

A Characterization of Box $\frac{1}{d}$ -Integral Binary Clutters*

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Let Q_6 denote the port of the dual Fano matroid F_7^* and let Q_7 denote the clutter consisting of the circuits of the Fano matroid F_7 that contain a given element. Let \mathcal{L} be a binary clutter on E and let $d \geq 2$ be an integer. We prove that all the vertices of the polytope $\{x \in \mathbb{R}_+^E \mid x(C) \geq 1 \text{ for } C \in \mathcal{L}\} \cap \{x \mid a \leq x \leq b\}$ are $\frac{1}{d}$ -integral, for any $\frac{1}{d}$ -integral a, b , if and only if \mathcal{L} does not have Q_6 or Q_7 as a minor. This includes the class of ports of regular matroids. Applications to graphs are presented, extending a result from Laurent and Poljak [7]. © 1995 Academic Press, Inc.

1. THE MAIN RESULT

Let \mathcal{L} be a collection of subsets of a set E . \mathcal{L} is called a *clutter* if, for all $A, B \in \mathcal{L}$, $A = B$ whenever $A \subseteq B$. Given an integer $d \geq 1$, a vector is $\frac{1}{d}$ -integral if all its components belong to $\frac{1}{d}\mathbb{Z} := \{\frac{i}{d} \mid i \in \mathbb{Z}\}$.

DEFINITION 1.1. *Let \mathcal{L} be a clutter on E . We say that \mathcal{L} is box $\frac{1}{d}$ -integral if $\mathcal{L} = \{\emptyset\}$ or, for all $a, b \in (\frac{1}{d}\mathbb{Z})^E$, each vertex of the polyhedron*

$$Q(\mathcal{L}, a, b) := \{x \in \mathbb{R}_+^E \mid x(C) \geq 1 \text{ for } C \in \mathcal{L}, a_e \leq x_e \leq b_e \text{ for } e \in E\}$$

is $\frac{1}{d}$ -integral. Equivalently, \mathcal{L} is box $\frac{1}{d}$ -integral if, for all subsets $I \subseteq E$ and all $a \in (\frac{1}{d}\mathbb{Z})^I$, each vertex of the polyhedron

$$Q(\mathcal{L}, a) := \{x \in \mathbb{R}_+^E \mid x(C) \geq 1 \text{ for } C \in \mathcal{L}, x_e = a_e \text{ for } e \in I\}$$

is $\frac{1}{d}$ -integral.

* This work was partially done while the authors visited DIMACS, New Brunswick, NJ.

We shall mostly use the second definition for box $\frac{1}{d}$ -integral clutters.

Given a clutter \mathcal{L} on E and a subset Z of E , set $\mathcal{L} \setminus Z = \{A \in \mathcal{L} \mid A \cap Z = \emptyset\}$ and let \mathcal{L}/Z consist of the minimal members of $\{A - Z \mid A \in \mathcal{L}\}$; both \mathcal{L}/Z and $\mathcal{L} \setminus Z$ are clutters. The operations are called, respectively, *deletion* and *contraction* of Z . A *minor* of \mathcal{L} is obtained from \mathcal{L} by a sequence of deletions and contractions.

Let \mathcal{M} be a matroid on a groundset $E \cup \{l\}$, where l is a distinguished element of the groundset, and let \mathcal{C} denote the family of circuits of \mathcal{M} . The l -*port* of \mathcal{M} is the clutter $\{C \mid C \cup \{l\} \in \mathcal{C}\}$. A clutter is *binary* if it is the port of some binary matroid.

The binary clutters Q_6 and Q_7 are defined, respectively, on six and seven elements. Q_6 is the clutter on the set $\{1, 2, 3, 4, 5, 6\}$ consisting of the sets $\{1, 3, 5\}$, $\{1, 2, 6\}$, $\{2, 3, 4\}$, and $\{4, 5, 6\}$. Q_7 is the clutter on the set $\{1, 2, 3, 4, 5, 6, 7\}$ consisting of the sets $\{1, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 6, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 3, 5, 7\}$, $\{2, 3, 4, 7\}$, and $\{4, 5, 6, 7\}$.

The following result is the main result of the paper. Applications to graphs are given in Section 5.

THEOREM 1.2. *Let \mathcal{L} be a binary clutter on a set E , $\mathcal{L} \neq \{\emptyset\}$. The following assertions are equivalent:*

- (i) \mathcal{L} does not contain Q_6 or Q_7 as a minor,
- (ii) \mathcal{L} is box $\frac{1}{d}$ -integral for each integer $d \geq 1$,
- (iii) \mathcal{L} is box $\frac{1}{d}$ -integral for some integer $d \geq 2$.

Observe that, for $d=1$, \mathcal{L} is box $\frac{1}{d}$ -integral if and only if \mathcal{L} has the following *weak max-flow-min-cut property* (since the weak max-flow-min-cut property is closed under minors [10]): $\mathcal{L} = \{\emptyset\}$ or, for each $w \in \mathbb{Z}_+^E$, the program

$$\begin{aligned} \min \quad & w^T x \\ \text{subject to} \quad & x(C) \geq 1 \quad \text{for all } C \in \mathcal{L} \\ & x_e \geq 0 \quad \text{for all } e \in E \end{aligned}$$

has an integer optimizing vector.

A nonempty clutter \mathcal{L} is said to be *Mengerian* if $\mathcal{L} = \{\emptyset\}$, or both the above program and its dual

$$\begin{aligned} \max \quad & 1^T y \\ \text{subject to} \quad & \sum_{C \in \mathcal{L}} y_C \leq w_e \quad \text{for } e \in E \\ & y_C \geq 0 \quad \text{for } C \in \mathcal{L} \end{aligned}$$

have integer optimizing vectors for all $w \in \mathbb{Z}_+^E$. Seymour [10] showed that a clutter $\mathcal{L} \neq \{\emptyset\}$, which is a matroid port, is Mengerian if and only if \mathcal{L} is binary and does not have any Q_6 minor. Therefore, from Theorem 1.2, the class of the binary clutters which are box $\frac{1}{d}$ -integral for some integer $d \geq 2$ is strictly contained in the class of Mengerian binary clutters.

The characterization of the clutters with the weak max-flow-min-cut property is a hard and unsolved problem, even within the class of matroid ports (see [10], [4]).

Theorem 1.2 does not hold for ports of arbitrary matroids. For this, consider the matroid U_4^2 on four elements whose circuits are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$. (Recall that a matroid is binary if and only if it does not contain U_4^2 as a minor (Tutte [15]).) The 4-port of U_4^2 is the clutter C_3 consisting of the sets $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. It is easy to check that C_3 is box $\frac{1}{d}$ -integral if and only if d is even. Hence, the assertions (ii) and (iii) of Theorem 1.2 are not equivalent for the clutter C_3 .

PROPOSITION 1.3. *Let d be an odd integer and let \mathcal{L} be a matroid port. If \mathcal{L} is box $\frac{1}{d}$ -integral, then \mathcal{L} is a binary clutter.*

Proof. Let \mathcal{L} be the l -port of a matroid \mathcal{M} . We can suppose that \mathcal{M} is connected. Assume that \mathcal{L} is box $\frac{1}{d}$ -integral. Then \mathcal{L} does not have C_3 as a minor, see Proposition 3.2. Therefore, \mathcal{M} does not have a minor U_4^2 using the element l . This implies that \mathcal{M} does not have any minor U_4^2 (Bixby [3]). Therefore, \mathcal{M} is a binary matroid. Hence, \mathcal{L} is a binary clutter. ■

In order to prove Theorem 1.2, it suffices to show the implications (iii) \Rightarrow (i) and (i) \Rightarrow (ii). The implication (iii) \Rightarrow (i) is implied by the following facts:

- box $\frac{1}{d}$ -integrality is preserved under minors, see Proposition 3.2.
- Q_6 is not box $\frac{1}{d}$ -integral, for each integer $d \geq 2$, see Proposition 3.3.
- Q_7 is not box $\frac{1}{d}$ -integral, for each integer $d \geq 2$, see Proposition 3.4.

The most difficult part is to show the implication (i) \Rightarrow (ii). For this, we use as a main tool a decomposition result for matroids without minor F_7^* using a given element l , stated in Theorem 2.3 (Tseng and Truemper [14], Truemper [12]).

The proof of Theorem 1.2 is presented in Sections 3 and 4. In Section 2, we recall some results about matroids and the decomposition result that we need here. We present in Section 5 some applications of our main result.

We conclude with another, equivalent, definition for box $\frac{1}{d}$ -integral clutters, which is related to the “ \mathcal{F} -property” considered by Nobili

and Sassano [8]. Given a clutter \mathcal{L} on E , $\mathcal{L} \neq \{\emptyset\}$, consider the polyhedron

$$Q(\mathcal{L}) := \{x \in \mathbb{R}_+^E \mid x(C) \geq 1 \text{ for all } C \in \mathcal{L}\}.$$

Given a k -dimensional face F ($k \geq 0$) of $Q(\mathcal{L})$, a subset $J \subseteq E$ is said to be *basic* for F if there exist $|E| - k$ equations $x(C_i) = 1$ ($C_i \in \mathcal{L}$, for $1 \leq i \leq |E| - k$) defining F whose projections on \mathbb{R}^J are linearly independent. Then, one can check that \mathcal{L} is box $\frac{1}{d}$ -integral if and only if the following property holds: For each k -dimensional face F of $Q(\mathcal{L})$ ($k \geq 0$) for each basic set $J \subseteq E$ for F and for each $x \in F$, $x_e \in \frac{1}{d}\mathbb{Z}$ for all $e \in J$ whenever $x_e \in \frac{1}{d}\mathbb{Z}$ for all $e \in E - J$. This property corresponds to the “ \mathcal{F} -property” considered (in blocking terms and in a slightly more general setting) by Nobili and Sassano [8].

2. PRELIMINARIES ON MATROIDS

We recall here several well known results on matroids that we need for the paper. We refer to [17], [13] for details on the material covered in this section.

We use the following notation. Given a set A and elements $a \in A$, $b \notin A$, $A - a$, $A + b$ denote, respectively, $A - \{a\}$ and $A \cup \{b\}$. If x, y are two binary vectors, then $x \oplus y$ denotes the binary vector obtained by taking the componentwise sum of x and y modulo 2.

Representation Matrix

Let \mathcal{M} be a binary matroid on a set E , i.e., there exists a binary matrix M whose columns are indexed by E such that a subset of E is independent in \mathcal{M} if and only if the corresponding subset of columns of M is linearly independent over the field $GF(2)$. Such a matrix M is called a *representation matrix* of \mathcal{M} .

Let X be a base of \mathcal{M} and set $Y = E - X$. For $y \in Y$, let C_y denote the fundamental circuit of y in the base X , i.e., C_y is the unique circuit of \mathcal{M} such that $y \in C_y$ and $C_y \subseteq X + y$. Let B denote the $|X| \times |Y|$ matrix whose columns are the incidence of the sets $C_y - y$ for $y \in Y$. Then, the matrix $[I|B]$ is a representation matrix of \mathcal{M} and B is called a *partial representation matrix* of \mathcal{M} .

For $x \in X$, let Σ_x denote the fundamental cocircuit of x with respect to the base X , i.e., Σ_x is the unique cocircuit of \mathcal{M} such that $x \in \Sigma_x$ and $\Sigma_x \subseteq Y + x$. The row of B indexed by x is the incidence vector of the set $\Sigma_x - x$.

For $y \in Y$ and $x \in C_y$, the set $X' = X - x + y$ is also a base of \mathcal{M} . The partial representation matrix B' of \mathcal{M} in the base X' is easily obtained from

B by *pivoting* with respect to the (x, y) -entry of B . Let $R_{x'}$, $x' \in X$, denote the rows of B ; they are vectors in $\{0, 1\}^Y$. Pivoting with respect to the (x, y) -entry of B amounts to replacing $R_{x'}$ by $R_{x'} \oplus R_x \oplus (1, 0, \dots, 0)$ (where 1 is in the y -position) for each $x' \in C_y$, $x' \neq x, y$.

Let \mathcal{C} denote the family of circuits of \mathcal{M} . A set $C \subseteq E$ is called a *cycle* of \mathcal{M} if $C = \emptyset$ or C is a disjoint union of circuits of \mathcal{M} . Equivalently, if M is a representation matrix of \mathcal{M} , then the cycles are the subsets whose incidence vectors u satisfy $Mu \equiv 0 \pmod{2}$.

Minors

Let Z be a subset of E . The matroid $\mathcal{M} \setminus Z$, obtained by *deletion* of Z , is the matroid on $E - Z$ whose family of circuits is $\mathcal{C} \setminus Z$. The matroid \mathcal{M} / Z , obtained by *contraction* of Z , is the matroid on $E - Z$ whose circuits are the nonempty sets of \mathcal{C} / Z . Note that contracting a loop or coloop is the same as deleting it. A *minor* of \mathcal{M} is obtained by a sequence of deletions and contractions. Every minor of \mathcal{M} is of the form $\mathcal{M} \setminus Z / Z'$ for some disjoint subsets Z, Z' of E . Given $e \in E$, the minor $\mathcal{M} \setminus Z / Z'$ uses the element e if $e \notin Z \cup Z'$; in other words, e belongs to the groundset of $\mathcal{M} \setminus Z / Z'$.

Minors can be easily visualized in the partial representation matrix. Let B be the partial representation matrix of \mathcal{M} corresponding to the base X . If $Z \subseteq X$, then the matrix obtained from B by deleting its rows indexed by Z is a partial representation matrix of \mathcal{M} / Z . If $Z \subseteq Y$, then the matrix obtained from B by deleting its columns indexed by Z is a partial representation matrix of $\mathcal{M} \setminus Z$.

k-Sum

Let \mathcal{M}_i be a binary matroid on E_i , for $i = 1, 2$. Let \mathcal{M} be the binary matroid on $E = E_1 \Delta E_2$ whose cycles are the subsets of E of the form $C_1 \Delta C_2$, where C_i is a cycle of \mathcal{M}_i for $i = 1, 2$. We consider the cases:

- $E_1 \cap E_2 = \emptyset$, then \mathcal{M} is called the *1-sum* of \mathcal{M}_1 and \mathcal{M}_2
- $|E_1|, |E_2| \geq 3$, $E_1 \cap E_2 = \{e_0\}$ and e_0 is not a loop or a coloop of \mathcal{M}_1 or \mathcal{M}_2 , then \mathcal{M} is the *2-sum* of \mathcal{M}_1 and \mathcal{M}_2 .

k-Separation

Let $r(\cdot)$ denote the rank function of the matroid \mathcal{M} on E . Let $k \geq 1$ be an integer. A *k-separation* of \mathcal{M} is a partition (E_1, E_2) of the groundset E satisfying

$$\begin{cases} |E_1|, |E_2| \geq k, \\ r(E_1) + r(E_2) \leq r(E) + k - 1. \end{cases}$$

When $r(E_1) + r(E_2) = r(E) + k - 1$, the separation is called *strict*. The matroid \mathcal{M} is said to be *k-connected* if it has no j -separation for $j \leq k - 1$. Throughout the paper, 2-connected will be abbreviated as *connected*.

If \mathcal{M} has a strict k -separation (E_1, E_2) , then it admits a partial representation matrix of a special form. Indeed, let X_2 be a maximal independent subset of E_2 and let $X_1 \subseteq E_1$ such that $X = X_1 \cup X_2$ is a base of \mathcal{M} , so $|X_1| = r(E_1) - k + 1$ and $|X_2| = r(E_2)$. Set $Y_i := E_i - X_i$, for $i = 1, 2$. The partial representation matrix B of \mathcal{M} in the base X has the form shown in Fig. 1. The rank of the matrix D is equal to $k - 1$.

In the case of a strict 1-separation, the matrix D is identically zero. Then, \mathcal{M} is the 1-sum of \mathcal{M}_1 and \mathcal{M}_2 .

In the case of a strict 2-separation, the matrix D has rank 1 and, thus, has the form shown in Fig. 2.

The set \tilde{Y}_1 consists of the elements $y \in Y_1$ for which $X_1 + y$ is an independent set of \mathcal{M} . So, if $y \in \tilde{Y}_1$, then the fundamental circuit of y in the base X is of the form $\tilde{X}_2 \cup A_y \cup \{y\}$ with $A_y \subseteq X_1$.

Given two elements $e_1 \in \tilde{X}_2$ and $e_2 \in \tilde{Y}_1$, we consider the matroids $\mathcal{M}_1 = \mathcal{M}/(X_2 - e_1) \setminus Y_2$ and $\mathcal{M}_2 = \mathcal{M}/X_1 \setminus (Y_1 - e_2)$ defined, respectively, on $E_1 \cup \{e_1\}$ and $E_2 \cup \{e_2\}$. It follows from the next Proposition 2.1 that \mathcal{M} is the 2-sum of \mathcal{M}_1 and \mathcal{M}_2 (after renaming e_1 as e_0 in \mathcal{M}_1 and e_2 as e_0 in \mathcal{M}_2). A set $C \subseteq E$ is said to be *crossing* if $C \cap E_1 \neq \emptyset$ and $C \cap E_2 \neq \emptyset$.

PROPOSITION 2.1. (i) *Let C be a circuit of \mathcal{M} . Then,*

- *either $C \subseteq E_i$ and C is a circuit of \mathcal{M}_i , for some $i \in \{1, 2\}$,*
- *or C is crossing and $(C \cap E_i) + e_i$ is a circuit of \mathcal{M}_i , for $i = 1$ and 2. Moreover, $(C \cap E_1) \cup \tilde{X}_2$ and $(C \cap E_2) \Delta \tilde{X}_2$ are circuits of \mathcal{M} .*

Every circuit of \mathcal{M}_i arises in one of the two ways indicated above.

(ii) *Let C, C' be two crossing circuits of \mathcal{M} . Then, $(C \cap E_i) \Delta (C' \cap E_j)$ is a cycle of \mathcal{M} for any $i, j \in \{1, 2\}$.*

Proof. (ii) follows directly from (i) and (i) is easy to check after observing that, for a circuit C of \mathcal{M} , C is crossing if and only if $|C \cap \tilde{Y}_1|$ is odd. ■

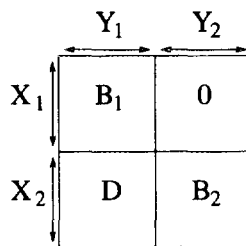


FIGURE 1

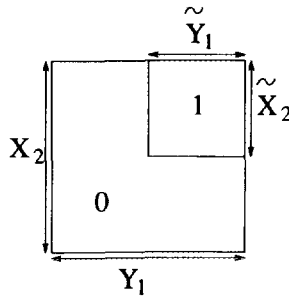


FIGURE 2

In the case of a strict 3-separation, the matrix D has rank 2. Moreover, if $|E_1|, |E_2| \geq 4$ and \mathcal{M} is 3-connected, it can be shown that \mathcal{M} has a partial representation matrix B of the form shown in Fig. 3, with $D_{12} = D_2 D_1$ (see [12]).

PROPOSITION 2.2. *Suppose \mathcal{M} has a strict 3-separation (E_1, E_2) with $|E_1|, |E_2| \geq 4$ and consider the partial representation matrix of \mathcal{M} from Fig. 3. If $\{y, z, l\}$ is a circuit of the matroid $\mathcal{M}/(X_1 - x) \setminus (Y_1 - \{y, z\})$, then the partition $(E_1, E_2 - l)$ of $E - l$ is a strict 2-separation of the matroid \mathcal{M}/l .*

Proof. Let a, b denote the rows of D_1 indexed, respectively, by e, f and let u, v denote the columns of D_2 indexed, respectively, by y, z . So, a, b are vectors indexed by the elements $y' \in Y_1 - \{y, z\}$ and u, v are indexed by the

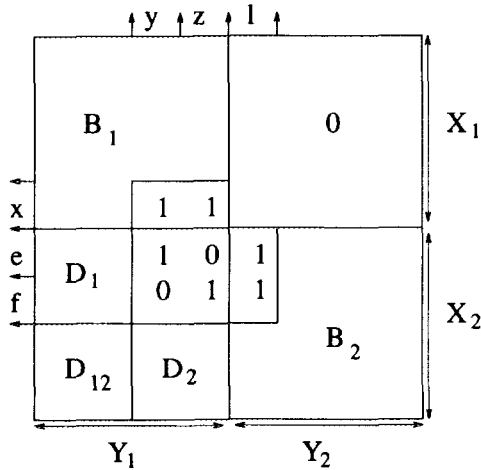


FIGURE 3

elements $x' \in X_2 - \{e, f\}$. Let w denote the vector whose components are the (x', l) -entries, for $x' \in X_2 - \{e, f\}$, of the first column of B_2 . Since the set $\{y, z, l\}$ is a circuit of the matroid $\mathcal{M}/(X_1 - x) \setminus (Y_1 - \{y, z\})$, we deduce that $w = u \oplus v$.

The (e, l) -entry of B is equal to 1, hence the set $X' = X - e + l$ is again a base of \mathcal{M} . Let B' denote the partial representation matrix of \mathcal{M} in the base X' . So B' can be obtained from B by pivoting with respect to its (e, l) -entry. Pivoting will affect only the rows of B indexed by $X_2 - e$. Let D' denote the submatrix of B' with row index set $X_2 - e + l$ and with column index set Y_1 . It is not difficult to check that the row of D' indexed by f is the vector $(a \oplus b, 1, 1)$ and that each other row of D' indexed by some element of $X_2 - \{e, f\}$ is one of the two vectors $(a \oplus b, 1, 1)$ or $(0, \dots, 0, 0, 0)$. Therefore, the submatrix of D' with row index set $X_2 - e$ has rank 1. This shows that the partition $(E_1, E_2 - l)$ of $E - l$ is a strict 2-separation of the matroid \mathcal{M}/l . ■

Fano Matroid

The Fano matroid F_7 is the matroid on $\{1, 2, 3, 4, 5, 6, 7\}$ whose circuits are the seven sets $\{1, 2, 3\}$, $\{1, 4, 7\}$, $\{1, 5, 6\}$, $\{2, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 5\}$ and $\{3, 6, 7\}$ (the lines of the Fano plane) together with their complements. The dual Fano matroid F_7^* is the dual of F_7 , its circuits are $\{4, 5, 6, 7\}$, $\{2, 3, 5, 6\}$, $\{2, 3, 4, 7\}$, $\{1, 3, 5, 7\}$, $\{1, 3, 4, 6\}$, $\{1, 2, 6, 7\}$ and $\{1, 2, 4, 5\}$ (the complements of the lines of the Fano plane).

By symmetry, there is only one port for F_7^* . The 7-port of F_7^* is the clutter Q_6 , already defined earlier, consisting of the sets $\{4, 5, 6\}$, $\{2, 3, 4\}$, $\{1, 3, 5\}$ and $\{1, 2, 6\}$.

Observe that every one-element contraction of F_7 has a 2-separation. For example, the sets $\{1, 4\}$ and $\{2, 3, 5, 6\}$ form a strict 2-separation of $F_7/7$.

We also consider the series-extension F_7^+ of the Fano matroid F_7 , obtained by adding a new element "8" in series with, say, the element "7" i.e., $\{7, 8\}$ is a cocircuit of F_7^+ . Hence, F_7^+ is the matroid defined on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ whose circuits are the sets C for which C is a circuit of F_7 with $7 \notin C$, and the sets $C \cup \{8\}$ for which C is a circuit of F_7 with $7 \in C$. Up to symmetry, there are two distinct l -ports of F_7^+ , depending whether l is one of the two series elements 7, 8, or not. We denote by Q_7 the l -port of F_7^+ when l is a series element of F_7^+ . Then, for $l = 8$, Q_7 consists of the sets $\{1, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 6, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 3, 5, 7\}$, $\{2, 3, 4, 7\}$ and $\{4, 5, 6, 7\}$, i.e., Q_7 consists of the circuits of F_7 containing the point 7.

We use the following facts about regular matroids ([13], [15], [17]). A matroid is *regular* if it does not have any F_7 , F_7^* , or U_4^2 minor. Let \mathcal{M} be a regular matroid and let $M = [I|B]$ be a binary matrix representing \mathcal{M} over $GF(2)$. Then the 1's of B can be replaced by ± 1 's so that the resulting

matrix \tilde{B} is totally unimodular, i.e., each square submatrix of \tilde{B} has determinant $0, \pm 1$. Moreover, $\tilde{M} = [I | \tilde{B}]$ represents \mathcal{M} over \mathbb{R} and every binary vector x such that $Mx \equiv 0 \pmod{2}$ corresponds to some $0, \pm 1$ -vector y such that $\tilde{M}y = 0$, where y is obtained from x by replacing its 1's by ± 1 's.

Decomposition Result

The following decomposition result was proved by Tseng and Truemper ([14], Theorem 4.3); see also ([12], Theorem 1.3) and ([13], Chap. 13) for a detailed exposition.

THEOREM 2.3. *Let \mathcal{M} be a matroid on the set $E \cup \{l\}$. Assume that \mathcal{M} does not have any minor F_7^* using the element l . Then, one of the following holds:*

- (i) \mathcal{M} has a 1-separation.
- (ii) \mathcal{M} is 2-connected and has a 2-separation.
- (iii) \mathcal{M} is a regular matroid.
- (iv) \mathcal{M} is the Fano matroid F_7 .
- (v) \mathcal{M} is 3-connected and has a 3-separation $(E_1, E_2 \cup \{l\})$ such that (E_1, E_2) is a strict 2-separation of \mathcal{M}/l .

Remark 2.4. Theorem 2.3 differs from Theorem 1.3 of [12] in the statement (v). However, the above formulation of (v) follows from Theorems 1.3 and 2.1 from [12] (the latter theorem states that the triple $\{y, z, l\}$ forms a circuit of $\mathcal{M}/(X_1 - x) \setminus (Y_1 - \{y, z\})$) and from the above Proposition 2.2.

We will use this decomposition result in the following form.

THEOREM 2.5. *Let \mathcal{M} be a binary matroid on the set $E \cup \{l\}$. Assume that \mathcal{M} does not have any minor F_7^* using the element l and that \mathcal{M} does not have any minor F_7^+ using the element l as a series element. Assume also that l is neither a loop nor a coloop of \mathcal{M} . Then, one of the following holds:*

- (a) \mathcal{M}/l has a 1-separation.
- (b) \mathcal{M}/l has a strict 2-separation.
- (c) \mathcal{M} is regular.

Proof. We apply Theorem 2.3. The statement (iii) coincides with (c). Moreover, (b) applies in cases (iv) and (v). In case (i), if $(E_1, E_2 \cup \{l\})$ is a 1-separation of \mathcal{M} , then (E_1, E_2) is a 1-separation of \mathcal{M}/l since l is not a (co)loop of \mathcal{M} ; hence, (a) applies. We suppose finally that we are in the

case (ii), i.e., $(E_1, E_2 \cup \{l\})$ is a strict 2-separation of \mathcal{M} . If $r_{\mathcal{M}}(E_1) = r_{\mathcal{M}/l}(E_1) + 1$, then (E_1, E_2) is a 1-separation of \mathcal{M}/l and, thus, (a) applies. Otherwise, $r_{\mathcal{M}}(E_1) = r_{\mathcal{M}/l}(E_1)$, implying that $r_{\mathcal{M}/l}(E_1) + r_{\mathcal{M}/l}(E_2) = r_{\mathcal{M}/l}(E) + 1$. Hence, in order to show that (b) applies, we need only to check that $|E_2| \geq 2$. Suppose, for contradiction, that $|E_2| = 1$, i.e., $E_2 = \{l'\}$. We deduce that $\{l, l'\}$ is a cocircuit of \mathcal{M} . Therefore, \mathcal{M} can be seen as the series-extension of \mathcal{M}/l obtained by adding l in series with l' . If \mathcal{M}/l is regular, then \mathcal{M} is regular too and, thus, (c) applies. Hence, we can suppose that \mathcal{M}/l is 2-connected and not regular. It follows from [9] that \mathcal{M}/l has a minor F_7 or F_7^* using l' . It is easy to see that, if \mathcal{M}/l has a minor F_7^* using l' , then \mathcal{M} has a minor F_7^* using l and, if \mathcal{M}/l has a minor F_7 using l' , then \mathcal{M} has a minor F_7^+ using l as a series element. We obtain a contradiction in both cases. ■

Remark 2.6. One can check that under the conditions of Theorem 2.5 (i.e., \mathcal{M} is a binary matroid having no minor F_7^* using l , no minor F_7^+ using l as a series element, and l is not a (co)loop of \mathcal{M}) \mathcal{M}/l is regular or \mathcal{M} has a 1-separation.

Signed Circuits

Let \mathcal{M} be a binary matroid on $E \cup \{l\}$ and let \mathcal{L} denote the l -port of \mathcal{M} . A convenient way to refer to the members of \mathcal{L} is in terms of odd circuits of \mathcal{M}/l with respect to some signing. Given a set $\Sigma \subseteq E + l$, a subset $A \subseteq E$ is called Σ -even (resp. Σ -odd) if $|A \cap \Sigma|$ is even (resp. odd). The following is easy to check.

PROPOSITION 2.7. *Let Σ be a cocircuit of \mathcal{M} such that $l \in \Sigma$ and let C be a subset of E . Then, $C \in \mathcal{L}$ if and only if C is a Σ -odd circuit of \mathcal{M}/l .*

3. Q_6 , Q_7 , AND REGULAR CASE

In this section we show the following results:

- It is sufficient to work with fully fractional vertices, see Proposition 3.1.
- Box $\frac{1}{d}$ -integrality is preserved under minors, see Proposition 3.2.
- Q_6 , the port of F_7^* , is not box $\frac{1}{d}$ -integral for any integer $d \geq 2$, see Proposition 3.3.
- Q_7 , the port of the series-extension of F_7 with respect to a series element, is not box $\frac{1}{d}$ -integral for any integer $d \geq 2$, see Proposition 3.4.
- Any port of a regular matroid is box $\frac{1}{d}$ -integral for each integer $d \geq 1$, see Theorem 3.5.

The following result is easy to check.

PROPOSITION 3.1. *Let $f \in E$, $I \subseteq E - f$, $a \in (\frac{1}{d}\mathbb{Z})^I$ and $x \in \mathbb{R}^{E - f}$. Then,*

(i) *x belongs to (resp. is a vertex of) $Q(\mathcal{L}|f, a)$ if and only if $(x, 0)$ belongs to (resp. is a vertex of) $Q(\mathcal{L}, (a, 0))$.*

(ii) *x belongs to (resp. is a vertex of) $Q(\mathcal{L} \setminus f, a)$ if and only if $(x, 1)$ belongs to (resp. is a vertex of) $Q(\mathcal{L}, (a, 1))$.*

As an immediate consequence, we have that

PROPOSITION 3.2. *Every minor of a box $\frac{1}{d}$ -integral clutter is box $\frac{1}{d}$ -integral.*

PROPOSITION 3.3. *The clutter Q_6 is not box $\frac{1}{d}$ -integral, for any integer $d \geq 2$.*

Proof. Consider the vector $u \in \mathbb{R}^6$ defined by $u_1 = 1 - \frac{1}{d}$, $u_2 = u_6 = \frac{1}{d}$, $u_3 = u_5 = \frac{1}{2d}$, $u_4 = 1 - \frac{3}{2d}$. Set $a_1 = 1 - \frac{1}{d}$, $a_2 = a_6 = \frac{1}{d}$. Then, u belongs to the polyhedron $Q(Q_6, a)$. In fact, it is a vertex of that polyhedron, since it satisfies the following six linearly independent equalities: $u_1 + u_3 + u_5 = 1$, $u_2 + u_3 + u_4 = 1$, $u_4 + u_5 + u_6 = 1$, $u_1 = a_1$, $u_2 = a_2$, and $u_6 = a_6$. ■

PROPOSITION 3.4. *The clutter Q_7 is not box $\frac{1}{d}$ -integral, for any integer $d \geq 2$.*

Proof. Consider the vector $u \in \mathbb{R}^7$ defined by $u_1 = u_3 = u_5 = \frac{1}{2d}$, $u_2 = u_4 = u_6 = \frac{1}{d}$, and $u_7 = 1 - \frac{3}{2d}$. Set $a_2 = a_4 = a_6 = \frac{1}{d}$. Then, u belongs to the polyhedron $Q(Q_7, a)$. In fact, it is a vertex of that polyhedron, since it satisfies the following seven linearly independent equalities: $u_1 + u_4 + u_7 = 1$, $u_2 + u_5 + u_7 = 1$, $u_3 + u_6 + u_7 = 1$, $u_1 + u_3 + u_5 + u_7 = 1$, $u_2 = a_2$, $u_4 = a_4$, and $u_6 = a_6$. ■

THEOREM 3.5. *Let \mathcal{M} be the port of a regular matroid. Then, \mathcal{M} is box $\frac{1}{d}$ -integral for each integer $d \geq 1$.*

Proof. Let \mathcal{M} be a regular matroid on $E \cup \{l\}$ and let \mathcal{L} be its l -port. If l is a loop then $\mathcal{L} = \{\emptyset\}$, so \mathcal{L} is box $\frac{1}{d}$ -integral. We suppose now that l is not a loop. Since \mathcal{M} is regular, we can find a totally unimodular matrix M which represents \mathcal{M} over \mathbb{R} and is of the form shown in Fig. 4. We can suppose that the matrix A has full row rank.

Moreover, each set $C \in \mathcal{L}$ corresponds to a vector $y_C \in \{0, 1, -1\}^E$ such that

$$\begin{cases} r^T y_C = 1 \\ Ay_C = 0. \end{cases}$$

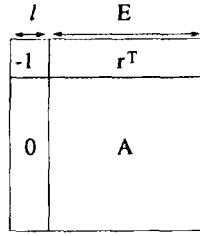


FIGURE 4

Each such y_C can be written as $y_C = y_C^1 - y_C^2$, where $y_C^1, y_C^2 \in \{0, 1\}^E$ and their supports $\{e \in E \mid (y_C^1)_e = 1\}, \{e \in E \mid (y_C^2)_e = 1\}$ partition the set C .

We define the polyhedron \mathcal{K} consisting of the vectors $(y_1, y_2) \in \mathbb{R}^E \times \mathbb{R}^E$ satisfying

$$\begin{cases} r^T y_1 - r^T y_2 = 1, \\ Ay_1 - Ay_2 = 0, \\ y_1, y_2 \geq 0. \end{cases}$$

Clearly, $(y_C^1, y_C^2) \in \mathcal{K}$ for each $C \in \mathcal{L}$. We state a preliminary result.

Claim 3.6. Let $u \in \mathbb{R}_+^E$. Then,

- (i) $\min(u(C) \mid C \in \mathcal{L}) = \min(u^T y_1 + u^T y_2 \mid (y_1, y_2) \in \mathcal{K})$.
- (ii) $u(C) \geq 1$ for all $C \in \mathcal{L}$ if and only if the system

$$\begin{cases} r^T + \pi^T A \leq u^T \\ -r^T - \pi^T A \leq u^T \end{cases}$$

(in the variable π) is feasible.

Proof. (i) The first minimum is greater or equal to the second one, since each $C \in \mathcal{L}$ corresponds to a pair $(y_C^1, y_C^2) \in \mathcal{K}$ such that $u(C) = u^T y_C^1 + u^T y_C^2$. Let (y_1, y_2) be a vertex of \mathcal{K} at which the second minimum is attained. Clearly, the supports of y_1, y_2 are disjoint. Since the matrix M is totally unimodular, we deduce that $y_1, y_2 \in \{0, 1\}^E$. Set $C = \{e \in E \mid (y_1)_e = 1 \text{ or } (y_2)_e = 1\}$. Then, $C \in \mathcal{L}$ and C corresponds to the vector $y_C = y_1 - y_2$ with $u^T y_1 + u^T y_2 = u(C)$. This shows that the second minimum is greater or equal to the first one.

- (ii) Observe that the system $\begin{cases} r^T + \pi^T A \leq u^T \\ -r^T - \pi^T A \leq u^T \end{cases}$ is feasible if and only if

$$\max(\rho \mid \rho r^T + \pi^T A \leq u^T, -\rho r^T - \pi^T A \leq u^T) \geq 1.$$

Moreover, by linear programming duality and Claim 3.6(i), we obtain:

$$\begin{aligned} & \max(\rho \mid \rho r^T + \pi^T A \leq u^T, -\rho r^T - \pi^T A \leq u^T) \\ & = \min(u^T y_1 + u^T y_2 \mid (y_1, y_2) \in \mathcal{X}) \\ & = \min(u(C) \mid C \in \mathcal{L}). \quad \blacksquare \end{aligned}$$

Let I be a subset of E and let $a \in (\frac{1}{d}\mathbb{Z})^I$. Let $\tilde{Q}(\mathcal{L}, a)$ denote the polyhedron consisting of the vectors $(\pi, u) \in \mathbb{R}^m \times \mathbb{R}^E$ (m denoting the number of rows of the matrix A) satisfying

$$\begin{cases} \pi^T A - u^T \leq -r^T, \\ -\pi^T A - u^T \leq r^T, \\ u_e = a_e \quad \text{for } e \in I. \end{cases}$$

Note that $\tilde{Q}(\mathcal{L}, a)$ has vertices as the matrix A has full row rank. By Claim 3.6(ii), $Q(\mathcal{L}, a)$ is the projection of $\tilde{Q}(\mathcal{L}, a)$ on the subspace \mathbb{R}^E .

Let u_0 be a vertex of $Q(\mathcal{L}, a)$. By Proposition 3.1, we can suppose that all components of u_0 are positive. Moreover, u_0 is the projection of a vertex (π_0, u_0) of $\tilde{Q}(\mathcal{L}, a)$. Since $\tilde{Q}(\mathcal{L}, a)$ is invariant under the multiplication of some columns of the matrix

$$\begin{bmatrix} r^T \\ A \end{bmatrix}$$

by -1 , we may assume that $\pi_0^T A + r^T \geq 0$ and, thus, that $-\pi_0^T A - u_0^T < r^T$. Therefore, (π_0, u_0) is a vertex of the polyhedron

$$\{(\pi, u) \mid \pi^T A - u^T \leq -r^T, u_e = a_e \text{ for } e \in I\}.$$

As the matrix defining this polyhedron is totally unimodular, we deduce that (π_0, u_0) is $\frac{1}{d}$ -integral. This shows that u_0 is $\frac{1}{d}$ -integral. (Note that the constraint matrix for $\tilde{Q}(\mathcal{L}, a)$ is *not* totally unimodular.) \blacksquare

4. PROOF OF THE MAIN RESULT

Let \mathcal{M} be a binary matroid on $E \cup \{l\}$ and let \mathcal{L} be the l -port of \mathcal{M} , i.e., $\mathcal{L} = \{C \subseteq E \mid C + l \text{ is a circuit of } \mathcal{M}\}$. Let $d \geq 1$ be an integer. We assume that \mathcal{L} does not have Q_6 or Q_7 as a minor. Hence, \mathcal{M} does not have F_7^* as a minor using l and \mathcal{M} does not have F_7^+ as a minor using l as a series element.

Our goal is to show that \mathcal{L} is box $\frac{1}{d}$ -integral. The proof is by induction on $|E| \geq 0$ and the main tool we use is Theorem 2.5.

The result holds for $|E| = 0$. Indeed, then l is either a loop, yielding $\mathcal{L} = \{\emptyset\}$, or a coloop, yielding $\mathcal{L} = \emptyset$. In both cases, \mathcal{L} is box $\frac{1}{d}$ -integral.

We assume that the result holds for every groundset with less than $|E|$ elements, i.e., that every binary clutter without Q_6 or Q_7 minor on a set with less than $|E|$ elements is box $\frac{1}{d}$ -integral.

We can suppose that l is neither a loop nor a coloop of \mathcal{M} . We know from Theorem 3.5 that \mathcal{L} is box $\frac{1}{d}$ -integral if \mathcal{M} is regular. From Theorem 2.5, we can assume that \mathcal{M}/l has a 1-separation or a strict 2-separation.

PROPOSITION 4.1. *If \mathcal{M}/l has a 1-separation, then \mathcal{L} is box $\frac{1}{d}$ -integral.*

Proof. Let (E_1, E_2) be a 1-separation of \mathcal{M}/l . Let \mathcal{L}_1 (resp. \mathcal{L}_2) denote the l -port of the matroid $\mathcal{M} \setminus E_2$ (resp. $\mathcal{M} \setminus E_1$). Clearly, $\mathcal{L}_1 \cup \mathcal{L}_2 \subseteq \mathcal{L}$; in fact, \mathcal{L}_1 and \mathcal{L}_2 partition \mathcal{L} . By the induction assumption, \mathcal{L}_1 and \mathcal{L}_2 are box $\frac{1}{d}$ -integral.

Given $I \subseteq E$ and $a \in (\frac{1}{d}\mathbb{Z})^I$, set $a_i = (a_e)_{e \in I \cap E_i}$, for $i = 1, 2$. Then, $Q(\mathcal{L}, a)$ is the cartesian product of $Q(\mathcal{L}_1, a_1)$ and $Q(\mathcal{L}_2, a_2)$, which implies that all its vertices are $\frac{1}{d}$ -integral. ■

From now on we assume that \mathcal{M}/l is 2-connected and admits a 2-separation (E_1, E_2) . Let I be a subset of E , let $a \in (\frac{1}{d}\mathbb{Z})^I$, and let u be a vertex of $Q(\mathcal{L}, a)$. Our goal is to show that u is $\frac{1}{d}$ -integral. From Proposition 3.1 and the induction hypothesis, we can suppose that $u_e \neq 0, 1$, for all $e \in E$. Call an inequality *tight* for u if it is satisfied at equality by u .

The inequalities defining $Q(\mathcal{L}, a)$ are of three types:

Type I: $x_e = a_e$ for $e \in I$.

Type II: $x(C) \geq 1$ for each noncrossing $C \in \mathcal{L}$ (i.e., $C \subseteq E_i$ for $i \in \{1, 2\}$).

Type III: $x(C) \geq 1$ for each crossing $C \in \mathcal{L}$.

The case when no inequality of type III is tight for u is easy:

PROPOSITION 4.2. *If $u(C) > 1$ for each crossing $C \in \mathcal{L}$, then u is $\frac{1}{d}$ -integral.*

Proof. The proof is analogous to that of Proposition 4.1. ■

We now suppose that there exists some crossing $C \in \mathcal{L}$ with $u(C) = 1$.

DEFINITION 4.3. We call *path* every set of the form $C \cap E_i$ where $i \in \{1, 2\}$ and $C \in \mathcal{L}$ is crossing.

Let Σ be a cocircuit of \mathcal{M} which contains l . Set

$$u_o = \min(u(P) \mid P \text{ is a path with } |P \cap \Sigma| \text{ odd}),$$

$$u_e = \min(u(P) \mid P \text{ is a path with } |P \cap \Sigma| \text{ even}).$$

Both u_o, u_e are well defined.

PROPOSITION 4.4. *We have that $u_o + u_e = 1$. Moreover, for each tight crossing $C \in \mathcal{L}$ with, say, $C \cap E_1$ Σ -odd and $C \cap E_2$ Σ -even, $u(C \cap E_1) = u_o$ and $u(C \cap E_2) = u_e$.*

Proof. Take $C \in \mathcal{L}$ crossing and tight. Then, $1 = u(C) = u(C \cap E_1) + u(C \cap E_2) \geq u_o + u_e$. Conversely, suppose that $u_o = u(C \cap E_i)$ and $u_e = u(C' \cap E_j)$, where $C, C' \in \mathcal{L}$ are crossing with $C \cap E_i$ Σ -odd, $C' \cap E_j$ Σ -even and $i, j \in \{1, 2\}$. From Proposition 2.1(ii), $C'' = (C \cap E_i) \Delta (C' \cap E_j)$ is a cycle of \mathcal{M}/l . Hence, $C'' = \bigcup_h C_h$, where C_h are pairwise disjoint circuits of \mathcal{M}/l . Since C'' is Σ -odd, at least one of the C_h 's is Σ -odd, i.e., belongs to \mathcal{L} . This implies that $u(C'') = \sum_h u(C_h) \geq 1$. Therefore, $u_o + u_e \geq 1$. Hence, we have the equality $u_o + u_e = 1$. The last part of the proposition follows immediately. ■

Let \mathcal{B} be a base of equalities for u , i.e., \mathcal{B} is a maximal set of linearly independent inequalities chosen among the inequalities defining $Q(\mathcal{L}, a)$ that are satisfied at equality by u . Let \mathcal{B}_i denote the subset of \mathcal{B} consisting of the inequalities which are supported by E_i , for $i = 1, 2$. Hence, $\mathcal{B}_1 \cup \mathcal{B}_2$ consists of inequalities of Type I or II and $\mathcal{B} - \mathcal{B}_1 \cup \mathcal{B}_2$ of inequalities of Type III. We can partition $\mathcal{B} - \mathcal{B}_1 \cup \mathcal{B}_2$ as $\mathcal{B}_3 \cup \mathcal{B}_4$, where \mathcal{B}_3 consists of inequalities $x(C) \geq 1$ for $C \in \mathcal{L}$ crossing with $C \cap E_1$ Σ -odd, $C \cap E_2$ Σ -even, and \mathcal{B}_4 of such inequalities with $C \in \mathcal{L}$ crossing, $C \cap E_1$ Σ -even and $C \cap E_2$ Σ -odd.

PROPOSITION 4.5. *There exists a base \mathcal{B} of equalities for u for which $\mathcal{B}_3 = \emptyset$ or $\mathcal{B}_4 = \emptyset$.*

Proof. Let \mathcal{B} be a base of equalities for u for which $|\mathcal{B}_1 \cup \mathcal{B}_2|$ is maximum. Suppose, for contradiction, that $\mathcal{B}_3 \neq \emptyset$ and $\mathcal{B}_4 \neq \emptyset$. Let $C, C' \in \mathcal{L}$ be crossing and yielding equalities of \mathcal{B} with $C \cap E_1, C' \cap E_2$ Σ -even and $C \cap E_2, C' \cap E_1$ Σ -odd. By Proposition 2.1(ii), $D_i := (C \cap E_i) \Delta (C' \cap E_i)$ is a cycle of \mathcal{M}/l . Moreover, D_i is Σ -odd by construction. Hence, $D_i = \bigcup_h C_h$ where the C_h 's are pairwise disjoint circuits of \mathcal{M}/l and at least one of them is Σ -odd. Using Proposition 4.4, we obtain that $1 = u_e + u_o \geq u(D_i) \geq 1$ which implies that $C \cap C' = \emptyset$ and that D_1 and D_2 are (noncrossing) circuits of \mathcal{M}/l , each yielding a tight equality for u . The

base \mathcal{B} cannot contain both equations $x(D_1)=1$ and $x(D_2)=1$ since $C \cup C' = D_1 \cup D_2$. If \mathcal{B} contains $x(D_1)=1$ but not $x(D_2)=1$, then, by replacing the equation $x(C')=1$ by the equation $x(D_2)=1$, we obtain a new base \mathcal{B}' (this follows from the fact that \mathcal{B} is a base and the relation $x(C) + x(C') = x(D_1) + x(D_2)$). As \mathcal{B}' satisfies: $|\mathcal{B}'_1 \cup \mathcal{B}'_2| > |\mathcal{B}_1 \cup \mathcal{B}_2|$, we have a contradiction with the choice of \mathcal{B} . Therefore, \mathcal{B} contains none of the equations $x(D_1)=1, x(D_2)=1$. At least one of them can be added to \mathcal{B} after deleting the equation $x(C')=1$, still preserving linear independence. Again we obtain a contradiction with the maximality of $|\mathcal{B}_1 \cup \mathcal{B}_2|$. ■

By symmetry, we can suppose that we have a base \mathcal{B} of equalities for u with $\mathcal{B}_3 = \emptyset, \mathcal{B}_4 \neq \emptyset$. (If both \mathcal{B}_3 and \mathcal{B}_4 are empty, we can conclude in the same way as in Proposition 4.2.) In matrix form, the system \mathcal{B} can be written as $Px = \beta$, where β is the vector consisting of the right hand sides of the inequalities and P is the nonsingular matrix shown in Fig. 5.

Hence, there exists a tight equality $u(C^*) = 1$ where $C^* \in \mathcal{L}$ is crossing, $C^* \cap E_1$ is Σ -odd and $C^* \cap E_2$ is Σ -even. We can find two elements $e_1 \in C^* \cap E_2, e_2 \in C^* \cap E_1$ with $e_1 \notin \Sigma$ and $e_2 \in \Sigma$ (after eventually changing the cocircuit Σ). (Indeed, let $e_2 \in C^* \cap E_1, e_1 \in C^* \cap E_2$ and let X be a base of \mathcal{M} containing $(C^* - e_2) \cup \{l\}$. Let Σ' denote the fundamental cocircuit of l in the base X ; then, $e_2 \in \Sigma'$ since $C^* + l$ is the fundamental circuit of e_2 in the base X , and $e_1 \notin \Sigma'$ since $e_1 \in X$. Hence, it suffices to replace Σ by Σ').

Set $\mathcal{M}_1 = \mathcal{M} / ((C^* \cap E_2) - e_1) \setminus (E_2 - C^*)$ and $\mathcal{M}_2 = \mathcal{M} / ((C^* \cap E_1) - e_2) \setminus (E_1 - C^*)$, defined, respectively, on the sets $E_1 \cup \{e_1, l\}$ and $E_2 \cup \{e_2, l\}$. (Note that \mathcal{M}_1 coincides with $\mathcal{M} / (X_2 - e_1) \setminus Y_2$ and \mathcal{M}_2 coincides with $\mathcal{M} / X_1 \setminus (Y_1 - e_2)$, where $X_i = X \cap E_i, Y_i = E_i - X_i$ for $i = 1, 2$. Also, \mathcal{M} / l is the 2-sum of \mathcal{M}_1 / l and \mathcal{M}_2 / l . Recall Section 2.)

Let \mathcal{L}_i denote the l -port of \mathcal{M}_i . By the induction assumption, \mathcal{L}_i is box $\frac{1}{d}$ -integral, for $i = 1, 2$.

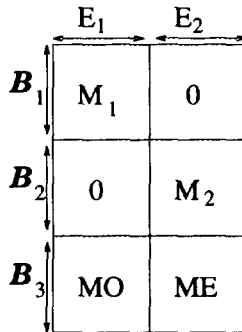


FIGURE 5

Let u_i denote the projection of u on \mathbb{R}^{E_i} and set $a_i = (a_e)_{e \in I \cap E_i}$, for $i = 1, 2$. We define $u_i^* \in \mathbb{R}^{E_i + e_i}$ by

$$\begin{cases} u_i^*(e) = u_i(e) & \text{for } e \in E_i, \quad i = 1, 2, \\ u_1^*(e_1) = u_e, \\ u_2^*(e_2) = u_o. \end{cases}$$

PROPOSITION 4.6. $u_i^* \in Q(\mathcal{L}_i, a_i)$, for $i = 1, 2$.

Proof. We give the proof for $i = 1$, the case $i = 2$ is identical. Take $C \in \mathcal{L}_1$. By Proposition 2.1(i), either $C \in \mathcal{L}$ and, thus, $u_1^*(C) = u(C) \geq 1$, or $C = C' \cap E_1 + e_1$ for some crossing circuit C' of \mathcal{M}/l . Then, $C' \cap E_1$ is Σ -odd, since C is Σ -odd and $e_1 \notin \Sigma$. By Proposition 2.1(ii), $(C' \cap E_1) \Delta (C^* \cap E_2)$ is a cycle of \mathcal{M}/l and it is Σ -odd since $C^* \cap E_2$ is Σ -even. Hence, $u(C' \cap E_1) + u(C^* \cap E_2) \geq 1$ which, together with $u(C^* \cap E_2) = u_e$, implies that $u(C' \cap E_1) \geq 1 - u_e = u_o$. Therefore, $u_1^*(C) = u(C' \cap E_1) + u_e \geq u_o + u_e = 1$. ■

We construct the set $\mathcal{B}^{(i)}$ of equalities for u_i^* consisting of

- the equalities of \mathcal{B}_i ,
- the equalities $x((C \cap E_i) + e_i) = 1$, one for each equality $x(C) = 1$ of \mathcal{B}_3 .

All equalities of $\mathcal{B}^{(i)}$ arise from those defining $Q(\mathcal{L}_i, a_i)$. Indeed, by Proposition 2.1, if $C \in \mathcal{L}$ with $C \subseteq E_i$, then $C \in \mathcal{L}_i$ and, if $C \in \mathcal{L}$ is crossing, then $(C \cap E_i) + e_i \in \mathcal{L}_i$, for $i = 1, 2$.

PROPOSITION 4.7. *The set $\mathcal{B}^{(i)}$ has rank $|E_i| + 1$ for at least one index $i \in \{1, 2\}$.*

Proof. We show that one of the two matrices in Figs. 6 and 7 has full rank $|E_i| + 1$.

As the matrix P of Fig. 5 has full rank $|E_1| + |E_2|$, it follows easily that the matrix displayed in Fig. 9 has full rank $|E_1| + |E_2| + 2$. This implies that

M_1	0
MO	1

FIGURE 6

0	M_2
1	ME

FIGURE 7

M_1	0	0	0
MO	1	0	0
0	0	0	M_2
0	0	1	ME
0	1	1	0

FIGURE 8

M_1	0	0	0
0	0	0	M_2
MO	0	0	ME
-MO	0	1	0
0	1	0	-ME

FIGURE 9

the matrix shown in Fig. 8 has also full rank $|E_1| + |E_2| + 2$, as it can be obtained by row and column operations from the matrix in Fig. 9. ■

By symmetry, we can suppose that $\mathcal{B}^{(1)}$ has full rank. This implies that u_1^* is a vertex of $Q(\mathcal{L}_1, a_1)$ and, thus, u_1^* is $\frac{1}{d}$ -integral, since \mathcal{L}_1 is box $\frac{1}{d}$ -integral. In particular, u_c is $\frac{1}{d}$ -integral, implying that $u_o = 1 - u_c$ is $\frac{1}{d}$ -integral. If we introduce the constraint $x(e_2) = u_o$, then u_2^* becomes a vertex of the polytope $Q(\mathcal{L}_2, a_2) \cap \{x \mid x(e_2) = u_o\}$ and, thus, u_2^* is $\frac{1}{d}$ -integral.

This shows that u is $\frac{1}{d}$ -integral and concludes the proof. ■

5. APPLICATIONS FOR GRAPHS

A *signed graph* is a pair (G, Σ) , where $G = (V, E)$ is a graph and Σ is a subset of the edge set E of G . The edges in Σ are called *odd* and the other edges *even*. An *odd circuit* C in (G, Σ) is a circuit C of G such that $|C \cap \Sigma|$ is odd. If $\delta(U)$ is a cut in G , then the two signed graphs (G, Σ) and $(G, \Sigma \Delta \delta(U))$ have the same collection of odd circuits. The operation $\Sigma \rightarrow \Sigma \Delta \delta(U)$ is called *resigning* (by the cut $\delta(U)$). We say that (G, Σ) *reduces* to (G', Σ') if (G', Σ') can be obtained from (G, Σ) by a sequence of the following operations:

- deleting an edge of G (and Σ),
- contradicting an even edge of G ,
- resigning.

The collection of odd circuits of a signed graph is a binary clutter. Indeed, given a signed graph (G, Σ) , let $\mathcal{S}(G, \Sigma)$ denote the binary matroid on $\{l\} \cup E$ represented over $GF(2)$ by the matrix

$$\left[\begin{array}{c|c} 1 & \sigma \\ \hline 0 & M_G \end{array} \right]$$

where M_G is the node-edge incidence matrix of G and σ is the incidence vector of the set Σ . Clearly, the l -port of $\mathcal{S}(G, \Sigma)$ coincides with the family of odd circuits of (G, Σ) . In particular, the collection of odd circuits of the signed graph $(K_4, E(K_4))$, i.e., K_4 with all edges odd, is the clutter \mathcal{Q}_6 , i.e. $\mathcal{S}(K_4, E(K_4))$ is F_7^* . One can check that (G, Σ) does not reduce to $(K_4, E(K_4))$ if and only if $\mathcal{S}(G, \Sigma)$ does not have an F_7^* minor using the element l . Moreover, $\mathcal{S}(G, \Sigma)$ does not have any minor F_7^+ using l as a series element, otherwise F_7 would be a minor of the graphic matroid $\mathcal{H}(G) = \mathcal{S}(G, \Sigma)/l$. (See [5] for details.)

The following result is an immediate application of Theorem 1.2.

THEOREM 5.1. *Let (G, Σ) be a signed graph and let \mathcal{L} denote its collection of odd circuits. The following assertions are equivalent.*

- (i) (G, Σ) does not reduce to $(K_4, E(K_4))$.
- (ii) \mathcal{L} is box $\frac{1}{d}$ -integral for any integer $d \geq 1$.
- (iii) \mathcal{L} is box $\frac{1}{d}$ -integral for some integer $d \geq 2$.

Given a graph $G = (V, E)$, we consider the polytope

$$R(G) = \{x \in \mathbb{R}^E \mid x(F) - x(C - F) \leq |F| - 1 \text{ (} C \text{ circuit of } G, F \subseteq C, |F| \text{ odd)}, \\ 0 \leq x_e \leq 1 \text{ (} e \in E)\}.$$

The polytope $R(G)$ is a relaxation of the cut polytope $P(G)$ (defined as the convex hull of the incidence vectors of the cuts of G). In general, $R(G)$ has fractional vertices. In fact, the 0, 1-vertices of $R(G)$ are the incidence vectors of the cuts of G , and $R(G)$ has only integral vertices, i.e. $R(G) = P(G)$, if and only if G does not have K_5 as a minor [2]. The fractional vertices of $R(G)$ have been studied in [6], [7].

The case $d = 3$ of the following Theorem 5.2. was proved in [7]. We will show how Theorem 5.2. follows from Theorem 5.1.

THEOREM 5.2. *Let $G = (V, E)$ be a graph. The following assertions are equivalent.*

- (i) G is series parallel, i.e., G does not have K_4 as a minor.
- (ii) For each $I \subseteq E$ and $a \in (\frac{1}{d}\mathbb{Z})^I$, all the vertices of the polytope $R(G) \cap \{x \mid x_e = a_e \text{ for } e \in I\}$ are $\frac{1}{d}$ -integral, for any integer $d \geq 1$.
- (iii) For each $I \subseteq E$ and $a \in (\frac{1}{d}\mathbb{Z})^I$, all the vertices of the polytope $R(G) \cap \{x \mid x_e = a_e \text{ for } e \in I\}$ are $\frac{1}{d}$ -integral, for some integer $d \geq 2$.

Proof. Let $G' = (V, E \cup E')$ denote the graph obtained from G by adding an edge e' in parallel with each edge e of G . We consider the signed graph (G', E') , so the edges of E are even and those of E' are odd. It is easy to see that G is series parallel if and only if (G', E') does not reduce to $(K_4, E(K_4))$. Let \mathcal{L}' denote the collection of odd circuits of (G', E') . From Theorem 5.1, \mathcal{L}' is box $\frac{1}{d}$ -integral if G is series parallel. For $x \in \mathbb{R}^E$, define $x' \in \mathbb{R}^{E'}$ by $x'_e = 1 - x_e$ for $e \in E$ and, for $a \in (\frac{1}{d}\mathbb{Z})^I$ with $I \subseteq E$, set $a'_e = 1 - a_e$ for $e \in I$.

Observe that $R(G) \cap \{x \mid x_e = a_e \text{ for } e \in I\} = \{x \mid (x, x') \in Q(\mathcal{L}', (a, a'))\}$. As $\{e, e'\} \in \mathcal{L}'$ for each $e \in E$, $Q(\mathcal{L}', (a, a')) \cap \{(x, y) \in \mathbb{R}^E \times \mathbb{R}^{E'} \mid y_e = 1 - x_e \text{ for } e \in E\}$ is a face of $Q(\mathcal{L}', (a, a'))$. Therefore, $R(G) \cap \{x \mid x_e = a_e \text{ for } e \in I\}$ is the projection of a face of $Q(\mathcal{L}', (a, a'))$. Hence, all its vertices are $\frac{1}{d}$ -integral if G is series parallel. This proves (i) \Rightarrow (ii).

It is easy to check that (iii) is closed under graph minors. Moreover, K_4 does not have the property (iii). Indeed, consider K_4 with its edges labeled 1, 2, 3, 4, 5, 6 in such a way that the triangles of K_4 are $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, $\{4, 5, 6\}$ (i.e., the members of Q_6). Set $x_2 = x_4 = x_6 = \frac{1}{d}$ and $x_1 = x_3 = x_5 = \frac{1}{2d}$. Then, x is a non $\frac{1}{d}$ -integral vertex of the polytope $R(K_4) \cap \{x \mid x_i = \frac{1}{d} \text{ for } i = 2, 4, 6\}$. This shows (iii) \Rightarrow (i). ■

More generally, given a binary matroid \mathcal{M} on a set E , consider the polytope $R(\mathcal{M})$ in \mathbb{R}^E defined by the inequalities $0 \leq x_e \leq 1$ for $e \in E$, and $x(F) - x(C - F) \leq |F| - 1$ for $F \subseteq C$ with $|F|$ odd and C circuit of \mathcal{M} . Hence, $R(\mathcal{M})$ coincides with $R(G)$ when \mathcal{M} is the graphic matroid $\mathcal{M}(G)$ of G . The 0, 1-vertices of $R(\mathcal{M})$ are the incidence vectors of the cocycles of \mathcal{M} . The matroids \mathcal{M} for which all vertices of $R(\mathcal{M})$ are integral have been characterized in [1] using a result of [11]. A natural question to ask is what are the matroids \mathcal{M} for which $R(\mathcal{M})$ is box $\frac{1}{d}$ -integral. Actually, this class is not larger than in the graphic case. To see this, observe that $F_7^* / I = \mathcal{M}(K_4)$ and that $F_7^* / I = F_7$ has an $\mathcal{M}(K_4)$ minor. On the other hand, a binary matroid \mathcal{M} has no $\mathcal{M}(K_4)$ minor if and only if \mathcal{M} is the graphic matroid of a series parallel graph. The latter follows easily from Tutte's forbidden minor characterization of graphic matroids ([16]).

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