A connection between positive semidefinite and Euclidean distance matrix completion problems

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Abstract

The positive semidefinite and Euclidean distance matrix completion problems have received a lot of attention in the literature. Interestingly, results have been obtained for these two problems that are very similar in their formulations. Although there is a strong relationship between positive semidefinite matrices and Euclidean distance matrices, it was not clear (as noted in [11]) how to link the two completion problems. The purpose of this note is twofold. First, we show how the results for the Euclidean distance matrix completion problem can be derived from the corresponding results for the positive semidefinite completion problem, using a functional transform introduced by Schoenberg [17]. Second, we introduce a new set of necessary conditions that are stronger than some previously known ones and we identify the graphs for which these conditions suffice for ensuring completability.

1 Introduction

A *partial (symmetric) matrix*[1] $A = (a_{ij})$ is a matrix whose entries are specified only on a subset of the positions, but in such a way that a_{ji} is specified and equal to a_{ij} whenever a_{ij} is specified. The *positive semidefinite completion problem* (*PSD completion problem*, for short) can be formulated as follows: Given a partial symmetric matrix A, can the unspecified entries of A be chosen in such a way that the resulting matrix is positive semidefinite ? Similarly, the *Euclidean distance matrix completion problem* (*EDM completion problem*, for short) asks whether a given partial symmetric matrix can be completed to a Euclidean distance matrix.

We remind that an $n \times n$ matrix X is said to be *positive semidefinite* if $x^T X x \geq 0$ for all $x \in \mathbb{R}^n$ and that an $n \times n$ matrix $D = (d_{ij})$ is called a *Euclidean distance matrix* if there exist vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ (for some $m \geq 1$) such that $d_{ij} = (\| u_i - u_j \|_2)^2$ for all $i, j = 1, \ldots, n$. (Here, $\| x \|_2 := \sqrt{\sum_{i=1}^m (x_i)^2}$ denotes the Euclidean norm of $x \in \mathbb{R}^m$.) We let PSD_n and EDM_n denote, respectively, the set of positive semidefinite matrices and Euclidean distance matrices of size $n \times n$.

For the EDM completion problem, one may obviously consider only partial matrices whose diagonal entries are all specified and equal to 0. For the PSD completion problem,

[1] All matrices here are assumed to have real entries and to be symmetric.

it can be easily observed that one can restrict oneself to the case of partial matrices whose diagonal entries are all specified and equal to 1. Thus we consider the set

$$\mathcal{E}_n := \mathrm{PSD}_n \cap \{X = (x_{ij}) \mid x_{ii} = 1 \ \forall i = 1, \ldots, n\},$$

called an *elliptope* in [13]; matrices in $\mathcal{E}_n$ are known as *correlation matrices* (cf. [10, 14, 15]).

The PSD and EDM completion problems can then be reformulated in the following way. Let $G = (V_n, E)$ be a graph with node set $V_n := \{1, \ldots, n\}$ and edge set E. We let $\mathcal{E}(G)$ (resp. $\mathrm{EDM}(G)$) denote the projection of $\mathcal{E}_n$ (resp. of EDM_n) on the subspace $\mathbb{R}^E$ indexed by the edge set of G. Therefore, the sets $\mathcal{E}(K_n)$ and $\mathcal{E}_n$ are in one-to-one correspondance, as well as the sets $\mathrm{EDM}(K_n)$ and EDM_n. Then, the PSD (resp. EDM) completion problem amounts to the question of deciding whether a given vector $x \in \mathbb{R}^E$ belongs to the projection $\mathcal{E}(G)$ (resp. $\mathrm{EDM}(G)$), the edge set of G representing the set of specified positions.

Two sets of necessary conditions for membership in $\mathcal{E}(G)$ and in $\mathrm{EDM}(G)$ have been introduced in the literature. Moreover, the classes of graphs for which these necessary conditions are also sufficient have been identified. Interestingly, the classes of graphs turn out to be identical in both cases. Yet, it was not clear how to derive the set of results concerning one completion problem from the corresponding results for the other completion problem. One first purpose of this note is to show that the results concerning the EDM completion problem can be derived from those concerning the PSD completion problem. The main tool for establishing this link is a functional transform introduced by Schoenberg [17]. Schoenberg showed how the sets EDM_n and $\mathcal{E}_n$ are linked via this transformation; we observe here that this connection extends to the projections $\mathrm{EDM}(G)$ and $\mathcal{E}(G)$.

We also introduce a new set of necessary conditions (the so-called 'cut conditions') for the PSD and EDM completion problems. It turns out however that, although these new conditions are stronger than some previously known ones, they do not permit to characterize the elliptope or the Euclidean distance matrix cone for larger classes of graphs (see Theorem 4.2).

The paper is organized as follows. In Section 2 we introduce some tools permitting to relate positive semidefinite matrices and Euclidean distance matrices. In Section 3, after recalling some necessary conditions for the two completion problems, we show how the results for the EDM completion problem can be derived from those for the PSD completion problem. We present in Section 4 new necessary conditions: the cut conditions (PSD3) and (EDM3) and study the corresponding classes of graphs.

Some definitions about graphs.

In what follows we set $V_n := \{1, \ldots, n\}$ and $E_n := \{ij \mid 1 \leq i < j \leq n\}$. Hence, $K_n = (V_n, E_n)$ is the complete graph on n nodes. Let $G = (V_n, E)$ be a graph, where $E \subseteq E_n$. Its *suspension graph* ∇G is defined as the graph with node set $V_{n+1} := V_n \cup \{n+1\}$ and with edge set $E(\nabla G) := E \cup \{(i, n+1) \mid i \in V_n\}$.

Let $G = (V_n, E)$ be a graph. Given a subset $U \subseteq V_n$, $G[U]$ denotes the subgraph of G *induced* by U, with node set U and with edge set $\{uv \in E \mid u, v \in U\}$. One says that U is a *clique* in G when $G[U]$ is a complete graph. Let C be a circuit in G; an edge $e \in E$

is called a *chord* of C if it joins two nodes of C that are not consecutive on the circuit C. Then, G is said to be *chordal* if every circuit of length ≥ 4 in G has a chord. Let C_{n-1} be a circuit on $n-1$ nodes; then, $W_n := \nabla C_{n-1}$ denotes the *wheel* on n nodes, obtained by adding a new node (the *center* of the wheel) adjacent to all nodes on the circuit.

Splitting a node u (of degree ≥ 2) in G means replacing u by two adjacent nodes u' and u'' and replacing every edge uv in an arbitrary manner, either by $u'v$, or by $u''v$ (but in such a way that each of u' and u'' is adjacent to at least one node). Figure 1.1 (a), (b) shows the wheel on 7 nodes and a splitting of it, while (c) shows the graph $\widehat{W_4}$ obtained by splitting one node in $W_4 = K_4$.

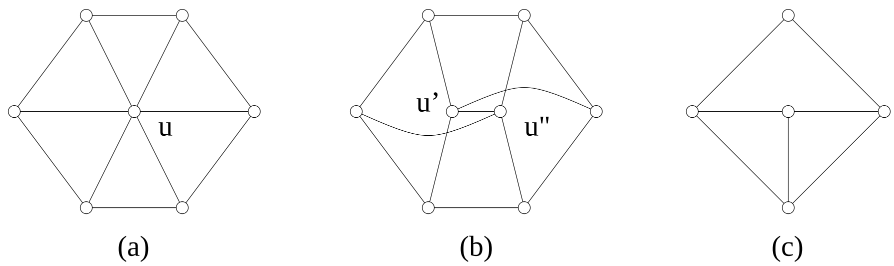

Figure 1.1: (a) The wheel W_7; (b) Splitting node u in W_7; (c) The graph $\widehat{W_4}$

Finally, we introduce three classes of graphs that will play an important role in this paper. The class $\mathcal{G}_{ch}$ consists of all chordal graphs; the class $\mathcal{G}_{K4}$ consists of the graphs that do not contain K_4 or a splitting of K_4 as a subgraph; and the class $\mathcal{G}_{wh}$ consists of the graphs that do not contain a wheel W_n ($n \geq 5$) or a splitting of a wheel W_n ($n \geq 4$) as an induced subgraph.

2 Tools

EDM and PSD matrices. We recall here a well-known correspondance between PSD and EDM matrices. For convenience, we present it in the general setting of graphs. Let $G = (V_n, E)$ be a graph and let ∇G be its suspension graph. One can define a one-to-one linear correspondance ξ between the spaces $\mathbb{R}^{E(\nabla G)}$ and $\mathbb{R}^{E \cup V_n}$ in the following manner: For $d \in \mathbb{R}^{E(\nabla G)}$, $p \in \mathbb{R}^{E \cup V_n}$, $p = \xi(d)$ if

(2.1)
$$\begin{array}{ll} p_{ii} = d_{i,n+1} & \text{for } i \in V_n, \\ p_{ij} = \frac{1}{2}(d_{i,n+1} + d_{j,n+1} - d_{ij}) & \text{for } ij \in E. \end{array}$$

(Here, we identify a pair (i,i) with an element $i \in V_n$.) Then,

(2.2)
$$d \in \mathrm{EDM}(\nabla G) \Longleftrightarrow \begin{array}{l} \xi(d) \text{ can be completed to a} \\ \text{positive semidefinite matrix.} \end{array}$$

In the case of the complete graph $G = K_n$, we obtain the well-known correspondance of Schoenberg [16, 17] between matrices in EDM_{n+1} and PSD_n. Namely,

(2.3)
$$D \in \mathrm{EDM}_{n+1} \Longleftrightarrow P \in \mathrm{PSD}_n,$$

where D is an $(n+1) \times (n+1)$ symmetric matrix D with an all-zero diagonal and P is the $n \times n$ symmetric matrix whose entries are given by (2.1).

The Schoenberg transform. Given $\lambda > 0$, we consider the function

$$F: \quad \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$
$$t \mapsto \exp(-\lambda t),$$

called *Schoenberg transform*. As observed by Schoenberg [17], it permits to make another link between the cone EDM_n and the cone PSD_n or, more precisely, its subset $\mathcal{E}_n$. We use the following notation: For a matrix $D = (d_{ij})$, $F(D)$ denotes the matrix $(F(d_{ij}))$.

LEMMA 2.4. [17] *Let D be an $n \times n$ symmetric matrix with an all-zero diagonal. The following assertions are equivalent.*

(i) $D \in \mathrm{EDM}_n$.

(ii) $\exp(-\lambda D) \in \mathcal{E}_n$ *for all* $\lambda > 0$.

(iii) *The matrix* $1 - \exp(-\lambda D) := (1 - \exp(-\lambda d_{ij}))_{i,j=1}^n$ *belongs to* EDM_n *for all* $\lambda > 0$. ∎

COROLLARY 2.5. *Let $G = (V_n, E)$ be a graph and let $d \in \mathbb{R}^E$. The following assertions are equivalent.*

(i) $d \in \mathrm{EDM}(G)$.

(ii) $\exp(-\lambda d) \in \mathcal{E}(G)$ *for all* $\lambda > 0$.

(iii) $1 - \exp(-\lambda d) \in \mathrm{EDM}(G)$ *for all* $\lambda > 0$.

PROOF. (i) $\Longrightarrow$ (ii) follows from the corresponding implication in Lemma 2.4 and taking projections. The implication (ii) $\Longrightarrow$ (iii) follows from the 'if' part in relation (2.2).
(iii) $\Longrightarrow$ (i) Assume that $1 - \exp(-\lambda d) \in \mathrm{EDM}(G)$ for all $\lambda > 0$. We show that $d \in \mathrm{EDM}(G)$. Since the set $\mathrm{EDM}(G)$ is a cone, it can be expressed as

$$\mathrm{EDM}(G) = \{x \in \mathbb{R}^E \mid v^T x \leq 0 \quad \forall v \in \mathcal{V}\},$$

for some set $\mathcal{V} \subset \mathbb{R}^E$. We show that $v^T d \leq 0$ for every vector $v \in \mathcal{V}$. By assumption,

$$v^T(1 - \exp(-\lambda d)) \leq 0.$$

Expanding in series the exponential function, we obtain:

$$v^T(1 - \exp(-\lambda d)) = \sum_{e \in E} v_e(1 - \exp(-\lambda d_e))$$
$$= \sum_{e \in E} v_e\Big(\sum_{p \geq 1} \frac{(-1)^{p-1}}{p!} \lambda^p (d_e)^p\Big) = \sum_{p \geq 1} \frac{(-1)^{p-1}}{p!} \lambda^p \Big(\sum_{e \in E} v_e (d_e)^p\Big) \leq 0.$$

Dividing by λ and, then, letting $\lambda \longrightarrow 0$ yields the desired inequality: $\sum_{e \in E} v_e d_e \leq 0$. ∎

3 Results for the two completion problems

3.1 Necessary conditions

We present here some known necessary conditions for membership in $\mathcal{E}(G)$ and in $\mathrm{EDM}(G)$. Let $G = (V_n, E)$ be a graph and let K be a clique in G. For a vector $x \in \mathbb{R}^E$, x_K denotes its projection on the subspace indexed by the edges in K; that is, $x_K = (x_{ij})_{ij \in E, i,j \in K}$. A first obvious necessary condition can be derived by noting that every principal submatrix of a positive semidefinite matrix (or of a Euclidean distance matrix) remains positive semidefinite (a Euclidean distance matrix). In other words,

(PSD1) $$x_K \in \mathcal{E}(K) \text{ for every clique } K \text{ in } G,$$

(EDM1) $$d_K \in \mathrm{EDM}(K) \text{ for every clique } K \text{ in } G$$

are, respectively, necessary conditions for $x \in \mathcal{E}(G)$ and $d \in \mathrm{EDM}(G)$. (Here, we use the same letter K for denoting a clique as a node set or as a graph.)

We need a definition in order to formulate the other type of necessary condition. Let $K_n = (V_n, E_n)$ be the complete graph on n nodes. The polyhedra

$$\mathrm{MET}(K_n) := \{x \in \mathbb{R}^{E_n} \mid x_{ij} - x_{ik} - x_{jk} \leq 0 \;\; \forall i,j,k \in V_n\},$$

$$\mathrm{MET}^{\square}(K_n) := \mathrm{MET}(K_n) \cap \{x \in \mathbb{R}^{E_n} \mid x_{ij} + x_{ik} + x_{jk} \leq 2 \;\; \forall i,j,k \in V_n\}$$

are called, respectively, the *metric cone* and *metric polytope* of K_n. For a graph $G = (V_n, E)$, $\mathrm{MET}(G)$ and $\mathrm{MET}^{\square}(G)$ are defined[2], respectively, as the projection of $\mathrm{MET}(K_n)$ and $\mathrm{MET}^{\square}(K_n)$ on the subspace $\mathbb{R}^E$. Necessary conditions for membership in $\mathcal{E}(G)$ and in $\mathrm{EDM}(G)$ can then be formulated in terms of the metric cone and polytope. Namely,

(PSD2) $$\frac{1}{\pi} \arccos x \in \mathrm{MET}^{\square}(G),$$

(EDM2) $$\sqrt{d} \in \mathrm{MET}(G)$$

are, respectively, necessary conditions for $x \in \mathcal{E}(G)$ and $d \in \mathrm{EDM}(G)$; this is obvious for (EDM2) and (PSD2) was formulated in Barrett, Johnson and Tarazaga [6].

In fact, the conditions (PSD2) and (EDM2) suffice for the description of $\mathcal{E}(K_3)$ and $\mathrm{EDM}(K_3)$, respectively. Assertion (i) in the following lemma is a classical geometrical fact (already formulated, e.g., by Blumenthal [7]) and (ii) is observed in Johnson, Jones and Kroschel [11].

Lemma 3.1.

(i) $\mathcal{E}(K_3) = \{x \in [-1,1]^{E_3} \mid \frac{1}{\pi} \arccos x \in \mathrm{MET}^{\square}(K_3)\}$.

(ii) $\mathrm{EDM}(K_3) = \{d \in \mathbb{R}_+^{E_3} \mid \sqrt{d} \in \mathrm{MET}(K_3)\}$. ∎

[2]Explicit linear inequality descriptions for these polyhedra can be found in Barahona [3].

3.2 Linking the conditions (PSD2) and (EDM2)

We establish here an intermediary result[3] which will enable us later to make a connection between the two necessary conditions (PSD2) and (EDM2).

LEMMA 3.2. *Let $G = (V_n, E)$ be a graph and $d \in \mathbb{R}_+^E$. Then,*

$$\sqrt{d} \in \mathrm{MET}(G) \Longrightarrow \frac{1}{\pi}\arccos(e^{-\lambda d}) \in \mathrm{MET}^{\square}(G) \quad \textit{for all } \lambda > 0.$$

PROOF. Note first that it suffices to show the result in the case when $G = K_n$ (as the general result will then follow by taking projections). Next, observe that it suffices to show the result in the case $n = 3$ (as $\mathrm{MET}(K_n)$ and $\mathrm{MET}^{\square}(K_n)$ are defined by inequalities that involve only three points). Now, we have: $\sqrt{d} \in \mathrm{MET}(K_3) \Longleftrightarrow d \in \mathrm{EDM}(K_3)$ (by Lemma 3.1 (ii)); $d \in \mathrm{EDM}(K_3) \Longleftrightarrow \exp(-\lambda d) \in \mathcal{E}(K_3)$ for all $\lambda > 0$ (by Corollary 2.5); finally, $\exp(-\lambda d) \in \mathcal{E}(K_3) \Longleftrightarrow \frac{1}{\pi}\arccos(e^{-\lambda d}) \in \mathrm{MET}^{\square}(K_3)$ (by Lemma 3.1 (i)). ∎

3.3 Results

Let $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_{12}$) denote the class of graphs G for which the condition (PSD1) (resp. the condition (PSD2), the two conditions (PSD1) and (PSD2) taken together) is sufficient for the description of $\mathcal{E}(G)$. In other words, $G \in \mathcal{G}_1$ if and only if

$$\mathcal{E}(G) = \{x \in \mathbb{R}^E \mid x_K \in \mathcal{E}(K)\ \forall K \text{ clique in } G\};$$

$G \in \mathcal{G}_2$ if and only if

$$\mathcal{E}(G) = \{x \in [-1,1]^E \mid \frac{1}{\pi}\arccos x \in \mathrm{MET}^{\square}(G)\};$$

and $G \in \mathcal{G}_{12}$ if and only if

$$\mathcal{E}(G) = \{x \in [-1,1]^E \mid x_K \in \mathcal{E}(K)\ \forall K \text{ clique in } G, |K| \geq 4,\ \frac{1}{\pi}\arccos x \in \mathrm{MET}^{\square}(G)\}.$$

Similarly, let $\mathcal{G}'_1$ (resp. $\mathcal{G}'_2$, $\mathcal{G}'_{12}$) denote the class of graphs G for which the condition (EDM1) (resp. the condition (EDM2), the two conditions (EDM1) and (EDM2) taken together) suffices for the description of $\mathrm{EDM}(G)$.

The classes $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_{12}$ have been characterized, respectively, in Grone, Johnson, Sá and Wolkowicz [9], Laurent [12], and Barrett, Johnson and Loewy [5]. The classes $\mathcal{G}'_1$ and $\mathcal{G}'_{12}$ are characterized, respectively, in Bakonyi and Johnson [2] and Johnson, Jones and Kroschel [11].

It turns out that $\mathcal{G}_1 = \mathcal{G}'_1$, $\mathcal{G}_{12} = \mathcal{G}'_{12}$. One can also identify the class $\mathcal{G}'_2$ (using the same techniques as in [12]) and show that $\mathcal{G}_2 = \mathcal{G}'_2$.

[3] This result holds, in fact, as an equivalence but we will need here only the present implication.

Our objective here is to supply an easy argument for showing the inclusions: $\mathcal{G}_1 \subseteq \mathcal{G}'_1$, $\mathcal{G}_2 \subseteq \mathcal{G}'_2$, and $\mathcal{G}_{12} \subseteq \mathcal{G}'_{12}$. As the reverse inclusions are easy to verify, this implies that the description of each class $\mathcal{G}'_i$ follows from that of $\mathcal{G}_i$.

For showing the inclusion $\mathcal{G}_i \subseteq \mathcal{G}'_i$, it is not important to know which graphs belong to $\mathcal{G}_i$. We use here the results from Corollary 2.5 on the Schoenberg transform and Lemma 3.2.

PROPOSITION 3.3. *$\mathcal{G}_1 \subseteq \mathcal{G}'_1$, $\mathcal{G}_2 \subseteq \mathcal{G}'_2$, and $\mathcal{G}_{12} \subseteq \mathcal{G}'_{12}$.*

PROOF. Let $G \in \mathcal{G}_1$ and let $d \in \mathbb{R}^E$ such that $d_K \in \text{EDM}(K)$ for every clique K in G; we show that $d \in \text{EDM}(G)$. For $\lambda > 0$, set $x(\lambda) := \exp(-\lambda d) \in \mathbb{R}^E_+$. Then, by Corollary 2.5, $x(\lambda)_K \in \mathcal{E}(K)$ for every clique K in G (as $d_K \in \text{EDM}(K)$). This implies that $x(\lambda) \in \mathcal{E}(G)$, since $G \in \mathcal{G}_1$. Therefore, using again Corollary 2.5, we obtain that $d \in \text{EDM}(G)$. This shows that $G \in \mathcal{G}'_1$.
Suppose now that $G \in \mathcal{G}_2$. Let $d \in \mathbb{R}^E_+$ such that $\sqrt{d} \in \text{MET}(G)$; we show that $d \in \text{EDM}(G)$. Then, by Lemma 3.2, $\frac{1}{\pi}\arccos(e^{-\lambda d}) \in \text{MET}^{\square}(G)$ for all $\lambda > 0$. Hence, $e^{-\lambda d} \in \mathcal{E}(G)$ for every $\lambda > 0$, since $G \in \mathcal{G}_2$. This implies that $d \in \text{EDM}(G)$, using again Corollary 2.5.
The inclusion $\mathcal{G}_{12} \subseteq \mathcal{G}'_{12}$ follows by combining the above arguments. ∎

In order to show the reverse inclusions: $\mathcal{G}'_1 \subseteq \mathcal{G}_1$, $\mathcal{G}'_2 \subseteq \mathcal{G}_2$, $\mathcal{G}'_{12} \subseteq \mathcal{G}_{12}$, we need to know the explicit description of the classes $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_{12}$. We remind that the classes $\mathcal{G}_{ch}$, $\mathcal{G}_{K4}$, $\mathcal{G}_{wh}$ have been defined in the Introduction. It is shown in [9], [12], [5], respectively, that

$$\mathcal{G}_1 = \mathcal{G}_{ch},\ \mathcal{G}_2 = \mathcal{G}_{K4},\ \mathcal{G}_{12} = \mathcal{G}_{wh}.$$

The two inclusions: $\mathcal{G}'_2 \subseteq \mathcal{G}_{K4}$, $G'_{12} \subseteq \mathcal{G}_{wh}$ will follow from relation (4.1) and Theorem 4.2 in Section 4 and the inclusion $\mathcal{G}'_1 \subseteq \mathcal{G}_{ch}$ is given in Lemma 3.4 (i) below.

Therefore, as we just saw, the description of class $\mathcal{G}'_1$ follows from that of $\mathcal{G}_1$. In fact, Lemma 3.4 shows that the description of $\mathcal{G}_1$ also follows from that of $\mathcal{G}'_1$. In other words, the two theorems from [9] and [2] concerning the PSD and EDM completion problems for chordal graphs are equivalent.

LEMMA 3.4.

(i) ([9],[2]) $\mathcal{G}_1,\ \mathcal{G}'_1 \subseteq \mathcal{G}_{ch}$.

(ii) *If $\mathcal{G}_{ch} \subseteq \mathcal{G}'_1$, then $\mathcal{G}_{ch} \subseteq \mathcal{G}_1$.*

PROOF. (i) Let $G = (V, E)$ be a nonchordal graph and let $C = (V(C), E(C))$ be a chordless circuit of length ≥ 4 in G. For the inclusion $\mathcal{G}_1 \subseteq \mathcal{G}_{ch}$, we exhibit a vector $x \in \mathbb{R}^E$ satisfying (PSD1) and such that $x \notin \mathcal{E}(G)$. Namely, define $x \in \mathbb{R}^E$ be setting $x_e := 1$ for all edges $e \in E(C)$ except $x_{e_0} := -1$ for one edge $e_0 \in E(C)$, and $x_e := 0$ for all remaining edges. For the inclusion $\mathcal{G}'_1 \subseteq \mathcal{G}_{ch}$, we define a vector $d \in \mathbb{R}^E$ satisfying (EDM1) and such that $d \notin \text{EDM}(G)$ by setting $d_e := 0$ for all edges $e \in E(C)$ except

$d_{e_0} := 1$ for one edge e_0 in C; $d_e := 1$ for every edge e joining a node of C to a node of $V \setminus V(C)$; and $d_e := 0$ for every edge e joining two nodes of $V \setminus V(C)$.
(ii) Let $G = (V_n, E)$ be a chordal graph; we show that $G \in \mathcal{G}_1$. For this, let $x \in \mathbb{R}^E$ such that $x_K \in \mathcal{E}(K)$ for every clique K in G. Consider the suspension graph ∇G and the vector $d \in \mathbb{R}^{E(\nabla G)}$ defined in the following manner: $d_{i,n+1} := 1$ for every $i \in V_n$ and $d_{ij} := 2(1 - x_{ij})$ for all $ij \in E$. Then, by relation (2.2), $d_{\nabla K} \in \mathrm{EDM}(\nabla K)$ for every clique K in G. Observe now that $\nabla G \in \mathcal{G}_1'$, as ∇G remains a chordal graph. Therefore, $d \in \mathrm{EDM}(\nabla G)$, which implies that $x \in \mathcal{E}(G)$. ∎

Finally, observe that the above reasoning does not extend to the other classes. For instance, it is not clear whether the description of the class $\mathcal{G}_2$ can be derived from that of the class $\mathcal{G}_2'$, the reason for that being that the graph property in question is not preserved by taking graph suspensions. Indeed, the suspension graph ∇G may contain a splitting of K_4 as a subgraph even if G itself does not.

4 Further necessary conditions

We formulate here further necessary conditions for membership in $\mathcal{E}(G)$ and $\mathrm{EDM}(G)$. For this we need a definition. Let $G = (V_n, E)$ be a graph and let $S \subseteq V_n$ be a subset of the nodes. The *cut* $\delta_G(S)$ consists of the edges $e \in E$ having one end node in S and the other one in $V_n \setminus S$. Then, $\mathrm{CUT}(G)$ and $\mathrm{CUT}^\square(G)$ denote the cone and polytope generated by the cuts of G; that is,

$$\mathrm{CUT}(G) := \{ \sum_{S \subseteq V_n} \lambda_S \chi^{\delta_G(S)} \mid \lambda_S \geq 0 \text{ for all } S \subseteq V_n \},$$

$$\mathrm{CUT}^\square(G) := \{ \sum_{S \subseteq V_n} \lambda_S \chi^{\delta_G(S)} \mid \lambda_S \geq 0 \text{ for all } S \subseteq V_n \text{ and } \sum_S \lambda_S = 1 \}.$$

(Here, $\chi^{\delta_G(S)} \in \{0,1\}^E$ denotes the incidence vector of the cut $\delta_G(S)$.) $\mathrm{CUT}(G)$ is called the *cut cone* of G and $\mathrm{CUT}^\square(G)$ its *cut polytope*. Clearly, $\mathrm{CUT}(G)$ and $\mathrm{CUT}^\square(G)$ are, respectively, the projections of $\mathrm{CUT}(K_n)$ and $\mathrm{CUT}^\square(K_n)$ on the subspace indexed by the edge set of G. (See, e.g., [8] for a survey on these polyhedra.) Necessary conditions for membership in $\mathcal{E}(G)$ and $\mathrm{EDM}(G)$ can be formulated in terms of $\mathrm{CUT}(G)$ and $\mathrm{CUT}^\square(G)$. Namely,

(PSD3) $$\frac{1}{\pi} \arccos x \in \mathrm{CUT}^\square(G),$$

(EDM3) $$\sqrt{d} \in \mathrm{CUT}(G)$$

are, respectively, necessary conditions for $x \in \mathcal{E}(G)$ and $d \in \mathrm{EDM}(G)$. The condition (PSD3) was formulated in [12]. That (EDM3) is indeed a necessary condition for membership in $\mathrm{EDM}(G)$ relies on a well-known connection between ℓ_1- and ℓ_2-spaces as we now explain. First, observe that it suffices to consider the case when $G = K_n$ is a complete graph (as the general result will follow by taking projections). So, let us consider

$d \in \mathrm{EDM}(K_n)$; that is, $\sqrt{d}$ is ℓ_2-embeddable (by the definition of $\mathrm{EDM}(K_n)$). This implies[4] that $\sqrt{d}$ is ℓ_1-embeddable and, therefore[5], that $\sqrt{d} \in \mathrm{CUT}(K_n)$.

The conditions (PSD3) and (EDM3) are respectively stronger than the conditions (PSD2) and (EDM2), because the following inclusions[6] hold:

$$\mathrm{CUT}(G) \subseteq \mathrm{MET}(G),\ \mathrm{CUT}^{\square}(G) \subseteq \mathrm{MET}^{\square}(G).$$

(Again, it suffices to check it for the complete graph, which is easy.)

Hence, arises the question of characterizing the graphs for which these new stronger conditions are sufficient. Let $\mathcal{G}_3$ (resp. $\mathcal{G}'_3$) denote the class of graphs G for which (PSD3) (resp. (EDM3)) suffices for ensuring membership in $\mathcal{E}(G)$ (resp. $\mathrm{EDM}(G)$). Similarly, let $\mathcal{G}_{13}$ (resp. $\mathcal{G}'_{13}$) consist of the graphs G for which the two conditions (PSD1) and (PSD3) (resp. (EDM1) and (EDM3)) together suffice for ensuring membership in $\mathcal{E}(G)$ (resp. $\mathrm{EDM}(G)$). Clearly,

$$\mathcal{G}_2 \subseteq \mathcal{G}_3,\ \mathcal{G}_{12} \subseteq \mathcal{G}_{13},\ \mathcal{G}'_2 \subseteq \mathcal{G}'_3,\ \mathcal{G}'_{12} \subseteq \mathcal{G}'_{13}. \tag{4.1}$$

In fact, as we see below, these inclusions hold at equality. In other words, even though the cut condition (PSD3) (or (EDM3)) is stronger than the metric condition (PSD2) (or (EDM2)), it suffices for ensuring membership in the elliptope (or the Euclidean distance matrix cone) for the same class of graphs. Namely,

Theorem 4.2. *$\mathcal{G}_2 = \mathcal{G}_3$ $(= \mathcal{G}_{K4})$, $\mathcal{G}'_2 = \mathcal{G}'_3$ $(= \mathcal{G}_{K4})$, $\mathcal{G}_{12} = \mathcal{G}_{13}$ $(= \mathcal{G}_{wh})$, and $\mathcal{G}'_{12} = \mathcal{G}'_{13}$ $(= \mathcal{G}_{wh})$.*

Before giving the proof let us remark that we now have to treat the two classes $\mathcal{G}_{13}$ and $\mathcal{G}'_{13}$ separately. Indeed, we do not know whether the analogue of Lemma 3.2 holds if one replaces the metric polyhedra by the cut polyhedra. Thus we do not have 'for free' the inclusion $\mathcal{G}_{13} \subseteq \mathcal{G}'_{13}$. We start with a preliminary lemma.

LEMMA 4.3. *Let $W_n := \nabla C$ be a wheel on n nodes, with center u_0 and circuit C. Consider the vector d indexed by the edge set of W_n and defined by $d(u_0, u) := 1$ for each node u of C, $d(u, v) := 4$ for each edge uv of C. Then, $d \in \mathrm{EDM}(W_n) \iff n$ is odd.*

PROOF. Let x be the vector indexed by the edge set of C and taking value -1 on every edge. Then, by Corollary 2.2, $d \in \mathrm{EDM}(W_n)$ if and only if $x \in \mathcal{E}(C)$. The latter holds if and only if $\frac{1}{\pi} \arccos x \in \mathrm{MET}^{\square}(C)$, that is, if and only if C has an even length. ∎

PROOF OF THEOREM 4.2. The inclusion $\mathcal{G}_3 \subseteq \mathcal{G}_{K4}$ is proved in [12] and the inclusion $\mathcal{G}'_3 \subseteq \mathcal{G}_{K4}$ can be easily verified using the same techniques.

[4] It is a classical result in analysis that, for a distance d, d is ℓ_2-embeddable $\Longrightarrow$ d is ℓ_1-embeddable; see, e.g., [19].

[5] We use here the well-known fact that the set of ℓ_1-embeddable distances on an n-element set is a cone whose extreme rays are generated by the cuts of K_n; in other words, d is ℓ_1-embeddable if and only if $d \in \mathrm{CUT}(K_n)$ (see [1] or [8]).

[6] The class of graphs for which equality holds has been identified in [4, 18]. Namely, $\mathrm{CUT}(G) = \mathrm{MET}(G)$ or, equivalently, $\mathrm{CUT}^{\square}(G) = \mathrm{MET}^{\square}(G)$ if and only if G does not admit K_5 as a minor.

The inclusion $\mathcal{G}_{13} \subseteq \mathcal{G}_{wh}$ relies on the following claims: (i) the wheel W_n ($n \geq 5$) and the graph $\widehat{W_4}$ (recall Figure 1.1 (c)) do not belong to $\mathcal{G}_{13}$; (ii) the class $\mathcal{G}_{13}$ is preserved under the operations of taking induced subgraphs and (iii) contracting edges. Assertion (i) follows from the fact that W_n ($n \geq 5$), $\widehat{W_4} \notin \mathcal{G}_3$ (as they contain a splitting of K_4) and that these graphs have no clique of size 4. We leave the verification of (iii) to the reader and we now check (ii). For this let $G = (V, E)$ be a graph in $\mathcal{G}_{13}$ and let $H := G[U]$ ($U \subseteq V$) be an induced subgraph of G. We show that $H \in \mathcal{G}_{13}$. Let x be a vector indexed by the edge set of H satisfying (PSD1) and (PSD3); we show that $x \in \mathcal{E}(H)$. For this we extend x to a vector y indexed by the edge set of G by setting $y_{uv} := 0$ for an edge $uv \in E$ with $u \in U$, $v \in V \setminus U$ and $y_{uv} := 1$ for an edge $uv \in E$ contained in $V \setminus U$. It is clear that y satisfies (PSD1). By assumption, $a := \frac{1}{\pi} \arccos x \in \mathrm{CUT}^{\square}(H)$; we verify that $b := \frac{1}{\pi} \arccos y \in \mathrm{CUT}^{\square}(G)$. Indeed, say $a = \sum_{S \subseteq U} \lambda_S \delta_H(S)$ where $\lambda_S \geq 0$, $\sum_S \lambda_S = 1$. Then, $b = \frac{1}{2} \sum_{S \subseteq U} \lambda_S \left(\delta_G(S) + \delta_G(U \setminus S) \right)$, which shows that $b \in \mathrm{CUT}^{\square}(G)$. Hence, y satisfies (PSD3). Therefore, $y \in \mathcal{E}(G)$ which implies that $x \in \mathcal{E}(H)$.

We now show the inclusion $\mathcal{G}'_{13} \subseteq \mathcal{G}_{wh}$. Let $G = (V, E)$ be a graph in $\mathcal{G}'_{13}$ and let $H := G[U]$ be an induced subgraph of G. Suppose in a first step that H is a wheel $W_n := \nabla C$ ($n \geq 5$) with center u_0. Consider the vector d indexed by the edge set of G and defined in the following manner: d takes value 4 on every edge of the circuit C except value 0 on one edge if n is odd; d takes value 1 on every edge joining the center u_0 of the wheel to a node of C; d takes value 1 on an edge between a node of C and a node outside the wheel; d takes value 0 on every remaining edge (i.e., an edge joining u_0 to a node outside the wheel or an edge joining two nodes outside the wheel). Then d satisfies (EDM1) and $d \notin \mathrm{EDM}(G)$ (by Lemma 4.3). Moreover d satisfies (EDM3), i.e., $\sqrt{d} \in \mathrm{CUT}(G)$. Indeed, say C is the circuit $(u_1, \ldots, u_{n-1})$. Then, $\sqrt{d} = \sum_{i=1}^{n-1} \delta_G(u_i)$ if n is even and $\sqrt{d} = \delta_G(\{u_1, u_{n-1}\}) + \sum_{i=2}^{n-2} \delta_G(u_i)$ if n is odd and (u_1, u_{n-1}) is the edge of C on which d takes value 0. Finally, if H is a splitting of a wheel W_n ($n \geq 4$), extend the above vector d by assigning value 0 to every new edge created during the splitting process. This concludes the proof. ∎

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