

ℓ_1 -Rigid Graphs

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Abstract. An ℓ_1 -graph is a graph whose nodes can be labeled by binary vectors in such a way that the Hamming distance between the binary addresses is, up to scale, the distance in the graph between the corresponding nodes. We show that many interesting graphs are ℓ_1 -rigid, i.e., that they admit an essentially unique such binary labeling.

Keywords: ℓ_1 -graph, cut cone, rigidity

1. Introduction

In this paper, we consider ℓ_1 -graphs, i.e., graphs whose path metric is isometrically embeddable in some ℓ_1 -space. In fact, ℓ_1 -graphs are exactly those admitting a binary addressing such that, up to scale, the Hamming distance between the binary addresses of two nodes coincides with their distance in the graph. Namely, a graph $G = (V, E)$ is an ℓ_1 -graph if, for some integer $m \geq 1$, there exists a map $\varphi: V \rightarrow \{0, 1\}^m$ and an integer scalar λ such that

$$\lambda d_G(u, v) = d_H(\varphi(u), \varphi(v))$$

for all nodes $u, v \in V$, where d_G denotes the path metric of G and d_H the Hamming distance, i.e., the path metric of the m -dimensional hypercube $H(m, 2)$. Such map φ is called an *embedding with scale* λ , or a λ -*embedding* of G into the hypercube $H(m, 2)$. Hence, isometric embeddings into the hypercube correspond to the case $\lambda = 1$. The minimum λ for which there exists a λ -embedding of G into a hypercube is called the *scale* of G . Here, we consider more specifically ℓ_1 -rigid graphs, i.e., ℓ_1 -graphs admitting essentially a unique ℓ_1 -embedding.

The ℓ_1 -graphs have been characterized in [24] and later in [10]; namely, a graph is an ℓ_1 -graph if and only if it is an isometric subgraph of a product of half-cubes and cocktail party graphs. The proof of [10] is shorter, since it relies on the theory of Delaunay polytopes in lattices (see, e.g., [11] for details); moreover, it applies to a larger class of metrics than graphic ones. The proof of [24] provides some insight on the structure of ℓ_1 -graphs. Moreover, it permits to establish that ℓ_1 -graphs can be recognized in polynomial time. Note that, for general metrics, the problem of testing ℓ_1 -embeddability is *NP-hard* [20].

More precisely, Shpectorov [24] proved the following result.

THEOREM 1.1. *Let G be an ℓ_1 -graph on n vertices. Then, there exist a graph Γ and an isometric embedding φ from G into Γ such that*

- (i) Γ is a Cartesian product of complete graphs, cocktail party graphs and half-cubes,
- (ii) if ψ is a λ -embedding from G into the hypercube $H(m, 2)$, then there exists a λ -embedding $\tilde{\psi}$ from Γ into the same hypercube $H(m, 2)$ such that $\psi = \tilde{\psi}\varphi$.

Moreover, the scale λ of G is either equal to 1, or is even, with $\lambda \leq n - 2$ if $n \geq 4$.

As a consequence of Theorem 1.1, G is ℓ_1 -rigid if and only if the graph Γ is ℓ_1 -rigid. Since Γ is a Cartesian product, it is ℓ_1 -rigid if and only if all its factors are ℓ_1 -rigid (see Fact (2.2) below). This is equivalent to the condition that no factor is isomorphic to a complete graph or a cocktail party graph on at least 4 vertices. Therefore,

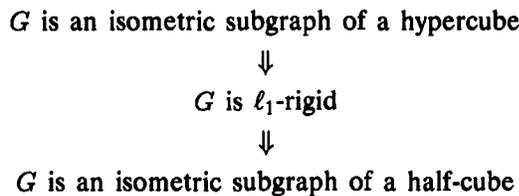
COROLLARY 1.1 [24]. *Every ℓ_1 -rigid graph G is an isometric subgraph of a half-cube, i.e., G has scale 1 or 2.*

Moreover, if G is not ℓ_1 -rigid, then the variety of its ℓ_1 -embeddings arises from that of the complete graph (since the variety of the ℓ_1 -embeddings of the cocktail party graph comes from that of the complete graph, see Facts (2.6) and (2.9)).

It is proved in [23] that, if G is bipartite, then G is an ℓ_1 -graph if and only if G is an isometric subgraph of a hypercube, i.e., G has scale 1.

On the other hand, every isometric subgraph of a hypercube is ℓ_1 -rigid; see Proposition 3.1. Actually, this result is also implied by the following result from [22] (see Section 4.1 for more details). Let G be a bipartite graph, then its path metric d_G is *metrically rigid*, i.e., there is a unique way of writing d_G as a sum of metrics.

The above discussion can be summarized by the following (strict) implications:



Therefore, ℓ_1 -rigidity is an important concept lying in between the well-studied concept of isometric hypercube embeddability (scale 1) and its relaxation to isometric half-cube embeddability (scale 1 or 2).

The questions of isometric embedding into a hypercube have been extensively studied; they have many applications, in particular, for addressing problems in communication networks (see, e.g., [19] and references there). Actually, the

question of isometric ℓ_1 -embedding (i.e., binary addressing up to scale) was already considered in [3].

The isometric subgraphs of hypercubes have been characterized by Djoković [18]. Given a graph $G = (V, E)$ and an edge $e = (i, j)$ of G , set

$$G(i, j) = \{u \in V : d_G(i, u) < d_G(j, u)\},$$

i.e., $G(i, j)$ consists of the nodes which are closer to i than to j . If G is bipartite, then the sets $G(i, j)$ and $G(j, i)$ form a partition of V , for each edge $e = (i, j)$ of G .

THEOREM 1.2 [18]. *The graph G is an isometric subgraph of a hypercube if and only if G is bipartite and, for every edge $e = (i, j)$ of G , both sets $G(i, j)$ and $G(j, i)$ are closed under taking shortest paths.*

On the other hand, no similar characterization is known for the isometric subgraphs of half-cubes.

In this paper, we present several classes of ℓ_1 -rigid graphs. This permits us to obtain some classifications for ℓ_1 -rigid graphs within several interesting classes of graphs, especially within distance regular graphs.

For definitions, notation and quoting of results on graphs, we rely entirely on the monography [4].

The paper is organized as follows. In Section 2, we give a catalog of graphs for which we group information on their ℓ_1 -embeddings, mainly, on rigidity and scale. We also present some operations which preserve ℓ_1 -embeddability and ℓ_1 -rigidity. Section 3 deals with bipartite graphs and some weighted graphs. We group in Section 4 all the proofs and tools. Actually, the main tool we use is the fact that ℓ_1 -metrics are exactly the points of the cut cone and ℓ_1 -rigid metrics are those lying on a simplicial face of this cone. In fact, all the examples of ℓ_1 -rigid graphs that we present here share the property that they are lying on a simplicial face of the cut cone, which is entirely defined by triangle inequalities. In Section 5, via some classification theorems from [4], and as application of Sections 2 and 3, we specify ℓ_1 -graphs and their rigidity within some important classes of graphs.

2. A catalog of ℓ_1 -graphs

Let d be a function defined on the pairs of $V = \{1, \dots, n\}$; d is a *metric* on V if d satisfies the following triangle inequality

$$d(i, j) - d(i, k) - d(j, k) \leq 0 \tag{1}$$

for all $i, j, k \in V$. The metric cone \mathcal{M}_n is the cone defined by the triangle inequalities (1).

Then, d is an ℓ_1 -metric if d is isometrically ℓ_1 -embeddable, i.e., there exist an integer $m \geq 1$ and n vectors $x_1, \dots, x_n \in \mathbb{R}^m$ such that $d(i, j) = \|x_i - x_j\|$ for all $i, j \in V$. We say that x_1, \dots, x_n form an ℓ_1 -embedding of d in \mathbb{R}^m . (For $y = (y_h)_{1 \leq h \leq m} \in \mathbb{R}^m$, $\|y\| = \sum_{1 \leq h \leq m} |y_h|$ denotes the ℓ_1 -norm.) When the embedding is binary, i.e., $x_1, \dots, x_n \in \{0, 1\}^m$, then the ℓ_1 -distance $\|x_i - x_j\|$ coincides with the Hamming distance $d_H(x_i, x_j)$.

Let d be an ℓ_1 -metric and let x_1, \dots, x_n be an ℓ_1 -embedding of d in \mathbb{R}^m . Consider the $n \times m$ matrix M whose rows are the vectors x_1, \dots, x_n ; M is called an ℓ_1 -realization matrix of d . Consider the following three operations on the realization matrix M :

- add to (or delete from) M a column with all identical entries,
- add an arbitrary vector $a \in \mathbb{R}^m$ to all the rows of M (translation),
- permute the columns of M .

Call two ℓ_1 -embeddings *equivalent* if their realization matrices can be obtained from one another via the above three operations. Then, a metric is ℓ_1 -rigid if it admits, up to equivalence, a unique ℓ_1 -embedding.

Fact (2.1) [2]. Let d be rational valued. Then, d is an ℓ_1 -metric if and only if λd admits a binary ℓ_1 -embedding for some integer $\lambda \geq 1$. We call the smallest such integer λ the *scale* of d .

For ℓ_1 -metrics with scale 1, i.e., metrics admitting a binary ℓ_1 -embedding, the following weaker notion of rigidity has been considered. Call d *h-rigid* if d admits a unique (up to equivalence) binary ℓ_1 -embedding. Hence, if d is ℓ_1 -rigid, then d is *h-rigid*, but not vice versa. For example, given an integer $t \geq 1$, Deza [9] proved that the equidistant metric $2td(K_n)$ (taking the value $2t$ on all pairs of $\{1, \dots, n\}$) is *h-rigid* for $n \geq t^2 + t + 3$. However, $2td(K_n)$ is not ℓ_1 -rigid for $n \geq 4$ and $t \geq 1$, while $2td(K_4)$ is already not *h-rigid* for $t \geq 1$. In fact, $2d(K_4)$ has exactly two (up to equivalence) binary ℓ_1 -embeddings.

We shall consider in this paper only metrics arising from graphs, or weighted graphs. Let $G(V, E)$ be a graph and let $w = (w_e)_{e \in E}$ be nonnegative weights assigned to the edges of G . The *path metric* $d_{(G, w)}$, or $d(G, w)$, of the weighted graph (G, w) is defined as follows: $d_{(G, w)}(i, j)$ is the shortest w -length $\sum_{e \in P} w_e$ of a path P joining i and j in G . In the unweighted case: $w_e = 1$ for all edges $e \in E$, we simply denote the path metric by d_G , or $d(G)$. When the path metric $d_{(G, w)}$ is an ℓ_1 -metric, we say that (G, w) is an ℓ_1 -graph and, when $d_{(G, w)}$ is ℓ_1 -rigid, we say that the graph (G, w) is ℓ_1 -rigid.

The ℓ_1 -graphs with scale 1, i.e., the graphs admitting a binary ℓ_1 -embedding, are precisely the isometric subgraphs of a cube. The ℓ_1 -graphs with scale 2 are the isometric subgraphs of a half-cube; they are called code graphs in [21].

A useful operation on metrics is the direct product operation. Let d_a be a

metric on V_a , for $a = 1, 2$. Consider the function d defined on $V_1 \times V_2$ by:

$$d((i_1, i_2), (j_1, j_2)) = d_1(i_1, j_1) + d_2(i_2, j_2)$$

for all $(i_1, i_2), (j_1, j_2) \in V_1 \times V_2$. Then, d is a metric on $V_1 \times V_2$, called the *direct product* of d_1, d_2 . Note that, for path metrics, this corresponds to the Cartesian product of graphs. It is well known and easy to see that ℓ_1 -embeddability is preserved by direct product. Moreover,

Fact (2.2). Let d be the direct product of d_1 and d_2 . Then, d is ℓ_1 -rigid if and only if d_1 and d_2 are ℓ_1 -rigid.

The following 1-sum operation also preserves ℓ_1 -metrics and ℓ_1 -rigidity. Let V_1 and V_2 be two sets which intersect in exactly one element, $V_1 \cap V_2 = \{k\}$. Let d_a be a metric defined on the set V_a , for $a = 1, 2$. Their 1-sum is the metric d defined on the set $V_1 \cup V_2$ by

$$\begin{cases} d(i, j) = d_a(i, j) & \text{if } i, j \in V_a \text{ for some } a = 1, 2 \\ d(i, j) = d_1(i, k) + d_2(j, k) & \text{if } i \in V_1, j \in V_2. \end{cases}$$

Fact (2.3). Let d be the 1-sum of d_1 and d_2 . Then, d is an ℓ_1 -metric if and only if d_1 and d_2 are ℓ_1 -metrics. Moreover, d is ℓ_1 -rigid if and only if d_1 and d_2 are ℓ_1 -rigid.

Another useful operation on metrics is the antipodal extension operation (for details, see [13]). Given a metric d on $V = \{1, \dots, n\}$ and $\alpha \in \mathbb{R}$, we define its *antipodal extension* $ant_\alpha(d)$ on $V \cup \{n+1\}$ as follows:

$$\begin{cases} ant_\alpha(d)_{1, n+1} = \alpha \\ ant_\alpha(d)_{i, n+1} = \alpha - d_{1i} & \text{for } 2 \leq i \leq n \\ ant_\alpha(d)_{ij} = d_{ij} & \text{for } 1 \leq i < j \leq n. \end{cases}$$

So, $ant_\alpha(d)$ is an extension of d obtained by adding a new node $n+1$ at distance α from node 1 and $\alpha - d_{1i}$ from the other nodes. This operation can be repeated, up to doubling of all the nodes; then, we obtain a metric on $2n$ nodes, denoted as $Ant_\alpha(d)$.

The nice feature of the antipodal extension operation is that, under some condition on α , it preserves ℓ_1 -embeddability and ℓ_1 -rigidity. Namely,

Fact (2.4) [2], [13]. $ant_\alpha(d)$ is an ℓ_1 -metric if and only if d is an ℓ_1 -metric and admits an ℓ_1 -embedding of size less than or equal to α ; moreover, $ant_\alpha(d)$ is ℓ_1 -rigid if and only if d is ℓ_1 -rigid. (See Fact (4.4), and Section 4.1 for the notion of size.)

In this paper, we study ℓ_1 -rigidity for the following classes of graphs:

- K_n , the complete graph (1-skeleton of the simplex α_{n-1})
- $K_{n \times 2}$, the cocktail party graph (1-skeleton of the cross polytope β_n)
- $H(n, d) \simeq (K_d)^n$, the Hamming graph and, in particular,
 - $H(n, 2)$, the cube (1-skeleton of the hypercube γ_n)
 - $H(2, d)$, the $d \times d$ -grid
- $\frac{1}{2}H(n, 2)$, the half-cube
- $J(n, d)$, the Johnson graph
- DO_{2n+1} , the double odd graph
- $\mathcal{A}(n, t)$ (considered in [21])
- O_3 , the Petersen graph
- the Shrikhande graph
- the (1-skeleton of the) dodecahedron
- the (1-skeleton of the) icosahedron
- C_n , the cycle of length n .

We will also consider in the next section ℓ_1 -rigidity for bipartite graphs and weighted cycles.

All the graphs listed above are ℓ_1 -graphs and all of them, except $K_{n \times 2}$ for $n \geq 5$, have scale 1 or 2. In fact, the ℓ_1 -graphs with scale 1 are the bipartite ℓ_1 -graphs, including even cycles, cubes, double odd graphs. We specify below which graphs are ℓ_1 -rigid or not.

Fact (2.5). K_n is not ℓ_1 -rigid, except for $n = 2, 3$. See [15] for the study of the variety of ℓ_1 -embeddings of K_n and connections with design theory.

Fact (2.6). $K_{n \times 2}$ is not ℓ_1 -rigid, except for $n = 2, 3$. In fact, its variety of ℓ_1 -embeddings entirely comes from the variety of ℓ_1 -embeddings of K_n (via the antipodal extension operation since $d(K_{n \times 2}) = \text{Ant}_2(d(K_n))$, see Section 4.3).

Fact (2.7). Any Doob graph (i.e., product of copies of K_4 and of the Shrikhande graph) is not ℓ_1 -rigid whenever it involves some K_4 ; its variety of ℓ_1 -embeddings comes from that of K_4 (by the direct product operation, since the Shrikhande graph is ℓ_1 -rigid).

Fact (2.8). The Hamming graph $H(n, d)$ is not ℓ_1 -rigid, except for $d \leq 3$ (again its variety of embeddings comes from that of K_d , by direct product).

Fact (2.9). The half-cube $\frac{1}{2}H(n, 2)$ is always ℓ_1 -rigid, except if $n = 3, 4$; indeed, $\frac{1}{2}H(3, 2) \simeq K_4$ and $\frac{1}{2}H(4, 2) \simeq K_{4 \times 2}$.

Fact (2.10). The Johnson graph $J(n, d)$ is always ℓ_1 -rigid, except in the case $d = 1$ and $n \geq 4$; indeed, $J(n, 1) \simeq K_n$.

Fact (2.11). The double odd graph DO_{2n+1} is always ℓ_1 -rigid (by Proposition 3.1, since it is bipartite).

Fact (2.12). The graph $\mathcal{A}(n, t)$ is ℓ_1 -rigid for $t \geq 3$.

Fact (2.13). The Petersen, Shrikhande graphs, the skeletons of the dodecahedron and of the icosahedron are all ℓ_1 -rigid, with scale 2.

The proofs of the statements presented in this section will be given in Section 4. We refer also to Section 4 for a precise description of all graphs and of their ℓ_1 -embeddings.

3. Bipartite and weighted ℓ_1 -graphs

Given a graph $G(V, E)$, consider the relation θ defined on the edge set E of G as follows: Given two edges $e = (i, j)$, $e' = (i', j')$ of G ,

$$e\theta e' \text{ if } d_G(i, i') + d_G(j, j') \neq d_G(i, j') + d_G(j, i').$$

The relation θ is reflexive, symmetric, but not transitive in general. An equivalent formulation of Theorem 1.2 is that G is an isometric subgraph of a hypercube if and only if G is bipartite and the relation θ is transitive. It is observed in [26] that every isometric subgraph of a hypercube admits (up to equivalence) a unique binary ℓ_1 -embedding (i.e., is h -rigid). In fact, we have the following stronger result.

PROPOSITION 3.1. *Every bipartite ℓ_1 -graph is ℓ_1 -rigid.*

This result can be extended to weighted bipartite graphs as follows. Let $w = (w_e)_{e \in E}$ be nonnegative edge weights assigned to the edges of G ; the weighting w is said to be *compatible with the relation θ* if $w_e = w_{e'}$ holds whenever $e\theta e'$.

PROPOSITION 3.2. *Let G be a bipartite ℓ_1 -graph and let w be a nonnegative weighting of the edges of G which is compatible with the relation θ . Then, the weighted graph (G, w) is ℓ_1 -rigid.*

As we shall explain in Section 4, a metric d on n points is ℓ_1 -rigid if it lies on a simplicial face of the cut cone \mathcal{C}_n . Hence, if G is a bipartite ℓ_1 -graph on n nodes, then its path metric d_G lies on a simplicial face \mathcal{F}_G of \mathcal{C}_n . The dimension of the face \mathcal{F}_G is equal to the number of equivalence classes of the relation θ . Moreover, every other metric d lying on \mathcal{F}_G is the path metric of (G, w) for some (compatible) nonnegative weighting w of G .

In particular, if G is a tree, then (G, w) is ℓ_1 -rigid for every nonnegative weighting w (since θ is the identity relation).

In fact, a similar result holds for cycles, as we now see. Note, however, that θ is not the identity relation for even cycles (any two opposite edges on the cycle are in relation by θ).

PROPOSITION 3.3. *Let $C = (V, E)$ be a cycle and let $w = (w_e)_{e \in E}$ be nonnegative integer edge weights. Then, (C, w) is ℓ_1 -rigid; moreover, (C, w) admits a binary ℓ_1 -embedding if and only if $\sum_{e \in E} w_e \equiv 0 \pmod{2}$.*

As example of application of the above results, we obtain that every connected graph with largest eigenvalue less than or equal to 2 is ℓ_1 -rigid. Indeed, such graphs are classified (see Theorem 3.2.5 in [4]) as subgraphs of extended Dynkin diagrams which are trees or cycles.

4. Tools and proofs

4.1. ℓ_1 -metrics and the cut cone

All the proofs use extensively the following correspondence between ℓ_1 -metrics and the points of the cut cone.

Given a subset S of $V = \{1, \dots, n\}$, let $\delta(S)$ denote the *cut metric* on V defined by: $\delta(S)(i, j) = 1$ if $|S \cap \{i, j\}| = 1$, and $\delta(S)(i, j) = 0$ otherwise, for $1 \leq i < j \leq n$. Hence, $\delta(S) = \delta(V - S)$ and $\delta(\emptyset) = \delta(V)$ is identically zero. The *cut cone* C_n is the cone in $\mathbb{R}^{\binom{V}{2}}$ generated by the cut metrics $\delta(S)$ for $S \subseteq V$. (For general information on the cut cone, see e.g., [14] and the survey [12].) A face \mathcal{F} of C_n is said to be *simplicial* if the nonzero cut metrics lying on \mathcal{F} are linearly independent. For any metric d on V , the following assertions hold:

Fact (4.1). d is an ℓ_1 -metric if and only if d belongs to the cut cone C_n , i.e., $d = \sum_S \lambda_S \delta(S)$, with $\lambda_S \geq 0$ for all $S \subseteq V$.

Fact (4.2). d is an ℓ_1 -metric with scale 1, i.e., d admits a binary ℓ_1 -embedding, if and only if $d = \sum_S \lambda_S \delta(S)$, with $\lambda_S \in \mathbb{Z}_+$ for all $S \subseteq V$.

Fact (4.3). d is ℓ_1 -rigid if and only if d lies on a simplicial face of the cut cone C_n , i.e., d admits a unique decomposition as a nonnegative sum of cut metrics.

For Facts (4.1) and (4.2), see [2] and for (4.3) see [16]. Note that Fact (2.1) follows immediately from (4.1) and (4.2). Any decomposition of d as a nonnegative sum of cut metrics is called a \mathbb{R}_+ -*realization* of d and any decomposition of d as a nonnegative integer sum of cut metrics is called a \mathbb{Z}_+ -*realization*. Moreover, if $d = \sum_S \lambda_S \delta(S)$ with $\lambda_S \geq 0$, then $\sum_S \lambda_S$ is called the *size* of the realization.

For instance, (4.2) can be checked as follows. Let d be an ℓ_1 -metric with scale 1 and let $x_1, \dots, x_n \in \{0, 1\}^m$ be a binary embedding of d . Consider the associated realization matrix M whose rows are x_1, \dots, x_n . Each column of M can be seen as the incidence vector of a subset S_j of V . Then, $d = \sum_{1 \leq j \leq m} \delta(S_j)$ holds. Therefore, the rows of M provide the binary labeling of the nodes while the columns of M provide the cut metrics whose sum gives d .

All the maximal simplicial faces of the cut cone \mathcal{C}_S are described in [17]; each of them contains a graphic metric. This yields a characterization on the ℓ_1 -rigid metrics on at most five points.

We can also precise the statement (2.4) on the antipodal extension operation. Namely, if d is a metric on V and $\alpha \in \mathbb{R}_+$, then:

Fact (4.4). $\text{ant}_\alpha(d)$ is an ℓ_1 -metric, i.e., $\text{ant}_\alpha(d) \in \mathcal{C}_{n+1}$, if and only if d is an ℓ_1 -metric, i.e., $d \in \mathcal{C}_n$, and d admits a \mathbb{R}_+ -realization of size less than or equal to α ; moreover, any \mathbb{R}_+ -realization of $\text{ant}_\alpha(d)$ is of the form:

$$\text{ant}_\alpha(d) = \sum_{S:1 \in S} \lambda_S \delta(S) + \sum_{S:1 \notin S} \lambda_S \delta(S \cup \{n+1\}) + (\alpha - \sum_S \lambda_S) \delta(\{n+1\})$$

where $d = \sum_S \lambda_S \delta(S)$ is a \mathbb{R}_+ -realization of d with $\sum_S \lambda_S \leq \alpha$.

Let us now indicate the pattern that our proofs of ℓ_1 -rigidity will follow.

Given a metric d on V , how to prove that d is ℓ_1 -rigid? By (4.3), this amounts to proving that d lies on a simplicial face of the cut cone \mathcal{C}_n . We proceed as follows.

Fact (4.5). First, we check that d is an ℓ_1 -metric.

For this, we exhibit a binary ℓ_1 -embedding for λd , for some scalar λ ; in all the cases treated here (except for $K_{n \times 2}$), λ is equal to 1 or 2. In other words, we exhibit a decomposition of λd as integer sum of cuts, say $\lambda d = \sum_{S \in \mathcal{S}} \lambda_S \delta(S)$ with $\lambda_S \in \mathbb{Z}$, $\lambda_S > 0$, for $S \in \mathcal{S}$.

Then, let \mathcal{F}_d denote the face of smallest dimension of the cut cone \mathcal{C}_n containing d . Hence, the cut metrics $\delta(S)$, $S \in \mathcal{S}$, belong to \mathcal{F}_d . We show that the face \mathcal{F}_d is simplicial by checking that the following Facts (4.6) and (4.7) hold.

Fact (4.6). The cut metrics $\delta(S)$, $S \in \mathcal{S}$, are linearly independent.

Fact (4.7). The only cut metrics lying on \mathcal{F}_d are the cut metrics $\delta(S)$ for $S \in \mathcal{S}$.

Actually, in all the cases treated here, we shall show the following property, which is stronger than (4.7): The only cut metrics which satisfy the same triangle equalities as d are the cut metrics $\delta(S)$, for $S \in \mathcal{S}$. This means that the face \mathcal{F}_d is completely determined by triangle inequalities.

More precisely, define also the smallest face \mathcal{F}_d^M of the metric cone \mathcal{M}_n containing d . Then, $\mathcal{F}_d \subseteq \mathcal{F}_d^M \cap \mathcal{M}_n$ holds. The equality $\mathcal{F}_d = \mathcal{F}_d^M \cap \mathcal{M}_n$ holds if the face \mathcal{F}_d is completely defined as an intersection of triangle faces. Clearly, if the face \mathcal{F}_d^M is simplicial, then the face \mathcal{F}_d too is simplicial. Lomonosov and Sebö [22] have shown that the path metric of a bipartite graph is metrically rigid, i.e., lies on a simplicial face of the metric cone, which, therefore, implies the result from Proposition 3.1. Note that $K_6 - P_3$, $K_5 - P_3$ are examples of ℓ_1 -rigid graphs which are not metrically rigid.

4.2. Proof of (2.2) and (2.3)

We first show the statement (2.2). Let d_a be a metric on V_a , for $a = 1, 2$, and let d denote the direct product of d_1, d_2 , defined on $V_1 \times V_2$. It is easy to see that:

- (i) If d_1, d_2 are ℓ_1 -metrics with \mathbb{R}_+ -realizations, $d_1 = \sum_{S \in \mathcal{S}} \alpha_S \delta(S)$, $d_2 = \sum_{T \in \mathcal{T}} \beta_T \delta(T)$, respectively, then d is an ℓ_1 -metric with \mathbb{R}_+ -realization

$$d = \sum_{S \in \mathcal{S}} \alpha_S \delta(S \times V_2) + \sum_{T \in \mathcal{T}} \beta_T \delta(V_1 \times T).$$

- (ii) Conversely, if d is an ℓ_1 -metric with \mathbb{R}_+ -realization $d = \sum_A \lambda_A \delta(A)$, then

$$\begin{cases} d_1 = \sum_A \lambda_A \delta(A_1) \\ d_2 = \sum_A \lambda_A \delta(A_2) \end{cases}$$

where $A_1 = \{i \in V_1 : (i, i_2) \in A\}$, $A_2 = \{i \in V_2 : (i_1, i) \in A\}$, for some fixed $i_1 \in V_1, i_2 \in V_2$.

Therefore, from (i), if d is ℓ_1 -rigid, then both d_1, d_2 are ℓ_1 -rigid. Conversely, assume that d_1, d_2 are ℓ_1 -rigid; we show that d is ℓ_1 -rigid. The proof follows the steps (4.6) and (4.7). We suppose that the \mathbb{R}_+ -realizations of d_1, d_2 are as in (i).

First, it is immediate to see that the cut metrics $\delta(S \times V_2)$, $S \in \mathcal{S}$, and $\delta(V_1 \times T)$, $T \in \mathcal{T}$, are linearly independent.

We check (4.7). Take $A \subseteq V_1 \times V_2$ such that $\delta(A)$ satisfies the same triangle equalities as d and, thus, the following equalities:

$$d((i_1, i_2), (j_1, j_2)) = d((i_1, i_2), (j_1, i_2)) + d((j_1, i_2), (j_1, j_2))$$

for all $i_1, j_1 \in V_1, i_2, j_2 \in V_2$. We show that A is one of $S \times V_2$ for $S \in \mathcal{S}$ or $V_1 \times T$ for $T \in \mathcal{T}$. Suppose, e.g., that, for some $i_2 \in V_2$, $A \cap (V_1 \times \{i_2\})$ is a proper nonempty subset of $V_1 \times \{i_2\}$. Then, by the above triangle equalities, we deduce that $\{i\} \times V_2 \subseteq A$ for any $i \in V_1$ such that $(i, i_2) \in A$, and $A \cap (\{i\} \times V_2) = \emptyset$ for any $i \in V_1$ such that $(i, i_2) \notin A$. Therefore, A is of the form $S \times V_2$ and, since d_1 is ℓ_1 -rigid, $S \in \mathcal{S}$. This concludes the proof.

We now show that the statement (2.3) holds. Let d_a be a metric on V_a , for $a = 1, 2$, where V_1 and V_2 have exactly one common element, $V_1 \cap V_2 = \{k\}$. Let d denote the 1-sum of d_1 and d_2 . It is easy to check that:

- (i) If d_1, d_2 are ℓ_1 -metrics with \mathbb{R}_+ -realizations, $d_1 = \sum_{S \in \mathcal{S}} \alpha_S \delta(S)$, $d_2 = \sum_{T \in \mathcal{T}} \beta_T \delta(T)$, respectively, where we can suppose that $k \notin S, T$ for all $S \in \mathcal{S}$ and $T \in \mathcal{T}$, then d is an ℓ_1 -metric with \mathbb{R}_+ -realization

$$d = \sum_{S \in \mathcal{S}} \alpha_S \delta(S) + \sum_{T \in \mathcal{T}} \beta_T \delta(T).$$

- (ii) Conversely, if d is an ℓ_1 -metric with \mathbb{R}_+ -realization $d = \sum_{A \in \mathcal{A}} \lambda_A \delta(A)$, where we can suppose that $k \notin A$ for all $A \in \mathcal{A}$, then, we have $A \subseteq V_1$, or $A \subseteq V_2$ for each $A \in \mathcal{A}$ and

$$\begin{cases} d_1 = \sum_{A \in \mathcal{A}, A \subseteq V_1} \lambda_A \delta(A) \\ d_2 = \sum_{A \in \mathcal{A}, A \subseteq V_2} \lambda_A \delta(A) \end{cases}$$

and, therefore, d_1, d_2 are ℓ_1 -metrics.

An immediate consequence of (i), (ii) above is that d is ℓ_1 -rigid if and only if d_1 and d_2 are ℓ_1 -rigid.

4.3. Proof of (2.5)–(2.8)

Let us first observe that the complete graph K_n is an ℓ_1 -graph. Indeed, a binary ℓ_1 -embedding of $2d(K_n)$ is obtained by labeling the nodes of K_n by the unit vectors of \mathbb{R}^n or, equivalently, $2d(K_n) = \sum_{1 \leq i < j \leq n} \delta(\{i, j\})$. For $n = 2, 3$, the cone C_n is simplicial and, thus, K_n is ℓ_1 -rigid. However, for $n \geq 4$, K_n admits many other ℓ_1 -embeddings. For example, for $1 \leq q \leq \frac{n}{2}$, $\sum_{S \subseteq \{1, \dots, n\}, |S|=q} \delta(S) = 2 \binom{n-2}{q-1} d(K_n)$; for $q = \lfloor \frac{n}{2} \rfloor$, this gives an ℓ_1 -embedding of minimum possible size $\frac{n(n-1)}{2 \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil} < 2$.

Consider now the cocktail party graph $K_{n \times 2}$ defined on the $2n$ nodes of $\{1, \dots, 2n\}$. Its path metric has all its values equal to 1 except values 2 on the n pairs $(i, n+i)$ for $1 \leq i \leq n$. Therefore, $d(K_{n \times 2})$ can be expressed in terms of $d(K_n)$ using the antipodal extension operation; namely, $d(K_{n \times 2}) = \text{Ant}_2(d(K_n))$. Therefore, by (2.4), $K_{n \times 2}$ is an ℓ_1 -graph and is ℓ_1 -rigid if and only if K_n is ℓ_1 -rigid, i.e., for $n = 2, 3$.

The statements (2.7) and (2.8) follow by application of (2.2).

4.4. Proof of (2.9) and (2.10)

We first show that the half-cube $\frac{1}{2}H(n, 2)$ is ℓ_1 -rigid for any $n \geq 5$.

Let us fix some notation. Let E_n denote the family of the subsets of $V = \{1, \dots, n\}$ of even cardinality. Then, $\frac{1}{2}H(n, 2)$ is the graph with node set E_n and with edges the pairs (A, B) such that $A, B \in E_n$ and $|A \Delta B| = 2$.

For any $i \in V$, define the set S_i consisting of all the sets $A \in E_n$ with $i \in A$. It is easy to see that $2d(\frac{1}{2}H(n, 2)) = \sum_{1 \leq i \leq n} \delta(S_i)$ holds; this corresponds to the binary ℓ_1 -embedding of $2d(\frac{1}{2}H(n, 2))$ obtained by assigning to each node $A \in E_n$ its incidence vector.

Observe that the subgraph of $\frac{1}{2}H(n, 2)$ induced by any subset S_i is isomorphic to $\frac{1}{2}H(n-1, 2)$. This simple observation will permit us to use induction on n .

For showing that the half-cube $\frac{1}{2}H(n, 2)$ is ℓ_1 -rigid for $n \geq 5$, we follow the steps (4.6) and (4.7). It is easy to check that the cuts $\delta(S_i)$, for $1 \leq i \leq n$, are linearly independent.

Then, by induction on $n \geq 5$, we show that the only nonzero cut metrics on E_n satisfying the same triangle equalities as $d(\frac{1}{2}H(n, 2))$ are the n cut metrics $\delta(S_i)$ for $1 \leq i \leq n$. The case $n = 5$ can be checked directly. We describe only the induction step. We suppose that the property holds for $n-1$ and we show that it holds for n .

Take $S \subseteq E_n$ such that the cut metric $\delta(S)$ satisfies the same triangle equalities as $d(\frac{1}{2}H(n, 2))$. For any $i \in V$, by looking at the local structure on S_i and applying the induction assumption, we deduce that $S \cap S_i = \emptyset$, S_i , $S_i \cap S_j$ or $S_i - S_j$ for some $j \in V$.

We shall use the fact that $d := d(\frac{1}{2}H(n, 2))$ satisfies the following triangle equalities:

$$d(A, C) = d(A, B) + d(B, C)$$

for any $A, B, C \in E_n$ such that $|A \Delta C| = 4$, $|A \Delta B| = |B \Delta C| = 2$.

Let us first suppose, for example, that $S_1 \subseteq S$. We show that, if $S \neq S_1$, then $S = E_n$. We use the following assertions (i)–(ii).

- (i) If $A \notin S$ with $2 \leq |A| \leq n-2$, then $A - \{a, b\} \notin S$ for all distinct $a, b \in A$.
Indeed, take $c \in V - A$, $c \neq 1$, and use the triangle equality: $d(A - \{a, b\}, A \cup \{1, c\}) = d(A - \{a, b\}, A) + d(A, A \cup \{1, c\})$.
- (ii) If $A \notin S$ with $|A| \leq n-4$, then $A \cup \{a, b\} \notin S$ for all distinct $a, b \in V - A$, $a, b \neq 1$.
Indeed, take $c \in V - A$, $c \neq a, b, 1$, and use the triangle equality: $d(A \cup \{a, b\}, A \cup \{1, c\}) = d(A, A \cup \{a, b\}) + d(A, A \cup \{1, c\})$.

Let $A \in E_n$ of minimum cardinality such that $A \notin S$. If $A = \emptyset$, then, by (ii), $S = S_1$ holds. If $|A| \leq n-2$, then we have a contradiction with (i) since $\emptyset \in S$. Hence, $|A| = n-1$, $A = [2, n] \notin S$ while $A - \{2, 3\}$, $A - \{n-1, n\} \in S$, contradicting the triangle equality: $d(A - \{2, 3\}, A - \{n-1, n\}) = d(A, A - \{2, 3\}) + d(A, A - \{n-1, n\})$.

Therefore, S_i is not contained in S for all i . We can suppose, for instance, that $S \cap S_1 = S_1 \cap S_2$. We deduce that $S \cap S_2 = S_1 \cap S_2$ and $S \cap S_3 = S_3 \cap S_i$ for some $i \neq 3$. By taking the intersection of both sides of $S \cap S_3 = S_3 \cap S_i$ with S_1 and S_2 , we obtain that i should coincide with 2 and 1, respectively, hence a contradiction.

This concludes the proof.

We now show that the Johnson graph $J(n, d)$ is ℓ_1 -rigid for $d \geq 2$.

Let \mathcal{P}_n^d denote the family of the subsets of $V = \{1, \dots, n\}$ of cardinality d . The Johnson graph $J(n, d)$ is the graph with node set \mathcal{P}_n^d and with edges the pairs (A, B) for $A, B \in \mathcal{P}_n^d$ and $|A \Delta B| = 2$.

A binary ℓ_1 -embedding of $2d(J(n, d))$ is obtained, simply, by labeling each node A by its incidence vector. This corresponds to the following \mathbb{Z}_+ -realization of $2d(J(n, d))$: $2d(J(n, d)) = \sum_{1 \leq i \leq n} \delta(S_i)$, where S_i is the subset of \mathcal{P}_n^d consisting of all $A \in \mathcal{P}_n^d$ with $i \in A$, for $1 \leq i \leq n$.

The subgraph of $J(n, d)$ induced by the node set S_i is isomorphic to $J(n - 1, d - 1)$. This observation permits us, as in the case of the half-cube, to apply induction.

The proof of ℓ_1 -rigidity of the Johnson graph $J(n, d)$ (for $d \geq 2$) is similar to the proof of ℓ_1 -rigidity for the half-cube $\frac{1}{2}H(n, 2)$ (for $n \geq 5$), so we omit the details.

4.5. Proof of (2.12)

Let us recall the definition of the graph $\mathcal{A}(n, t)$, taken from [21].

Let $V = \{1, \dots, n\}$ and $n \geq 2t, t \geq 3$. We define the families:

- X_{00} , consisting of the sets A , for $A \subseteq V, |A| = t$
- X_{11} , consisting of the sets $A \cup \{n + 1, n + 2\}$, for $A \subseteq V, |A| = t$
- X_{10} , consisting of the sets $A \cup \{n + 1\}$, for $A \subseteq V, |A| = t - 1$
- X_{01} , consisting of the sets $A \cup \{n + 2\}$, for $A \subseteq V, |A| = t - 1$

$\mathcal{A}(n, t)$ is the graph with node set $X_{00} \cup X_{11} \cup X_{10} \cup X_{01}$ and with edges the pairs (C, D) with $|C \Delta D| = 2$.

The subgraph of $\mathcal{A}(n, t)$ induced by any of the sets X_{00}, X_{11} (resp. X_{01}, X_{10}) is isomorphic to the Johnson graph $J(n, t)$ (resp. $J(n, t - 1)$). Hence, it is ℓ_1 -rigid since we assume that $t \geq 3$. For $1 \leq i \leq n$, denote by S_i (resp. T_i, U_i, W_i) the subset of X_{00} (resp. X_{11}, X_{10}, X_{01}) consisting of the members containing the element i .

A binary ℓ_1 -embedding of $2d(\mathcal{A}(n, t))$ is obtained, simply, by labeling each node by its incidence set. Correspondingly, we have the following \mathbb{Z}_+ -realization of $2d(\mathcal{A}(n, t))$:

$$2d(\mathcal{A}(n, t)) = \delta(X_{00} \cup X_{01}) + \delta(X_{00} \cup X_{10}) + \sum_{1 \leq i \leq n} \delta(S_i \cup T_i \cup U_i \cup W_i).$$

We show that $\mathcal{A}(n, t)$ is ℓ_1 -rigid.

First, it is easy to check that the cut metrics $\delta(X_{00} \cup X_{01}), \delta(X_{00} \cup X_{10})$ and $\delta(S_i \cup T_i \cup U_i \cup W_i)$, for $1 \leq i \leq n$, are linearly independent (using, for instance, the corresponding independence result for the Johnson graph).

Let $\delta(S)$ be a nonzero cut metric satisfying the same triangle equalities as $d(\mathcal{A}(n, t))$. We show that $\delta(S)$ is one of the cut metrics $\delta(X_{00} \cup X_{01})$, $\delta(X_{00} \cup X_{10})$ or $\delta(S_i \cup T_i \cup U_i \cup W_i)$, for $1 \leq i \leq n$.

We shall use, in particular, the following triangle equalities:

- (i) $d(A, B \cup \{n+1, n+2\}) = d(A, A \cup \{n+1, n+2\}) + d(A \cup \{n+1, n+2\}, B \cup \{n+1, n+2\})$
- (ii) $d(A, B \cup \{n+1, n+2\}) = d(A, B) + d(B, B \cup \{n+1, n+2\})$
for $A, B \subseteq V$, $|A| = |B| = t$, and $|A \Delta B| = 2$,
- (iii) $d(A, (A-a) \cup \{b, n+1, n+2\}) = d(A, (A-a) \cup \{n+1\})$
 $+ d((A-a) \cup \{n+1\}, (A-a) \cup \{b, n+1, n+2\})$
- (iv) $d(A, (A-a) \cup \{b, n+1, n+2\}) = d(A, (A-a) \cup \{n+2\})$
 $+ d((A-a) \cup \{n+2\}, (A-a) \cup \{b, n+1, n+2\})$ for $A \subseteq V$, $|A| = t$,
 $a \in A$, $b \in V - A$,
- (v) $d((A-a) \cup \{n+1\}, (A-b) \cup \{n+2\}) = d((A-a) \cup \{n+1\}, A)$
 $+ d(A, (A-b) \cup \{n+2\})$.
- (vi) $d((A-a) \cup \{n+1\}, (A-b) \cup \{n+2\}) = d((A-a) \cup \{n+1\},$
 $A \cup \{n+1, n+2\}) + d(A \cup \{n+1, n+2\}, (A-b) \cup \{n+2\})$

The subgraph induced by $\mathcal{A}(n, t)$ on X_{00} is the ℓ_1 -rigid graph $J(n, t)$; hence, from the fact that the ℓ_1 -embedding of $J(n, t)$ uses precisely the cut metrics $\delta(S_i)$ (see Section 4.4), we deduce that $S \cap X_{00} = \emptyset$, X_{00} , S_i , or $X_{00} - S_i$ for some $1 \leq i \leq n$. Similarly, $S \cap X_{11} = \emptyset$, X_{11} , T_i , or $X_{11} - T_i$ for some i ; $S \cap X_{10} = \emptyset$, X_{10} , U_i , or $X_{10} - U_i$ for some i ; $S \cap X_{01} = \emptyset$, X_{01} , W_i , or $X_{01} - W_i$ for some i .

We can suppose, e.g., that $S \cap X_{00} = X_{00}$ or S_i . We assume first that $S \cap X_{00} = X_{00}$. If $X_{11} \cap S \neq \emptyset$, then $S = X_{00} \cup X_{11} \cup X_{10} \cup X_{01}$, using the triangle equalities (i)–(iv) above. So we can suppose that $S \cap X_{11} = \emptyset$. From the triangle equalities (v), (vi), S contains exactly one of $A - a \cup \{n+1\}$, $A - b \cup \{n+2\}$, for all $a, b \in A$, $|A| = t$. One deduces easily that $S = X_{00} \cup X_{01}$, or $X_{00} \cup X_{10}$.

We now assume that $S \cap X_{00} = S_i$ for some i . Using again some triangle equalities, one can check that $S = S_i \cup T_i \cup U_i \cup W_i$.

This concludes the proof.

4.6. Proof of (2.13)

Here, we show that the Petersen graph, the Shrikhande graph, the dodecahedron and the icosahedron are ℓ_1 -rigid.

Figures 1 and 2 show, respectively, the Petersen graph O_3 and the Shrikhande graph G_{Sh} together with an isometric embedding in the half-cube $\frac{1}{2}H(6, 2)$ (up to reflection in the first coordinate), i.e., a binary ℓ_1 -embedding of $2d(G)$ for $G = O_3$ and G_{Sh} . Actually, the binary ℓ_1 -embeddings we describe in Figures 1 and 2 are equivalent to the ones given in Section 3.11 from [4]. Observe that, in

1	000111
2	010110
3	011010
4	011001
5	110001
6	100011
7	001101
8	110100
9	101010
10	101100

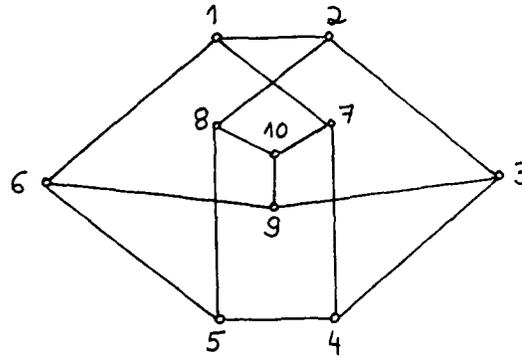


Figure 1. The Petersen graph.

1	000000
2	001001
3	111001
4	110000
5	010100
6	000101
7	101101
8	111100
9	011110
10	010111
11	100111
12	101110
13	001010
14	011011
15	110011
16	100010

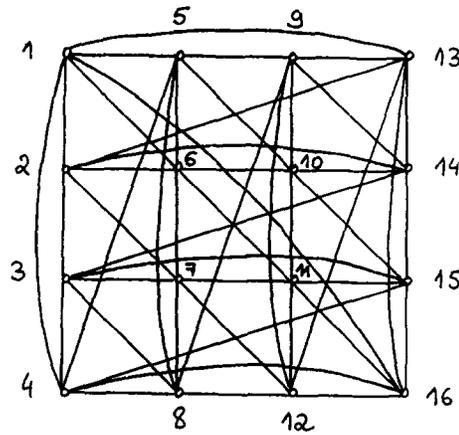
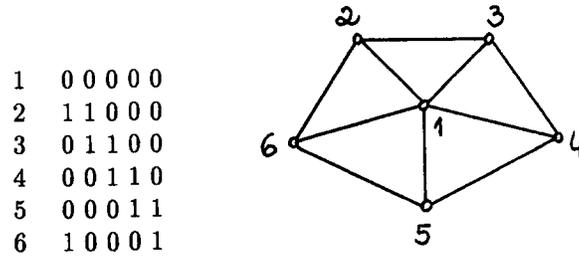


Figure 2. The Shrikhande graph.

Figure 3. ∇C_5 .

both cases, the binary labels are the blocks of a design, namely, of $S_2(2, 3, 6)$ for the Petersen graph, and $S_4(2, \{2, 4\}, 6)$ (where the blocks of size 2 form a cycle) for the Shrikhande graph. The corresponding \mathbb{Z}_+ -realizations are given by:

$$2d(O_3) = \delta(\{5, 6, 8, 9, 10\}) + \delta(\{2, 3, 4, 5, 8\}) + \delta(\{3, 4, 7, 9, 10\}) \\ + \delta(\{1, 2, 7, 8, 10\}) + \delta(\{1, 2, 3, 6, 9\}) + \delta(\{1, 4, 5, 6, 7\})$$

and

$$2d(G_{Sh}) = \delta(\{3, 4, 7, 8, 11, 12, 15, 16\}) + \delta(\{3, 4, 5, 8, 9, 10, 14, 15\}) \\ + \delta(\{2, 3, 7, 8, 9, 12, 13, 14\}) + \delta(\{5, 6, 7, 8, 9, 10, 11, 12\}) \\ + \delta(\{9, 10, 11, 12, 13, 14, 15, 16\}) \\ + \delta(\{2, 3, 6, 7, 10, 11, 14, 15\}).$$

Again, one shows that O_3 , G_{Sh} are ℓ_1 -rigid by checking that the conditions (4.6) and (4.7) hold. We omit the details.

We consider now the icosahedron and the dodecahedron, whose path metrics are denoted, respectively, by d_{ico} and d_{dod} ; these two graphs can be found, e.g., in chapter 1 from [4]. A useful observation is that both graphs can be expressed in terms of smaller graphs (one-half number of nodes) using the antipodal extension operation.

Namely, consider the 5-wheel ∇C_5 , i.e., the graph obtained by adding a node adjacent to all nodes of the 5-cycle; ∇C_5 and a binary ℓ_1 -embedding of $2d(\nabla C_5)$ are shown in Figure 3. Then, $d_{ico} = Ant_3(d(\nabla C_5))$, or $2d_{ico} = Ant_6(2d(\nabla C_5))$. It is easy to check that ∇C_5 is ℓ_1 -rigid. Therefore, by (2.4), the icosahedron is ℓ_1 -rigid. A binary ℓ_1 -embedding for $2d_{ico}$ follows from that of $2d(\nabla C_5)$ using (4.4).

Note that there are other ways to express the icosahedron as antipodal extension of some smaller ℓ_1 -rigid graph. For instance, $d_{ico} = Ant_3(d(G_i))$ where G_1 and G_2 are the graphs shown in Figures 4 and 5, respectively (G_1 is weighted, with weights 1 or 2, the edges with weight 2 are indicated by a double line).

Consider also the weighted graph G_3 shown in Figure 6; its edges have weight 1 or 2 (edges with weight 2 being indicated by a double line). Then,

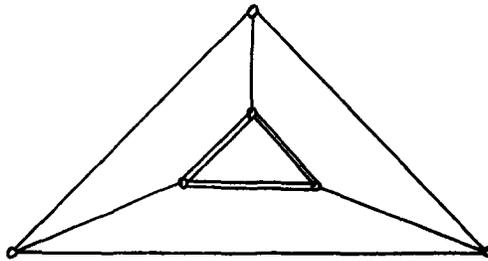


Figure 4. G_1 .

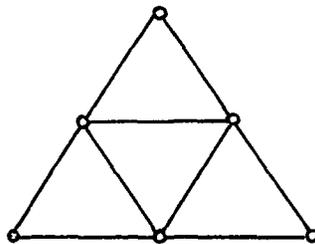


Figure 5. G_2 .

$d_{dod} = Ant_5(d(G_3))$ holds. Using the binary ℓ_1 -embedding of $2d(G_3)$ shown in Figure 6, it is easy to check that the graph G_3 is ℓ_1 -rigid. Therefore, the dodecahedron is ℓ_1 -rigid.

4.7. Proof of Propositions 3.1 and 3.2

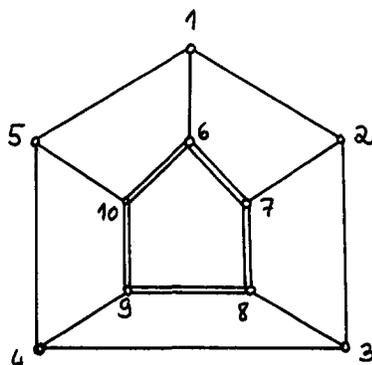
We prove that every bipartite ℓ_1 -embeddable graph is ℓ_1 -rigid.

We first recall (from [18]) how the binary ℓ_1 -embedding is constructed.

Let $G = (V, E)$ be a bipartite ℓ_1 -graph. Let \bar{E} denote the set of equivalence classes of the relation θ introduced in Section 3. Recall that, given two edges $e = (i, j)$ and $e' = (i', j')$ of G , $e\theta e'$ holds if and only if the two partitions of V into $G(i, j) \cup G(j, i)$ and $G(i', j') \cup G(j', i')$ are identical.

A binary ℓ_1 -embedding of d_G is constructed as follows. Choose a node u_0 in V . For each node $u \in V$, let $F(u)$ denote the subset of \bar{E} consisting of the classes \bar{e} for which u and u_0 belong to distinct sets in the partition $V = G(i, j) \cup G(j, i)$, if $e = (i, j)$. Then, labeling the node u_0 by the zero vector and each other node u by the incidence vector of the set $F(u)$, we obtain a binary ℓ_1 -embedding of d_G of dimension $|\bar{E}|$. This corresponds to the following \mathbb{Z}_+ -realization of d_G :

1	1	1	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0
3	0	0	1	1	0	0	0	0	0
4	0	0	0	1	1	0	0	0	0
5	1	0	0	0	1	0	0	0	0
6	1	1	0	0	0	1	1	0	0
7	0	1	1	0	0	0	1	1	0
8	0	0	1	1	0	0	0	1	1
9	0	0	0	1	1	0	0	0	1
10	1	0	0	0	1	1	0	0	1

Figure 6. G_3 .

$$d_G = \sum_{\bar{e} \in \bar{E}} \delta(S_{\bar{e}})$$

where $S_{\bar{e}} = G(i, j)$ if $e = (i, j)$ is any representant of the class \bar{e} .

We now show that G is ℓ_1 -rigid. First, the family of the cut metrics $\delta(S_{\bar{e}})$, $\bar{e} \in \bar{E}$, is trivially linearly independent. Then, consider a nonzero cut metric $\delta(S)$ which satisfies the same triangle equalities as d_G ; we show that $S = S_{\bar{e}}$ for some $\bar{e} \in \bar{E}$. Consider a shortest path P between two nodes i and j in G , $P = (i_1 = i, i_2, \dots, i_k = j)$, and suppose that $i \notin S$ and $j \in S$. Let $2 \leq h \leq k-1$ be the largest index such that $i_h \notin S$, so $i_{h+1} \in S$. Since $\delta(S)$ satisfies the triangle equality: $d(i_h, i_p) = d(i_h, i_{h+1}) + d(i_{h+1}, i_p)$ for $h+2 \leq p \leq k$, we deduce that $i_p \in S$ for all $h+1 \leq p \leq k$. Hence, the nodes of P belonging to S , as well as the nodes of P not belonging to S , form an interval on P . From this, it is easy to obtain that S is necessarily of the form $S_{\bar{e}}$ for some $e \in E$.

So we have shown that G is ℓ_1 -rigid. In fact, we have also shown that the smallest face \mathcal{F}_G of the cut cone \mathcal{C}_n containing d_G is generated by the cut metrics $\delta(S_{\bar{e}})$, $\bar{e} \in \bar{E}$. Every metric $d \in \mathcal{F}_G$ is of the form: $d = \sum_{\bar{e} \in \bar{E}} w_{\bar{e}} \delta(S_{\bar{e}})$ for some scalars $w_{\bar{e}} \geq 0$. Let us extend w to E by setting $w_f = w_{\bar{e}}$ for $f \in E$, $f \theta e$. By construction, w is compatible with θ and d coincides with the path metric of the weighted graph (G, w) . This shows Proposition 3.2.

4.8. Proof of Proposition 3.3

Let $C = (1, 2, \dots, n)$ be a cycle on $V = \{1, \dots, n\}$ and w_i be a nonnegative integer weight assigned to the edge $(i, i+1)$, for $1 \leq i \leq n$, the indices being taken modulo n .

We first show that (C, w) is an ℓ_1 -graph. Let C^* denote the cycle obtained by replacing each edge e of C by a path of length w_e . Hence, the projection of

$d(C^*)$ on V coincides with $d(C, w)$. If $\sum_e w_e$ is even, then C^* is an even cycle and C^* is ℓ_1 -embeddable with scale 1, otherwise C^* is ℓ_1 -embeddable with scale 2. In both cases, an ℓ_1 -embedding for $d(C, w)$ is obtained from an ℓ_1 -embedding for $d(C^*)$ by projection. We now show that $d(C, w)$ is ℓ_1 -rigid, i.e., lies on a simplicial face of C_n .

If $w_n > w_1 + \dots + w_{n-1}$, then no shortest path in C uses the edge $(1, n)$. Hence, $d(C, w)$ coincides with the path metric of the path $P = (1, \dots, n)$ with weights w_1, \dots, w_{n-1} on its edges; therefore, by Proposition 3.2, (C, w) is ℓ_1 -rigid.

We can now assume that $\max(w_i : 1 \leq i \leq n) \leq \frac{1}{2} \sum_{1 \leq i \leq n} w_i$. Hence, $d(i, i+1) = w_i$ for all $1 \leq i \leq n$. Define i^* as the largest index $i, 2 \leq i \leq n$, such that $w_1 + \dots + w_{i-1} \leq w_i + \dots + w_n$. Hence, $w_1 + \dots + w_{i^*-1} \leq w_{i^*} + \dots + w_n$ and $w_1 + \dots + w_{i^*} > w_{i^*+1} + \dots + w_n$. Let (P, w) (resp. (Q, w)) denote the path $(1, \dots, i^*)$ (resp. $(i^* + 1, \dots, n, 1)$) with the weights w_e being assigned to its edges. By construction of i^* , the projection of $d(C, w)$ on P (resp. Q) coincides with $d(P, w)$ (resp. $d(Q, w)$). From Section 4.7, $d(P, w) = \sum_{2 \leq h \leq i^*} w_h \delta(\{h, h+1, \dots, i^*-1, i^*\})$ and $d(Q, w) = \sum_{i^*+1 \leq k \leq n} w_k \delta(\{i^*+1, i^*+2, \dots, k-1, k\})$ and they are ℓ_1 -rigid.

Let $\delta(S)$ be a cut metric satisfying the same triangle equalities as $d(C, w)$. Since the paths (P, w) and (Q, w) are ℓ_1 -rigid, we obtain that $S \cap \{1, \dots, i^*\} = \emptyset, \{1, \dots, i^*\}, \{1, 2, \dots, h-2, h-1\}$, or $\{h, h+1, \dots, i^*\}$ for some $2 \leq h \leq i^*$, and $S \cap \{i^*+1, \dots, n-1, n, 1\} = \emptyset, \{i^*+1, \dots, n-1, n, 1\}, \{i^*+1, \dots, k-1, k\}$, or $\{k+1, \dots, n-1, n, 1\}$ for some $i^*+1 \leq k \leq n$. We can suppose, for instance, that $1 \notin S$. Then, S is one of the following sets: $\emptyset, A_k := \{i^*+1, \dots, k-1, k\}$ for $i^*+1 \leq k \leq n$, $B_h := \{h, h+1, \dots, i^*\}$ for $2 \leq h \leq i^*$, and $S_{h,k} := \{h, h+1, \dots, i^*\} \cup \{i^*+1, \dots, k-1, k\}$ for $2 \leq h \leq i^*$ and $i^*+1 \leq k \leq n$.

Let \mathcal{F} denote the smallest face of C_n containing $d(C, w)$. Every nonzero cut metric lying on \mathcal{F} must be one of the cut metrics $\delta(A_k), \delta(B_h), \delta(S_{h,k})$, for $2 \leq h \leq i^*$ and $i^*+1 \leq k \leq n$. If we show that the cut metrics $\delta(A_k), \delta(B_h)$ and $\delta(S_{h,k})$ are linearly independent, then this will imply that the face \mathcal{F} is simplicial and, thus, that (C, w) is ℓ_1 -rigid.

Indeed, suppose that

$$0 = \sum_k a_k \delta(A_k) + \sum_h b_h \delta(B_h) + \sum_{h,k} s_{h,k} \delta(S_{h,k}).$$

Computing the value on the edges $(h_0 - 1, h_0), (k_0, k_0 + 1)$ and (h_0, k_0) , for $2 \leq h_0 \leq i^*, i^* + 1 \leq k_0 \leq n$, yields, respectively, the relations:

- (i) $b_{h_0} + \sum_k s_{h_0,k} = 0$,
- (ii) $a_{k_0} + \sum_h s_{h,k_0} = 0$,
- (iii) $\sum_{k_0 \leq k \leq n} a_k + \sum_{2 \leq h \leq h_0} b_h + \sum_{2 \leq h \leq h_0, i^*+1 \leq k \leq k_0-1} s_{h,k} + \sum_{h_0+1 \leq h \leq i^*, k_0 \leq k \leq n} s_{h,k} = 0$.

Using (i) and (ii), (iii) can be rewritten as

$$\sum_{2 \leq h \leq h_0, k_0 \leq k \leq n} s_{h,k} = 0,$$

from which one obtains that $s_{h,k} = 0$ for all h, k and, thus, $a_k = 0, b_h = 0$ for all h, k .

This concludes the proof.

5. Applications

In application of the results from Sections 2 and 3, we group here information on ℓ_1 -rigidity for some important classes of graphs, whose classification is provided, in particular, in [4].

Fact (5.1). The only strongly regular ℓ_1 -graphs are $K_{n \times 2}$, $H(2, d)$, $J(n, 2)$, C_5 , the Petersen graph O_3 and the Shrikhande graph [21, Corollary 3.9]. All, except $K_{n \times 2}$ and the grid $H(2, n)$ for $n \geq 4$, are ℓ_1 -rigid.

Fact (5.2). The 1-skeletons of all regular polytopes, except the 24-cell, the 600-cell, and the undecided case of the 120-cell, are ℓ_1 -graphs [1]. Among them, all are ℓ_1 -rigid except for the simplex α_n for $n \geq 3$ and the cross-polytope β_n for $n \geq 4$.

Remark that the Johnson graph and the half-cube are also the 1-skeletons of some polytopes, in fact, of some L -polytopes (or Delaunay polytopes). (Recall that an L -polytope is the convex hull of the lattice points lying on the boundary of a hole in a lattice L , see, e.g., [5].) For $J(n, 1)$ and $\frac{1}{2}H(n, 2)$ with $n = 2, 3$, the L -polytope is regular (it is a simplex); else, it is not regular. For $J(5, 2) \simeq T(5)$, the L -polytope is 0_{21} , and for $\frac{1}{2}H(5, 2)$ (the Clebsch graph), it is 1_{21} , which are both semiregular (see [6]).

Using Section 3.15 from [4], we obtain (5.3) and (5.4) below.

Fact (5.3). The only distance regular ℓ_1 -graphs with $\mu \geq 2$ and $d \geq 3$ are $\frac{1}{2}H(n, 2)$ for $n \geq 6$, $J(n, d)$, $H(n, d)$ for $n \geq 3$, the icosahedron and the Doob graphs.

Fact (5.4). The only amply regular ℓ_1 -graphs with $\mu \geq 2$ are $K_{n \times 2}$, $\frac{1}{2}H(n, 2)$, $J(n, d)$, $H(n, d)$ and direct products of icosahedra and Doob graphs.

Fact (5.5). Using Table 10.6 from [4], among distance regular finite Coxeter graphs, the only ℓ_1 -graphs are $K_{n \times 2}$, $\frac{1}{2}H(n, 2)$, $J(n, d)$, $H(n, 2)$, C_n for $n \geq 5$, the icosahedron and the dodecahedron.

Fact (5.6). Using Theorem 7.5.1 from [4], all distance regular cubic ℓ_1 -graphs are K_4 , the Petersen graph O_3 , $H(3, 2)$, the double odd graph DO_5 and the dodecahedron.

Fact (5.7). We consider now ℓ_1 -graphs among distance regular antipodal graphs. Koolen ([21], Lemma 5.7) observed that any distance regular antipodal ℓ_1 -graph is a double cover and, moreover, classified them among the double covers of complete graphs, i.e., among Taylor graphs (Lemma 3.13); namely, they are $J(6, 3)$, $\frac{1}{2}H(6, 2)$, the icosahedron and the two bipartite graphs: $C_6 \simeq DO_3$, $H(3, 2)$, and, thus, they are all ℓ_1 -rigid. Other examples of distance regular antipodal ℓ_1 -graphs include the dodecahedron (double cover of the Petersen graph O_3) and the three bipartite graphs: C_{2n} (double cover of C_n), $H(n, 2)$ (double cover of the folded cube) and DO_{2n+1} (double cover of the odd graph O_{2n+1}). Actually, the only distance regular graphs that are ℓ_1 -graphs with scale 1 are C_{2n} , DO_{2n+1} and $H(n, 2)$ ([21], [25]).

Fact (5.8). Using Theorem 2.8 from [21], the only root graphs with $\mu \geq 3$ and root representation in \mathbb{Z}^n which are ℓ_1 -graph are: complete multipartite graphs with classes of size 1 or 2, $\frac{1}{2}H(n, 2)$, the suspension of $J(n, 2) \simeq T(n)$, $J(n, d)$, $\mathcal{A}(n, t)$, the suspension of the $p \times q$ -grid and $L_{n,t}$. Moreover, except the undecided case of $L_{n,t}$, we know which ones are ℓ_1 -rigid. Namely, $\frac{1}{2}H(n, 2)$, $\mathcal{A}(n, t)$, the suspension of $T(n)$ or of a $p \times q$ -grid are ℓ_1 -rigid (this is easy to check in the last two cases). A complete multipartite graph with p classes of size 1 or 2 is ℓ_1 -rigid if and only if $p \leq 3$ (this is easy to see using (2.4)).

Fact (5.9). The complement of $J(n, 2)$ is not an ℓ_1 -graph if $n \geq 6$ (since it contains $K_{2,3}$ as an isometric subgraph) and it is ℓ_1 -rigid if $n = 5$ (since it is isomorphic to the Petersen graph).

Fact (5.10). Let Q be a generalized quadrangle and let G_Q denote its collinearity graph.

- (i) If Q is degenerate, then G_Q is an ℓ_1 -graph with scale 1 if all lines have size 2 and with scale 2 otherwise. Moreover, G_Q is ℓ_1 -rigid if and only if all the lines of Q have size at most 3.
- (ii) If Q is a generalized quadrangle with all lines of size 3, then G_Q is an ℓ_1 -graph if and only if Q is degenerate or Q is the 3×3 -grid $H(2, 3)$. In both cases, G_Q is ℓ_1 -rigid.

Proof. (ii) follows from the classification of generalized quadrangles with line size 3 (see Section 1.15 in [4]) and from (i). (i) follows by applying (2.3). Indeed, let Q be a degenerate generalized quadrangle with point set X with lines L_1, \dots, L_k and let x_0 denote their common point. Observe that $d(G_Q)$ arises as the 1-sum of the metrics d_a , where d_a is the path metric of the complete graph on L_a , for $1 \leq a \leq k$. Therefore, from (2.3), G_Q is an ℓ_1 -graph with scale 1 if all lines have size 2, and with scale 2 otherwise. Moreover, G_Q is ℓ_1 -rigid if and only if all d_a are ℓ_1 -rigid, i.e., all lines have size at most 3. \square

Fact (5.11). Given $p, q \geq 2$, the complement of the $p \times q$ -grid $K_p \times K_q$ is not an ℓ_1 -graph if $q \geq 5$, or $p \geq 3$ and $q \geq 4$ (since it contains $K_{2,3}$ as isometric subgraph) and it is ℓ_1 -rigid for $(p, q) = (2, 3), (2, 4), (3, 3)$ (since it is isomorphic to $C_6, H(3, 2), K_3 \times K_3$, respectively).

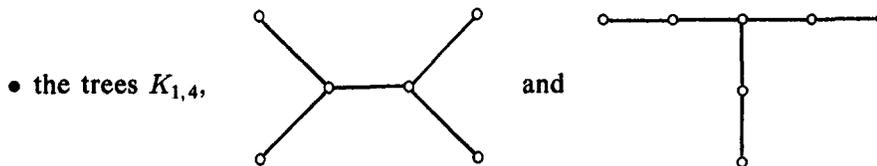
Fact (5.12). There are 30 connected graphs on $2 \leq n \leq 5$ nodes (see [8]). Among them, there are

- three graphs which are not ℓ_1 -graphs, namely, $\nabla K_{1,3}, K_5 - (P_2 \cup P_3), K_{2,3}$
- four nonrigid ℓ_1 -graphs with scale 2, namely, $K_4, K_5, K_5 - P_2, K_4$ plus a pendant edge
- nine ℓ_1 -rigid graphs with scale 1, namely, C_4, C_4 plus a pendant edge and all seven trees.

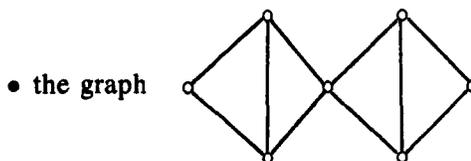
The remaining 15 graphs are ℓ_1 -rigid with scale 2.

Fact (5.13). Among the integral graphs on $n \leq 7$ nodes (i.e., the eigenvalues of their adjacency matrix are all integral; see [7] for their list), the ℓ_1 -graphs are:

- K_4, K_5, K_6
and the following ones which are ℓ_1 -rigid:
- $C_2, C_3, C_4, C_6, K_{3 \times 2}$, the 2×3 -grid,



- the degenerate generalized quadrangle on 7 nodes with line size 3 (i.e., three triangles sharing a common vertex),



Fact (5.14). We conclude with a remark on (k, g) -cages. All known (k, g) -cages of even girth are bipartite; therefore, by Proposition 3.1, they are ℓ_1 -rigid whenever they are ℓ_1 -graphs. Actually, all known ℓ_1 -graphs among (k, g) -cages are the $(2, g)$ -cage C_g , the $(k, 3)$ -cage K_{k+1} and the $(3, 5)$ -cage O_3 (Petersen graph).

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References

1. P. Assouad, "Embeddability of regular polytopes and honeycombs in hypercubes," *The Geometric Vein*, Springer, 1981, pp. 141–147.
2. P. Assouad and M. Deza, "Metric subspaces of L^1 ," *Publications mathématiques d'Orsay* 3 (1982).
3. I.F. Blake and J.H. Gilchrist, "Addresses for graphs," *IEEE Trans. Inform. Theory* 5 (1973), 683–688.
4. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer Verlag, 1989.
5. J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, *Grundlehren der mathematischen Wissenschaften*, Springer Verlag, Berlin, 290, 1987.
6. H.S.M. Coxeter, *Regular Polytopes*. Dover Publications, 1973.
7. D. Cvetković, M. Doob, I. Gutman, and A. Torgasev, "Recent results in the theory of graph spectra," *Ann. Discrete Math.* 36, (1988).
8. D. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs—Theory and Applications*, Academic Press, 1980.
9. M. Deza, "Une propriété extrémale des plans projectifs finis dans une classe de codes equidistants," *Discrete Math.* 6 (1973), 343–352.
10. M. Deza and V.P. Grishukhin, "Hypermetric graphs," *Q. J. Math. Oxford* (2) 44 (1993), 399–433.
11. M. Deza, V.P. Grishukhin, and M. Laurent, "Extreme hypermetrics and L -polytopes," in *Sets, Graphs and Numbers, Budapest, 1991, Colloquia Mathematica Societatis János Bolyai*, vol. 60, 1992, 157–209.
12. M. Deza and M. Laurent, "Applications of cut polyhedra," LIENS-92-18, Ecole Normale Supérieure, Paris, 1992 (to appear in *J. Comp. Appl. Math.*).
13. M. Deza and M. Laurent, "Extension operations for cuts," *Discrete Math.* 106–107 (1992), 163–179.
14. M. Deza and M. Laurent, "Facets for the cut cone I," *Math. Prog.* 56 (1992), 121–160.
15. M. Deza and M. Laurent, "Variety of hypercube embeddings of the equidistant metric and designs," *J. Combin. Information System Sci.* 18 (1993).
16. M. Deza and M. Laurent, "The cut cone; Simplicial faces and linear dependencies." *Bull. Inst. Math. Academia Sinica*, 21 (1993), 143–182.
17. M. Deza and N.M. Singhi, "Rigid pentagons in hypercubes," *Graphs and Combin.* 4 (1988), 31–42.
18. D.Ž. Djoković, "Distance preserving subgraphs of hypercubes," *J. Combin. Theory Series B* 14 (1973), 263–267.
19. R.L. Graham and P.M. Winkler, "On isometric embeddings of graphs," *Trans. Amer. Math. Society* 288 (1985), 527–536.
20. A.V. Karzanov, "Metrics and undirected cuts," *Math. Prog.* 32 (1985), 183–198.
21. J. Koolen, "On metric properties of regular graphs," Master's thesis, Eindhoven University of Technology, 1990.
22. M. Lomonossov and A. Sebö, "On the geodesic-structure of graphs: a polyhedral approach to metric decomposition," in *Integer Programming and Combinatorial Optimization*, Erice, 1993, 221–234.
23. R.L. Roth and P.M. Winkler, "Collapse of the metric hierarchy for bipartite graphs," *European J. Combin.* 7 (1986), 371–375.
24. S.V. Shpectorov, "On scale embeddings of graphs into hypercubes," *European J. Combin.* 14 (1993), 117–130.
25. P.M. Weichsel, "Distance regular subgraphs of a cube," *Discrete Math.* 109 (1992), 297–306.
26. P.M. Winkler, "The metric structure of graphs: theory and applications," in *Surveys in Combinatorics 1987*, C. Whitehead, ed., London Mathematical Society, Lecture Note Series 123, Cambridge University Press, Cambridge, 1987, pp. 197–221.