_1 - [V_1 + K_1B_1\ W_1 + K_1B_2]G^{-1}[V_1 + K_1B_1\ \ -(W_1 + K_1B_2)]^T = 0 \tag{9}$$

has a solution K_1 such that $\sigma(A - [B_1 \;\; B_2]G^{-1}\begin{bmatrix} V_1^T + B_1^T K_1 \\ -W_1^T - B_2^T K_1 \end{bmatrix}) \subset \mathbb{C}^-$. Notice that $G = \begin{bmatrix} R_{11} & N_1 \\ -N_1^T & R_{22} \end{bmatrix}$. Since R_{11} is invertible one can use, e.g., the expression for the inverse of a block matrix (see e.g. [14, p.656]) to verify that $G^{-1} =: \begin{bmatrix} G_{11}^{inv} & G_{12}^{inv} \\ G_{21}^{inv} & G_{22}^{inv} \end{bmatrix}$ has this property too, i.e., $G_{12}^{inv} = -G_{21}^{inv^T}$. Consequently,

$[V_1 + K_1 B_1 \; W_1 + K_1 B_2]G^{-1}\begin{bmatrix} V_1^T + B_1^T K_1 \\ -W_1^T - B_2^T K_1 \end{bmatrix} =$

$$\begin{aligned} &= [V_1 + K_1 B_1 \; W_1 + K_1 B_2]\begin{bmatrix} G_{11}^{inv} & -G_{21}^{inv^T} \\ G_{21}^{inv} & G_{22}^{inv} \end{bmatrix}\begin{bmatrix} V_1^T + B_1^T K_1 \\ -W_1^T - B_2^T K_1 \end{bmatrix} \\ &= [V_1 + K_1 B_1 \; W_1 + K_1 B_2]\begin{bmatrix} G_{11}^{inv} & G_{21}^{inv^T} \\ G_{21}^{inv} & -G_{22}^{inv} \end{bmatrix}\begin{bmatrix} V_1^T + B_1^T K_1 \\ W_1^T + B_2^T K_1 \end{bmatrix} \end{aligned}$$

is clearly symmetric too.
So (9) is an ordinary Riccati equation, from which we know that it has at most one stabilizing solution. Therefore, we conclude that the zero-sum game has a solution if and only if (9) has a stabilizing solution. Furthermore, in case the game has a solution the equilibrium actions are unique and given by

$$u_i(t) = F_i x(t) + g_i(t), \text{ where } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -G^{-1}\begin{bmatrix} B_1^T K_1 + V_1^T \\ -B_2^T K_1 + W_1^T \end{bmatrix}, \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = -G^{-1}\begin{bmatrix} B_1^T m_1(t) \\ B_2^T m_2(t) \end{bmatrix}$$

and $m_i(t),\; i = 1, 2$, are given by

$$m_1(t) = \int_t^\infty e^{-A_{cl}^T (t-s)} K_1 c(s) ds \text{ and } m_2(t) = -m_1(t).$$

This, since (see (3)) both $(K_1 B_2 + W_1 - F_2^T R_{22} + F_1^T N_1) = 0$ and $(K_1 B_1 + V_1 + F_1^T R_{11} + F_2^T N_1^T) = 0$.
□

4 Concluding Remarks

In this note we considered the affine regular indefinite infinite-planning horizon linear-quadratic differential game. Both necessary conditions and sufficient conditions were derived for the existence of an affine feedback Nash equilibrium. Since Q_i are assumed to be indefinite, the obtained results were directly used to solve the zero-sum game. We showed that this game has at most one equilibrium. Further, (assuming that the system is not corrupted by noise) the equilibrium actions coincide with those one obtains for the open-loop information case if the uncontrolled system is stable. However, since the open-loop result requires some additional conditions to be satisfied, we conclude that the realization of a zero-sum Nash equilibrium under an open-loop information setting will less often occur than under a feedback information setting.

Notation

The next shorthand notation will be used.
$k := n + \sum_{i=1}^{N} m_i$; $\bar{N} := \{1, \cdots, N\}$.
$\text{col}(D_i) := \left[D_1^T, \cdots, D_N^T\right]^T$; $\text{diag}(D_i)$ is the diagonal matrix where the i^{th} diagonal entry equals D_i.
If $D = [D_1, \ \cdots, \ D_N]$, D_{-i} is obtained from D by replacing the i^{th} entry by a zero entry which has the same size as D_i, i.e. $D_{-i} := [D_1, \cdots, D_{i-1}, \ 0 \ , D_{i+1}, \cdots, D_N]$.
$I_{N,-i}$ is obtained from the identity matrix by replacing the $i^{th} m_i \times m_i$ identity block matrix by the zero matrix of the same size, i.e. $I_{N,_i} := \text{diag}(I_{m_1}, \cdots, I_{m_{i-1}}, 0_{m_i}, I_{m_{i+1}}, \cdots, I_N)$.
$M_{i,/1}$ is obtained from M_i by dropping its first n columns, i.e. $M_{i,/1} = M_i[0_{(n-k)\times n} \ I_{n-k}]^T$.
$I_{N,-i}$ is obtained from the identity matrix by replacing the $i^{th} m_i \times m_i$ identity block matrix by the zero matrix of the same size, i.e. $I_{N,_i} := \text{diag}(I_{m_1}, \cdots, I_{m_{i-1}}, 0_{m_i}, I_{m_{i+1}}, \cdots, I_N)$.
$I_{N+1,-m_i}$ is obtained from the $k \times k$ identity matrix by replacing the $(1+i)^{th} m_i \times m_i$ identity block matrix by the zero matrix of the same size, i.e. $I_{N+1,-m_i} := \text{diag}(I_n, I_{m_1}, \cdots, I_{m_{i-1}}, 0_{m_i}, I_{m_{i+1}}, \cdots, I_N)$.
E_i is obtained from the column matrix containing $N+1$ zero blocks, where block i is replaced by the identity matrix, i.e. $E_i^T = [0 \cdots 0 \ I \ 0 \cdots 0]$.
$B := [B_1, \cdots, B_N]$; $\widetilde{B}^T := \text{diag}(B_1^T, B_2^T, \cdots, B_N^T)$.
$F := [F_1^T, \cdots, F_N^T]^T$; $g := [g_1^T, \cdots, g_N^T]^T$; $m := [m_1^T, \cdots, m_N^T]^T$; $K := [K_1^T, \cdots, K_N^T]^T$.
$S_i := B_i R_{ii}^{-1} B_i^T$; $\bar{M}_i := M_i - M_i E_{i+1}^T R_{ii}^{-1} E_{i+1}^T M_i$.
Row i of matrix G equals row i of matrix M_i, excluding its first entry, i.e.

$$
\begin{aligned}
G : \ &= \begin{bmatrix} [0 \ I_{m_1} \ 0 \cdots\cdots 0] \, M_1 \\ [0 \ 0 \ I_{m_2} \ 0 \ \cdots \ 0] \, M_2 \\ \ddots \\ [0 \ \cdots\cdots \ 0 \ I_{m_N}] \, M_N \end{bmatrix} \begin{bmatrix} 0_{n\times(k-n)} \\ I_{k-n} \end{bmatrix} \\
&= \begin{bmatrix} R_{11} & V_{122} & \cdots & \cdots & \cdots & V_{12N} \\ V_{222}^T & R_{22} & V_{233} & \cdots & \cdots & V_{23N} \\ V_{323}^T & V_{333}^T & R_{33} & V_{344} & \cdots & V_{34N} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ V_{(N-1)2(N-1)}^T & V_{(N-1)3(N-1)}^T & \cdots & V_{(N-1)(N-1)(N-1)}^T & R_{(N-1)(N-1)} & V_{(N-1)NN} \\ V_{N2N}^T & V_{N3N}^T & \cdots & \cdots & V_{NNN}^T & R_{NN} \end{bmatrix}.
\end{aligned}
$$

We assume throughout that this matrix G is invertible.
Entry i of matrix Z is the $(i+1)^{th}$ entry of the first column of matrix M_i, i.e.

$$
Z \ := \begin{bmatrix} [0 \ I \ 0 \ 0 \ \cdots \ 0] \, M_1 \\ [0 \ 0 \ I \ 0 \ \cdots \ 0] \, M_2 \\ \vdots \\ [0 \ 0 \ 0 \ 0 \ \cdots \ I] \, M_N \end{bmatrix} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} V_{111}^T \\ V_{212}^T \\ \vdots \\ V_{N1N}^T \end{bmatrix}.
$$

Appendix

Theorem 4.1 *Let $S := BR^{-1}B^T$. Consider the minimization of the linear quadratic cost function*

$$
\int_0^\infty x^T(t)Qx(t) + 2p^T(t)x(t) + u^T(t)Ru(t)dt \tag{10}
$$

subject to the state dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + c(t, x_0),\ x(0) = x_0, \tag{11}$$

and $u \in \mathcal{U}_s(x_0)$. *Then,*
1. with $c(.) = p(.) = 0$, *(10,11) has a solution for all* $x_0 \in I\!R^n$ *if and only if the algebraic Riccati equation*

$$A^T K + KA - KSK + Q = 0$$

has a symmetric stabilizing solution $K(.)$ *(i.e.* $A - SK$ *is a stable matrix).*
2. for every x_0, *(10,11) with* $c(., x_0),\ p(.) \in L_2$, *has a solution iff. item 1 has a solution. Moreover if this problem has a solution then the problem has the unique solution*

$$u^*(t) = -R^{-1}B^T(Kx^*(t) + m(t)).$$

Here $m(t)$ *is given by*

$$m(t) = \int_t^\infty e^{-(A-SK)^T(t-s)}(Kc(s) + p(s))ds,$$

and $x^*(t)$ *satisfies*

$$\dot{x}^*(t) = (A - SK)x^*(t) - Sm(t) + c(t),\ x^*(0) = x_0.$$

Proof. Similar to the proof of [6, Theorem 5.16]. □

Proof of Theorem 3.1.
"⇒ part" Suppose that u^* is a Nash solution. Then in particular we have that

$$J_1(x_0, u^*) \le J_1(x_0, u^*_{-1}(\alpha))$$

for every x_0 and input α such that $u^*_{-1}(\alpha) \in \mathcal{U}_s$ From this inequality we see that for every $x_0 \in I\!R^n$ the (nonhomogeneous) linear quadratic control problem to minimize

$$J_1 = \int_0^\infty \{\left(x^T[I\ F^{*^T}_{-1}] + [0\ g^{*^T}_{-1}] + u_1^T(t)E_2^T\right) M_1 \left(\begin{bmatrix} I \\ F^*_{-1}\end{bmatrix} x(t) + \begin{bmatrix} 0 \\ g^*_{-1}(t)\end{bmatrix} + E_2 u_1(t)\right)\}dt,$$

subject to the (nonhomogeneous) state equation

$$\dot{x}(t) = (A + B_{-1}F^*_{-1})x(t) + B_1u_1(t) + B_{-1}g^*_{-1}(t) + c(t),\ x(0) = x_0,$$

has a solution. Straightforward calculations show that $E_2^T M_1(I - E_2 R_{11}^{-1} E_2^T M_1) = 0$. Therefore, with

$$v_1(t) := u_1(t) + R_{11}^{-1}E_2^T M_1 \left(\begin{bmatrix} I \\ F^*_{-1}\end{bmatrix} x(t) + \begin{bmatrix} 0 \\ g^*_{-1}(t)\end{bmatrix}\right) \tag{12}$$

the above minimization problem can be rewritten as the minimization of

$$J_1 = \int_0^\infty \{x^T(t)[I\ F^{*^T}_{-1}]\bar{M}_1 \begin{bmatrix} I \\ F^*_{-1}\end{bmatrix} x(t) + v_1^T(t)R_{11}v_1(t) \tag{13}$$
$$+ 2[0\ g^{*^T}_{-1}(t)]\bar{M}_1 \begin{bmatrix} I \\ F^*_{-1}\end{bmatrix} x(t) + [0\ g^{*^T}_{-1}(t)]\bar{M}_1 \begin{bmatrix} 0 \\ g^*_{-1}(t)\end{bmatrix}\}dt,$$

subject to the (nonhomogeneous) state equation

$$\dot{x}(t) = (A - B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix} + B_{-1}F_{-1}^*)x(t) + B_1v_1(t) + B_{-1}g_{-1}^*(t) \qquad (14)$$
$$-B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} 0 \\ g_{-1}^*(t) \end{bmatrix} + c(t),\ x(0) = x_0,$$

This implies, see Theorem 4.1, that the algebraic Riccati equation

$$[A + B_{-1}F_{-1}^* - B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix}]^TK_1 + K_1[A + B_{-1}F_{-1}^* - B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix}] -$$
$$K_1S_1K_1 + [I\ F_{-1}^{*^T}]\bar{M}_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix} = 0 \qquad (15)$$

has a stabilizing solution.
According Theorem 4.1 the minimization problem (13,14) has a unique solution. Introducing for notational convenience $\bar{A}_1 := A + B_{-1}F_{-1}^* - B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix} - S_1K_1$, its solution is

$$\tilde{v}_1(t) = -R_{11}^{-1}B_1^T(K_1x(t) + m_1(t)) \qquad (16)$$

$$\text{with } m_1(t) = \int_t^\infty e^{-\bar{A}_1^T(t-s)}\{K_1n_1(s) + p_1(s)\}ds, \qquad (17)$$

where $p_1^T(s) = [0\ g_{-1}^{*^T}]\bar{M}_1\begin{bmatrix} 0 \\ g_{-1}^*(s) \end{bmatrix}$, $n_1(s) = B_{-1}g_{-1}^*(s) - B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} 0 \\ g_{-1}^*(s) \end{bmatrix} + c(s)$ and K_1 the stabilizing solution of the algebraic Riccati equation (20). Consequently, see (12),

$$\begin{aligned} \tilde{u}_1(t) &:= \tilde{v}_1(t) - R_{11}^{-1}E_2^TM_1\left(\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ g_{-1}^*(t) \end{bmatrix}\right) \\ &= -R_{11}^{-1}(B_1^TK_1 + E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix})x(t) - R_{11}^{-1}(B_1^Tm_1(t) + E_2^TM_1\begin{bmatrix} 0 \\ g_{-1}^*(t) \end{bmatrix}) \qquad (18) \end{aligned}$$

solves the original optimization problem. Since the optimal control for this problem is uniquely determined, and by definition the equilibrium control $u_1^* = F_1^*x(t) + g_1^*(t)$ solves the optimization problem, it follows from (18) that the next two equalities hold

$$\begin{aligned} F_1^* &= -R_{11}^{-1}(B_1^TK_1 + E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix}) \\ g_1^*(t) &= -R_{11}^{-1}(B_1^Tm_1(t) + E_2^TM_1\begin{bmatrix} 0 \\ g_{-1}^*(t) \end{bmatrix}). \end{aligned}$$

Analogously we obtain in general that

$$u_i^*(t) = -R_{ii}^{-1}(B_i^TK_i + E_{i+1}^TM_i\begin{bmatrix} I \\ F_{-i}^* \end{bmatrix})x(t) - R_{ii}^{-1}(B_i^Tm_i(t) + E_{i+1}^TM_i\begin{bmatrix} 0 \\ g_{-i}^*(t) \end{bmatrix}) \qquad (19)$$

with $\bar{A}_i := A + B_{-i}F^*_{-i} - B_iR_{ii}^{-1}E_{i+1}^TM_i \begin{bmatrix} I \\ F^*_{-i} \end{bmatrix} - S_iK_i$, $m_i(t) = \int_t^\infty e^{-\bar{A}_i^T(t-s)}(K_in_i(s) + p_i(s))ds$, $p_i^T(s) = [0 \ g^{*T}_{-i}(s)]\bar{M}_i \begin{bmatrix} 0 \\ g^*_{-i}(s) \end{bmatrix}$, $n_i(s) = B_{-i}g^*_{-i}(s) - B_iR_{ii}^{-1}E_{i+1}^TM_i \begin{bmatrix} 0 \\ g^*_{-i}(s) \end{bmatrix} + c(s)$ and K_i the stabilizing solution of the algebraic Riccati equation

$$[A + B_{-i}F^*_{-i} - B_iR_{ii}^{-1}E_{i+1}^TM_i \begin{bmatrix} I \\ F^*_{-i} \end{bmatrix}]^TK_i + K_i[A + B_{-i}F^*_{-i} - B_iR_{ii}^{-1}E_{i+1}^TM_i \begin{bmatrix} I \\ F^*_{-i} \end{bmatrix}] - K_iS_iK_i + [I \ F^{*T}_{-i}]\bar{M}_i \begin{bmatrix} I \\ F^*_{-i} \end{bmatrix} = 0.$$

Again, since the minimizing control is uniquely determined, we conclude from (19) and the fact that by definition $u_i^*(t) = F_i^*x(t) + g_i^*(t))$ that in general the next two equalities hold

$$\begin{aligned} F_i^* &= -R_{ii}^{-1}(B_i^TK_i + E_{i+1}^TM_i \begin{bmatrix} I \\ F^*_{-i} \end{bmatrix}) \quad (20) \\ g_i^*(t) &= -R_{ii}^{-1}(B_i^Tm_i(t) + E_{i+1}^TM_i \begin{bmatrix} 0 \\ g^*_{-i}(t) \end{bmatrix}). \end{aligned}$$

From this we have, in particular, that $\forall i$, $\bar{A}_i = A + BF^* =: A_{cl}$. Furthermore, we conclude from this that

$$\begin{aligned} R_{ii}F_i^* + E_{i+1}^TM_i \begin{bmatrix} I \\ F^*_{-i} \end{bmatrix} &= -(B_i^TK_i + E_{i+1}^TM_iE_1) \quad (21) \\ &= -(B_i^TK_i + V_{i1i}^T), \end{aligned}$$

or,

$$GF^* = -\begin{bmatrix} B_1^TK_1 + V_{111}^T \\ \vdots \\ B_N^TK_N + V_{N1N}^T \end{bmatrix} = -(\tilde{B}^TK + Z).$$

Whereas

$$R_{ii}g_i^* + E_{i+1}^TM_i \begin{bmatrix} 0 \\ g^*_{-i}(t) \end{bmatrix} = -B_i^Tm_i(t), \text{ yielding, } Gg^* = -\begin{bmatrix} B_1^Tm_1(t) \\ \vdots \\ B_N^Tm_N(t) \end{bmatrix} = -\tilde{B}^Tm(t).$$

Furthermore notice from (20) that $F_i^* + R_{ii}^{-1}(B_i^TK_i = -R_{ii}^{-1}E_{i+1}^TM_i \begin{bmatrix} I \\ F^*_{-i} \end{bmatrix})$. Substitution of this into (20) gives (5).

Next, reconsider $m_i(t)$. Substitution of $n_i(s)$ and $p_i(s)$ into this expression yields

$$\begin{aligned} m_i(t) &= \int_t^\infty e^{-A_{cl}^T(t-s)}\{K_ic(s) \\ &\quad + K_iB_{-i}g^*_{-i}(s) - K_iB_{-i}R_{ii}^{-1}E_{i+1}^TM_i \begin{bmatrix} 0 \\ g^*_{-i}(s) \end{bmatrix} + [I \ F^{*T}_{-i}]\bar{M}_i \begin{bmatrix} 0 \\ g^*_{-i}(s) \end{bmatrix}\}ds. \quad (22) \end{aligned}$$

From (21) we have that

$$
\begin{aligned}
-K_iB_iR_{ii}^{-1}E_{i+1}^TM_i &= (F_i^* + R_{ii}^{-1}E_{i+1}^TM_i\begin{bmatrix} I \\ F_{-i}^* \end{bmatrix} + R_{ii}^{-1}E_{i+1}^TM_iE_1)^TE_{i+1}^TM_i \\
&= (F_i^{*^T} + [I\ F_{-i}^{*^T}]M_iE_{i+1}R_{ii}^{-1})E_{i+1}^TM_i.
\end{aligned}
$$

Using this, (22) can be rewritten as

$$
\begin{aligned}
m_i(t) &= \int_t^\infty e^{-A_{cl}^T(t-s)}\{K_ic(s) \\
&\quad +K_iB_{-i}g_{-i}^*(s) + (F_i^{*^T}E_{i+1}^TM_i + [I\ F_{-i}^{*^T}]M_i)\begin{bmatrix} 0 \\ g_{-i}^*(s) \end{bmatrix}\}ds \\
&= \int_t^\infty e^{-A_{cl}^T(t-s)}\{K_ic(s) + K_iB_{-i}g_{-i}^*(s) + [I\ F^{*^T}]M_i\begin{bmatrix} 0 \\ g_{-i}^*(s) \end{bmatrix}\}ds \\
&= \int_t^\infty e^{-A_{cl}^T(t-s)}\{K_ic(s) + K_iB_{-i}g_{-i}^*(s) + [I\ F^{*^T}]M_{i/1}g_{-i}^*(s)\}ds.
\end{aligned}
$$

Since $g_{-i}^* = I_{N,-i}g^* = -I_{N,-i}G^{-1}\tilde{B}^Tm(t)$, the integral equation for m as advertized in (6) results. As $\sigma(A_{cl}) \subset \mathbb{C}^-$ and $c(.) \in L^2$ it follows from, e.g., [11][Theorem 2.1.1] that (6) has a unique solution.

"$\Leftarrow$ part" Let K be a stabilizing solution of (5) and define for $i \neq 1$, $u_i^* := (F_i^*, g_i^*)$ by (3,4,6). Next, without loss of generality, consider the minimization by player one of the cost functional

$$
J_1(x_0, u_1, u_2^*, \cdots, u_N^*) = \int_0^\infty \left\{ [x^T(t),\ u_1^T(t),\ x^T(t)F_{-1}^{*^T}(t) + g_{-1}^{*^T}]M_1 \begin{bmatrix} x(t) \\ u_1(t) \\ F_{-1}^*x(t) + g_{-1}^* \end{bmatrix} \right\} dt,
$$

subject to the system $\dot{x}(t) = (A + B_{-1}F_{-1}^*)x(t) + B_1u_1(t) + B_{-1}g_{-1}^* + c(t),\ x(0) = x_0$.
From the "$\Rightarrow$" part of the proof we have that the problem can be rewritten as the minimization of (13) subject to (14). From (3) it follows (see e.g. (20) again) that (5) can be rewritten as (20). In particular we obtain for $i = 1$ from (20) that the algebraic Riccati equation

$$
[A + B_{-1}F_{-1}^* - B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix}]^TK + K[A + B_{-1}F_{-1}^* - B_1R_{11}^{-1}E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix}] - KS_1K + [I\ F_{-1}^{*^T}]\bar{M}_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix} = 0
$$

has a stabilizing solution $K = K_1$. But this implies, according Theorem 4.1, that the minimization of (13) subject to (14) has a solution. From the "$\Rightarrow$" part of the proof we recall that its solution is given by (16,17). So, using (3) and (4) we see that the optimal control for player one is given by

$$
\begin{aligned}
u_1(t) &= v_1^*(t) - R_{11}^{-1}E_2^TM_1\left(\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ g_{-1}^*(t) \end{bmatrix}\right) \\
&= -R_{11}^{-1}(B_1^TK_1 + E_2^TM_1\begin{bmatrix} I \\ F_{-1}^* \end{bmatrix})x(t) - R_{11}^{-1}(B_1^Tm_1(t) + E_2^TM_1\begin{bmatrix} 0 \\ g_{-1}^*(t) \end{bmatrix}) \\
&= F_1^*x(t) + g_1^*.
\end{aligned}
$$

Or stated differently, (F_1^*, g_1^*) is the optimal response of player one in case all other players i use the control strategy (F_i^*, g_i^*). This proves that this set of control actions constitute a Nash equilibrium for the game. □

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