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A GENERALIZATION OF ANTIWEBS TO INDEPENDENCE SYSTEMS AND THEIR CANONICAL FACETS

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We consider independence system polytopes, i.e. polytopes whose extreme points are the incidence vectors of the sets of an independence system. We first give a sufficient condition for recognizing Boolean facets. Then, the notion of antiweb introduced by Trotter for graphs is generalized to independence systems and used for obtaining canonical facets of the associated polytopes. We also point out how our results relate with known ones for knapsack, set covering and matroid polytopes.

Key words: 0, 1 integer programming, independence system, facet, antiweb.

1. Introduction

Given a finite set $E=\{e_1,\ldots,e_n\}$, an *independence system* (IS for short) on E is a family $\mathcal{J}$ of subsets of E closed under inclusion, i.e. satisfying the following property:

(I1) If $J\in\mathcal{J}$ and $I\subseteq J$, then $I\in\mathcal{J}$.

A set belonging to $\mathcal{J}$ is called *independent* and a set that does not belong to $\mathcal{J}$ is called *dependent*. Minimal (for set inclusion) dependent sets are called *circuits* and we denote by $\mathscr{C}(\mathcal{J})$ the family of circuits of $\mathcal{J}$. The collection $\mathscr{C}(\mathcal{J})$ is a *clutter*, i.e., if $C, C'\in\mathscr{C}(\mathcal{J})$ and $C\subseteq C'$, then $C=C'$. An independence system is fully characterized by its family of circuits and, conversely, every clutter $\mathscr{C}$ determines a unique IS: $\mathcal{J}(\mathscr{C})=\{I\subseteq E\colon\ C\not\subseteq I$ for all $C\in\mathscr{C}\}$. The *rank function* of the IS $\mathcal{J}$ is the set function defined by $r(S)=\max(|I|\colon\ I\in\mathcal{J}$ and $I\subseteq S)$ for all $S\subseteq E$. We also define the *independence number* $\alpha(\mathcal{J})$ of the IS $\mathcal{J}$ as the maximum size of the independent sets, i.e. $\alpha(\mathcal{J})=r(E)$. Notice that, if all circuits have size two, then $\mathcal{J}$ is the family of stable sets of the graph G with E as nodeset and $\mathscr{C}$ as edgeset and the independence number is exactly the stability number of G as defined in [1]. For a subset $S\subseteq E$, the family $\mathcal{J}_S=\{I\in\mathcal{J}\colon\ I\subseteq S\}$ is clearly an IS on S whose family of circuits is given by $\mathscr{C}_S=\{C\in\mathscr{C}\colon\ C\subseteq S\}$ and whose independence number satisfies $\alpha(\mathcal{J}_S)=r(S)$ (note that, when $\mathcal{J}$ is a matroid, this is the classical notion of restriction, cf. [17, Chap. 4]).

Given an IS $\mathcal{J}$ on E, we define the *independence system polytope*: $P=P(\mathcal{J})=\operatorname{Conv}(\mathcal{J})$ of $\mathbb{R}^E$ to be the convex hull of the incidence vectors of the independent sets of $\mathcal{J}$. In many applications, a weight w_e is associated with each element $e\in E$

and one is interested in finding an independent set of maximum weight, which amounts to solving the following optimization problem:

$$\text{Maximize} \quad w \cdot x\left(\sum_{e \in E} w_e x_e \right)$$

$$\text{subject to} \quad x \in P(\mathscr{I}).$$

This problem can be solved, at least in theory, by linear programming techniques if one can describe the polytope $P(\mathscr{I})$ by a (minimal) system of linear inequalities. In practice, some efficient procedures for the above optimization problem can be found even if only a partial description of $P(\mathscr{I})$ is available. Hence, it is of fundamental interest to find some classes of valid inequalities defining facets for $P(\mathscr{I})$. Much work was done in this direction, mainly based on the study of special configurations of the family of circuits (for another approach, based on the study of structural properties of matroidal type of the family $\mathscr{I}$ of independent sets, see [5]).

For any IS $\mathscr{I}$ on E and any subset S of E, the inequality:

$$\sum_{e \in S} x_e \leq r(S) \tag{1.1}$$

is clearly a valid inequality for $P(\mathscr{I})$, called *rank inequality*; it is also said to be *Boolean* since its nonzero coefficients take all the same value. Hence, it is natural to ask for some conditions on $\mathscr{I}$ (or $\mathscr{C}(\mathscr{I})$) and S that ensure that (1.1) induces a facet; when $S = E$, the facet induced by (1.1) is also called *canonical facet* of $P(\mathscr{I})$. A subset S of E is called *closed* if $r(S \cup \{e\}) \geq r(S)+1$ holds for all elements $e \in E - S$ and S is called *nonseparable* if $r(S) \lneqq r(T) + r(S-T)$ holds for all non-empty subsets T of S with $T \neq S$. It is easy to see that a necessary condition for the rank inequality (1.1) to be facet inducing is that the set S be closed and nonseparable. For some classes of IS, this condition is also sufficient. For instance, when the IS $\mathscr{I}$ is a *matroid*, i.e. satisfies the condition:

(I2) For all $I, J \in \mathscr{I}$ with $|I| \lneqq |J|$, there exists $e \in J - I$ such that $I + e \in \mathscr{I}$;

then it is a well known result by Edmonds [7] (see also [10]) that the facets of $P(\mathscr{I})$ are generated by the rank inequalities (1.1) for closed and nonseparable subsets $S \subseteq E$. The same result holds when the IS $\mathscr{I}$ can be written as the intersection of two matroids [8, 10]. In general, however, there exist non-Boolean facets; one of the known techniques for producing some is by "lifting" known facets—for instance, Boolean ones—of a lower dimensional polytope (for references on lifting procedures, see, for instance, [3b, 11, 13, 14]). We give in Section 3 a sufficient condition for recognizing the Boolean facets (1.1); we also show how our result relates with known characterizations of Boolean facets for knapsack and matroid polytopes, as well as known conditions for the existence of canonical facets for set covering polytopes.

For the node packing polytope, various classes of graphs have been introduced that yield canonical facets; for instance, the cliques, odd holes and odd antiholes in [12], the webs and antiwebs in [16]. Some extensions of these results to independence systems can be found in [9, 11, 15] where the notions of generalized cliques, odd holes and antiholes are defined. In this paper, we give a further generalization of these notions; we introduce in Section 4 a class of IS: The generalized antiwebs and we characterize the ones having canonical facets. We also point out the relation existing between generalized antiwebs and the (q, t)-roses introduced by Sassano in [14] for the set covering polytope. For this, let us now recall explicitly the correspondence between IS and set covering polytopes.

Many IS arise as the family of solutions of integer programming problems of packing type. More precisely, let M be an $m \times n$ matrix with coefficients in $\{0, 1\}$ and $b \in \mathbb{R}^n$ be a vector with nonzero integer coordinates. The following polytopes are often considered:

$$Q_0 = \mathrm{Conv}(\{x \in \{0, 1\}^n:\ Mx \geqslant 1_n\}), \tag{1.2}$$

$$P_0 = \mathrm{Conv}(\{x \in \{0, 1\}^n:\ Mx \leqslant 1_n\}), \tag{1.3}$$

$$P = \mathrm{Conv}(\{x \in \{0, 1\}^n:\ Mx \leqslant b\}), \tag{1.4}$$

where we refer to (1.2) as the *set covering polytope*, to (1.3) as the *set packing polytope* (which can be transformed into an equivalent node packing polytope, using the notion of intersection graph, cf. [12]) and to (1.4) as the *generalized set packing polytope* or IS polytope; 1_n denoting the n-vector whose coordinates are all equal to 1. Note that, when $m = 1$, then P is a knapsack polytope. The family $\mathscr{I}$ of subsets of $[1, n]$ whose incidence vectors belong to the polytope P is clearly an IS and $P = P(\mathscr{I})$ holds.

In the particular case when, for all $j \in [1, m]$, $b_j + 1$ is equal to the number of nonzero coordinates of the jth row of M, the circuits of $\mathscr{I}$ correspond exactly to those rows of M that do not dominate any other row of M; furthermore, the polytope (1.4) can be transformed into the set covering polytope (1.2), and conversely, by using the substitution: $x' = 1_n - x$. Note also that, in this case, if $\mathscr{A}$ denotes the family of subsets of $[1, n]$ whose incidence vectors are the rows of the matrix M, then, the IS $\mathscr{I}$ can be described by:

$$\mathscr{I} = \mathscr{I}(\mathscr{A}) = \{I \subseteq [1, n]:\ |I \cap A| \leqslant |A| - 1 \text{ for all } A \in \mathscr{A}\}. \tag{1.5}$$

Generally, let $\mathscr{C}$ denote the family of circuits of the IS $\mathscr{I}$ associated to the polytope P; then one can alternatively describe P by:

$$P = \mathrm{Conv}\left\{x \in \{0, 1\}^n:\ \sum_{e \in C} x_e \leqslant |C| - 1 \text{ for all } C \in \mathscr{C}\right\}$$

and, if one considers the corresponding set covering polytope:

$$Q = \mathrm{Conv}\left\{x \in \{0, 1\}^n:\ \sum_{e \in C} x_e \geqslant 1 \text{ for all } C \in \mathscr{C}\right\}$$

then the two polytopes P, Q can always be mapped one into the other by the transformation: $x' = 1_n - x$. Consequently, set covering and IS polytopes are, at least theoretically (via the knowledge of the collection of circuits), equivalent—modulo the above transformation—and we will refer to this equivalence throughout the paper when comparing results for IS and set covering polyhedra. An obvious implication of this fact is that any result stated for the IS polytope can be translated and used for the set covering problem and conversely.

2. Definitions and notations

In the following, 1_n (resp. 0_n) denotes the vector of $\mathbb{R}^n$ whose coordinates are all equal to 1 (resp. 0). For all $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer less or equal to a. Given a set I, the vector: $x^I = (x_e^I)_{e \in E}$ denotes the *incidence vector* of I defined by: $x_e^I = 1$ if $e \in I$ and 0 otherwise. Also, given some elements $e \in I$, $e' \notin I$, we denote by $I + e'$ the set $I \cup \{e'\}$ and by $I - e$ the set $I - \{e\}$.

A *polyhedron* $P \subseteq \mathbb{R}^n$ is the intersection of finitely many closed halfspaces and, if P also is bounded, then P is a *polytope*. The *dimension* of a polyhedron P, denoted by dim P, is the maximum number of affinely independent points in P minus one and, if $0_n \notin P$, then dim P is equal to the maximum number of linearly independent points in P. The polyhedron P is called *full dimensional* if its dimension is equal to n. In the following, for any IS $\mathcal{I}$ on E, we can assume w.l.o.g. that $P(\mathcal{I})$ is full dimensional, i.e. that $\{e\} \in \mathcal{I}$ for all $e \in E$. Given a polyhedron P, $c \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, the inequality: $c \cdot x \leqslant \beta$ is called *valid* for P if it is satisfied by all points of P; then, the set: $V = \{x \in P\colon\ c \cdot x = \beta\}$ is called the *face* of P *induced* (or *defined*) by the inequality $c \cdot x \leqslant \beta$. A *facet* is a face of P having dimension dim $P - 1$; hence, if P is full dimensional, then each facet is determined by a unique, up to positive multiple, valid inequality.

For the terminology and the basic properties for graphs and hypergraphs, we refer for instance to [1] and, for matroid theory notions, to [17].

3. A sufficient condition for the existence of Boolean facets

Given an independence system $\mathcal{I}$ on E with $\mathscr{C}$ as family of circuits and $r(\cdot)$ as rank function and given a subset S of E, we define a graph $G_S(\mathcal{I})$, called *critical graph* of $\mathcal{I}$ on S, having S as node set and whose edges are defined as follows: two distinct elements e, e' of S are adjacent in $G_S(\mathcal{I})$ if and only if there exists an independent set I satisfying: $I \subseteq S$, $|I| = r(S)$, $e \in I$, $e' \notin I$ and $I - e + e' \in \mathcal{I}$. If we denote by $\mathscr{C} \backslash \{e, e'\}$ the subcollection of $\mathscr{C}$ formed by the circuits that do not contain the pair $\{e. e'\}$, then, the following can be easily observed:

Remark 3.1. Two distinct elements e, e' of S are adjacent in $G_S(\mathcal{I})$ if and only if $\alpha(\mathcal{I}(\mathscr{C}_S \backslash \{e, e'\})) \geqslant \alpha(\mathcal{I}_S) + 1$; i.e. the removal of all circuits of $\mathscr{C}_S$ containing both e, e' increases the rank of S.

Theorem 3.2. *Let S be a closed subset of E and assume that the critical graph $G_S(\mathscr{I})$ of $\mathscr{I}$ on S is connected. Then the rank inequality* (1.1), $\sum_{e\in S} x_e \leqslant r(S)$, *induces a facet of the polytope $P(\mathscr{I})$.*

Proof. Let V be the face of $P(\mathscr{I})$ induced by the valid inequality (1.1). Suppose that there exists another valid inequality $c\cdot x \leqslant \beta$ such that $V=\{x\in P\colon c\cdot x=\beta\}$. We prove that $c\cdot x$ and $\sum_{e\in S} x_e$ are identical linear forms up to positive multiple; i.e. that $c_e=0$ for all $e\in E-S$ and $c_e=c_{e'}$, for all $e, e'\in S$. Take first an element $e\in E-S$. Since S is a closed set, $r(S+e)\geqslant r(S)+1$; therefore, one can find an independent set I such that $e\in I$, $|I\cap S|=r(S)$. Hence the incidence vectors x^I, x^{I-e} of the sets I, $I-e$ belong to the face V, implying that $c\cdot x^I=c\cdot x^{I-e}=\beta$ and thus $c_e=0$. Take now two distinct elements $e, e'\in S$ and suppose that they are adjacent in $G_S(\mathscr{I})$. Then, there exists a set $I\in\mathscr{I}$ such that: $I\subseteq S$, $e\in I$, $e'\notin I$, $I'=I-e+e'\in\mathscr{I}$ and $|I|=|I'|=r(S)$. Thus, the incidence vectors x^I, $x^{I'}$ of I, I' lie on the face V and we have therefore that $c\cdot x^I=c\cdot x^{I'}=\beta$ which implies that $c_e=c_{e'}$. It now follows easily from the connectivity of $G_S(\mathscr{I})$ that $c_e=c_{e'}$ for all $e, e'\in E$. □

Remark 3.3. Theorem 3.2 extends Chvátal's result [4, Theorem 4.2] for the existence of canonical facets for the node packing polytope. Via the correspondence mentioned above between IS and set covering polytopes, Theorem 3.2 in the case $S=E$ corresponds to Lemma 3.1 from [14]. Also, Theorem 3.2 extends a closely related result given by Sekiguchi in [15] for IS defined by (1.5). For this, we reformulate the problem which was considered in [15].

Given a family $\mathscr{A}$ of subsets of E, we consider the IS $\mathscr{I}=\mathscr{I}(\mathscr{A})$ defined by (1.5) and, for all $B\in\mathscr{A}$, the IS $\mathscr{I}(\mathscr{A}-B)=\{I\subseteq E\colon\ |I\cap A|\leqslant|A|-1 \text{ for all } A\in\mathscr{A}, A\neq B\}$. In [15], a set $B\in\mathscr{A}$ is called *critical* if $\alpha(\mathscr{I}(\mathscr{A}-B))\geqslant\alpha(\mathscr{I})+1$. Let $\mathscr{A}^*$ be the collection of all critical sets of $\mathscr{A}$ and $\mathscr{H}(\mathscr{A})=(E,\mathscr{A}^*)$ be the hypergraph with E as nodeset and $\mathscr{A}^*$ as edgeset.

Proposition 3.4 [15]. *If the hypergraph $\mathscr{H}(\mathscr{I})$ is connected, then the inequality* (3.1) *induces a facet of $P(\mathscr{I})$.*

We now show that Proposition 3.4 is, in fact, implied by Theorem 3.2.

Proposition 3.5. *If the hypergraph $\mathscr{H}(\mathscr{I})$ is connected, then the critical graph $G_E(\mathscr{I})$ is connected.*

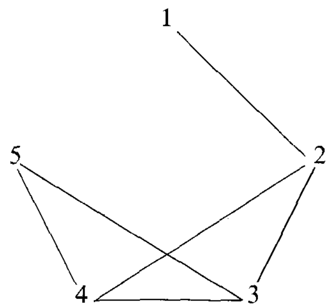
Proof. The assertion follows easily from the following claim:

Claim 3.6. *If two distinct elements of E are contained in a common critical set of $\mathscr{A}^*$, then they are adjacent in $G_E(\mathscr{I})$.*

Proof. Take two distinct elements e, e' of E and a set $B\in\mathscr{A}^*$ such that $\{e, e'\}\subseteq B$. By definition of $\mathscr{A}^*$, there exists a set I such that $|I|=\alpha(\mathscr{I})+1$ and $|I\cap A|\lneqq|A|$

for all $A \in \mathscr{A}$, $A \neq B$. We finish the proof by showing that I is indeed independent in $\mathscr{I}(\mathscr{C} \backslash \{e, e'\})$, i.e. that any circuit contained in I must contain both e, e'. Take $C \in \mathscr{C}$ such that $C \subseteq I$. It can be easily verified from the definition of $\mathscr{I}$ that C indeed belongs to $\mathscr{A}$. It follows from the definition of I that $C = B$ and thus $\{e, e'\} \subseteq C$. □

Remark 3.7. The converse of proposition 3.5 is false as shown by the following example. Take $E = \{1, 2, 3, 4, 5\}$ and $\mathscr{A} = \{123, 124, 235, 245, 345\}$ (where 123 stands for the set $\{1, 2, 3\}$, for short). It is easy to verify that there is only one critical set in $\mathscr{A}$: 345 and, thus, the hypergraph $\mathscr{H}(\mathscr{I})$ is not connected; however, the critical graph $G_E(\mathscr{I})$ is connected and more precisely it has the following configuration:



We now prove that, for matroid and knapsack polytopes, the necessary condition of Theorem 3.2 for the existence of Boolean facets is also a sufficient condition.

When the IS $\mathscr{I}$ is a matroid, we saw in Section 1 that the rank inequalities (1.1) yielding facets are exactly the ones obtained from closed and nonseparable sets $S \subseteq E$. In the following, we indicate precisely how this fact relates with the condition from Theorem 3.2.

Proposition 3.8. *Let $\mathscr{I}$ be a matroid on E and S be a subset of E. We have equivalence between the following assertions*:

(i) *The rank inequality* (1.1) *induces a facet of $P(\mathscr{I})$.*
(ii) *S is closed and nonseparable.*
(iii) *S is closed and the graph $G_S(\mathscr{I})$ is connected.*

Proof. It is enough to show that the implication (ii)⇒(iii). For this, we prove that if the graph $G_S(\mathscr{I})$ is not connected, then the set S is separable. Suppose that $G_S(\mathscr{I})$ is not connected and let $A \subseteq S$ denote the nodeset of one connected component of $G_S(\mathscr{I})$ and $B = S - A$; then, A, B are nonempty and any two nodes belonging respectively to A, B are not adjacent. We show that $r(S) = r(A) + r(B)$ holds. Let I be a maximal independent subset of A, then $|I| = r(A)$; one can complete I into a maximal independent subset K of S, thus $K = I \cup J$ with $J \subseteq B$ and $|K| = r(S)$. If $|J| \lneqq r(B)$, then, there exists an element $x \in B - J$ such that $J \cup x \in \mathscr{I}$. Applying axiom (I2) to the independent sets $J \cup x$ and $K = J \cup I$, one deduces that there exists an element $y \in K - J = I$ such that $K - y + x \in \mathcal{J}$; this implies by definition that x, y are adjacent in $G_S(\mathcal{J})$ contradicting the fact that $x \in B$, $y \in A$ belong to distinct connected components of $G_S(\mathcal{J})$. Therefore, $|J| = r(B)$ holds, from which one deduces that $r(S) = r(A) + r(B)$, i.e. S is separable. □

We now examine how Theorem 3.2 relates with known results concerning Boolean facets of the knapsack polytope. We adopt the notations from [2]. Let $E = [1, n]$ and $a_0, a_1, \ldots, a_n$ be positive numbers such that $a_1 \geq \cdots \geq a_n$. We consider the *knapsack polytope*:

$$\mathrm{KP} = \mathrm{Conv}(\{x \in \{0,1\}^n\colon\ a_1x_1 + \cdots + a_nx_n \leq a_0\}). \tag{3.9}$$

Let $\mathcal{J}$ be the IS on E associated with KP, hence $P(\mathcal{J}) = \mathrm{KP}$; let $r(\cdot)$ be its rank function and $\mathcal{C}$ be its family of circuits (note that the circuits are exactly the minimal covers as defined in [2]). Given a circuit $C = \{j_1, \ldots, j_c\}$ with $1 \leq j_1 \lneqq \cdots \lneqq j_c \leq n$, its extension is defined by: $E(C) = C \cup \{j \in E - C\colon\ a_j \geq a_{j_1}\} = \{1, \ldots, j_1\} \cup C$. It is well known that, for each circuit C, the inequality:

$$\sum_{e \in E(C)} x_e \leq |C| - 1 \tag{3.10}$$

is valid for KP (or $P(\mathcal{J})$) and that KP can be equivalently defined by:

$$\mathrm{KP} = \mathrm{Conv}\left(\left\{x \in \{0,1\}^n\colon\ \sum_{e \in E(C)} x_e \leq |C| - 1 \text{ for all } C \in \mathcal{C}\right\}\right).$$

In the next proposition, we recall how all Boolean facets can be generated from inequalities (3.10).

Proposition 3.11 [2]. *Let S be a subset of E. The rank inequality* (1.1) *induces a facet of $P(\mathcal{J})$* (=KP) *if and only if there exists a circuit $C = \{j_1, \ldots, j_c\}$ with $1 \leq j_1 \lneqq \cdots \lneqq j_c \leq n$ such that $S = E(C)$ and satisfying the following conditions*:

$$\text{When } S \neq E, \qquad \sum_{j \in C - j_1 + i_1} a_j \leq a_0 \quad \text{where } a_{i_1} = \max(a_j\colon\ j \in E - S), \tag{3.12}$$

$$\sum_{j \in C - \{j_1, j_2\} + 1} a_j \leq a_0. \tag{3.13}$$

Then $r(S) = |C| - 1$ holds.

Proposition 3.14. *We have equivalence between the following assertions*:

(i) *The rank inequality* (1.1) *induces a facet of the polytope $P(\mathcal{J})$* (=KP).
(ii) *S is the extension of a circuit C satisfying* (3.12) *and* (3.13).
(iii) *S is closed and the graph $G_S(\mathcal{J})$ is connected.*

Proof. It is enough to show the implication (ii)⇒(iii). Take a subset S such that $S = E(C) = [1, j_1] \cup C$ where $C = \{j_1, \ldots, j_c\}$ is a circuit satisfying (3.12) and (3.13). If $S \neq E$, we verify that S is closed. Let $e \in E - S$; then, we deduce from (3.12) and the fact that $a_e \leqslant a_{i_1}$ that the set $C - j_1 + e$ is independent which implies that $r(S \cup e) \geqslant |C| = r(S) + 1$. We now show that the graph $G_S(\mathscr{I})$ is connected. For this, we show that every element $e \in S$ is adjacent to j_1. Take first $e = j_k$ for $2 \leqslant k \leqslant c$. Since C is a circuit, both sets $C - j_1$, $C - j_k$ are independent sets of cardinality $r(S)$ which implies that j_1, j_k are adjacent. Consider now an element e, $1 \leqslant e \lneqq j_1$; then $a_e \leqslant a_1$ which, together with (3.13), implies that the set $C - \{j_1, j_2\} + e$ is independent and one deduces again that e, j_1 are adjacent. □

An additional example, where the sufficient condition of Theorem 3.2 for the existence of canonical facets is also necessary, is provided by set covering polytopes associated with β-maximal matrices, considered by Cornuejols and Sassano [6, Prop. 4].

We conclude this section by pointing out how the notion of critical cutset, defined by Balas and Zemel [3a] for the node packing polytope, by Sekiguchi [15] for general IS and by Cornuejols and Sassano [6] for the set covering polytope, relates with the notion of closed and nonseparable set we considered before. Given a family $\mathscr{A}$ of subsets of E, let $\mathscr{I} = \mathscr{I}(\mathscr{A})$ be the IS defined by (1.5) and $r(\cdot)$ be its rank function. For a nonempty subset $S \subsetneq E$, one defines its *cutset* $C(S) = (S, E - S)$ as the subcollection of $\mathscr{A}$ formed by all sets $A \in \mathscr{A}$ that meet both S and $E - S$. When removing $C(S)$ from $\mathscr{A}$, one obtains the IS $\mathscr{I}(\mathscr{A} - C(S))$ whose independence number is obviously given by:

$$\alpha(\mathscr{I}(\mathscr{A} - C(S))) = \alpha(\mathscr{I}_S) + \alpha(\mathscr{I}_{E-S}) = r(S) + r(E - S) \geqslant \alpha(\mathscr{I}).$$

In [15], the cutset $C(S)$ is called *critical* if $\alpha(\mathscr{I}(\mathscr{A} - C(S))) \geqslant \alpha(\mathscr{I}) + 1$ holds. Therefore, saying that the set E is nonseparable amounts to saying that every cutset of $\mathscr{I}$ is critical and, thus, Proposition 1 from [3a] (see also [15]) and, equivalently, Proposition 1 from [6] for the set covering polytope, are simply stating that a necessary condition for having a canonical facet is that the set E be nonseparable. Similarly, saying that the set S is closed amounts to saying that, for all $e \in E - S$, the cutset $C(S) = (S, e)$ of $\mathscr{I}_{S+e}$ is not critical; hence, Theorem 3 from [3a, 15] and, equivalently, Proposition 3 from [6] for the set covering polytope, can be rephrased as follows:

Proposition 3.15. *We have equivalence between the following assertions:*

(i) *The rank inequality* (1.1) *induces a facet of* $P(\mathscr{I})$.

(ii) *S is closed and the rank inequality* (1.1) *induces a facet of* $P(\mathscr{I}_S)$.

4. Generalized antiwebs and their canonical facets

Let n, t, q be some integers such that $n \geqslant t \geqslant q \geqslant 2$. We denote the groundset by $E = \{e_1, e_2, \ldots, e_n\}$, $|E| = n$ and we define the sets: $N = \{1, 2, \ldots, n\}$ and, for all $i \in N$, $E^i = \{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ (where the indices are taken modulo n) formed by t consecutive elements.

We call *(n, t, q)-generalized antiweb* on E the family of subsets of E denoted by $\mathcal{AW}(n, t, q)$ and defined by:

$$\mathcal{AW}(n, t, q) = \{C \subseteq E: \ |C| = q \text{ and } C \subseteq E^i \text{ for some } i \in N\}.$$

In the following, we will refer to the (n, t, q)-generalized antiweb as the independence system having $\mathcal{AW}(n, t, q)$ as family of circuits as well.

The (n, t, q)-generalized antiwebs contain as special cases the following structures:

-antiwebs with parameters n, t when $q = 2$, i.e. in the graph theoretical sense (cf. [16]).

-generalized cliques (cf. [9, 11, 15]) when $n = t$.

-generalized odd holes (cf. [9]) when $q = t$ and t does not divide n.

-generalized antiholes (cf. [9]) when $n = qt + 1$.

(Notice that the objects introduced in [9] are slightly more general; basically, in [9], the points are "blown up" and replaced by pairwise disjoint sets.)

In view of the correspondence between IS and set covering polytopes that we pointed out in Section 1, it can be easily verified that (n, t, q)-generalized antiwebs correspond, in fact, to the (q, t)-roses of order n introduced by Sassano in [14] in the set covering context. We became aware of this fact after reading Sassano's paper, but we felt that there were still some strong motivations for the study of generalized antiwebs. In particular, one of our purposes was to provide a new and more elegant, also constructive and simpler, proof for the characterization of generalized antiwebs having canonical facets (cf. Proposition 4.2 and Theorem 4.7 and, in parallel, Lemma 3.2 and Theorem 3.1 from [14]). We also wanted to show how the notion of roses introduced for the set covering polytope relates, in fact, to well known objects: the antiwebs in the graph theoretical context, henceforth unifying results that apparently apply to different contexts. Moreover, our approach suggests a natural extension of this work to the study of *generalized webs* $\mathcal{W}(n, t, q)$ which can be defined by:

$$\mathcal{W}(n, t, q) = \{C \subseteq E: \ |C| = q \text{ and } C \not\subseteq E^i \text{ for all } i \in N\}.$$

Remark 4.1. A set $I \subseteq E$ is independent in $\mathcal{AW}(n, t, q)$ if and only if $|I \cap E^i| \leq q - 1$ for all $i \in N$.

In the following, we give in Proposition 4.2 the numerical value of the independence number of $\mathcal{AW}(n, t, q)$ and, in Theorem 4.7, we characterize the generalized antiwebs having canonical facets.

Proposition 4.2. *The independence number of $\mathcal{AW}(n, t, q)$ is given by*:

$$\alpha(\mathcal{AW}(n, t, q)) = \left\lfloor \frac{n(q-1)}{t} \right\rfloor.$$

Proof. By using Remark 4.1, we have that, for any independent set I, $\sum_{i=1}^{n} |I \cap E^i| \leq n(q-1)$. Since each element of E belongs to exactly t distinct sets E^i, we deduce that $\sum_{i=1}^{n} |I \cap E^i| = t|I|$ and therefore $|I| \leqslant n(q-1)/t$ which implies that $\alpha(\mathscr{AW}(n, t, q)) \leqslant \lfloor n(q-1)/t \rfloor$.

In order to show the reverse inequality, we exhibit an independent set I of size $|I| = \lfloor n(q-1)/t \rfloor$. For this, consider the integers α, ρ obtained by Euclidean division of $n(q-1)$ by t, i.e.

$$n(q-1) = \alpha t + \rho \quad \text{with } 0 \leqslant \rho \leqslant t-1 \text{ and } \alpha = \left\lfloor \frac{n(q-1)}{t} \right\rfloor. \tag{4.3}$$

We define the following set I:

$$I = \{e_{u_k}:\ 0 \leqslant k \leqslant \alpha - 1\} \quad \text{with } u_k = 1 + \left\lfloor \frac{kn}{\alpha} \right\rfloor. \tag{4.4}$$

We now show that I is indeed an independent set of size $\alpha = \lfloor n(q-1)/t \rfloor$. It can be easily observed that, for all k, k' such that $0 \leqslant k \lneqq k' \leqslant \alpha - 1$, the inequality $1 \leqslant u_k \lneqq u_{k'} \leqslant n$ holds, which therefore implies that $|I| = \alpha$. We verify that I is independent, i.e., in view of Remark 4.1, that $|I \cap E^i| \leqslant q-1$ holds for all $i \in N$. Suppose for contradiction that, for some $i \in N$, we have $|I \cap E^i| \geqslant q$. Let k be the first integer such that $e_{u_k} \in E^i$, i.e.

$$u_k = 1 + \left\lfloor \frac{kn}{\alpha} \right\rfloor \geqslant i. \tag{4.5}$$

If $|I \cap E^i| \geqslant q$, then this implies that $e_{u_{k+q-1}} \in E^i$, i.e.

$$u_{k+q-1} = 1 + \left\lfloor \frac{(k+q-1)n}{\alpha} \right\rfloor \leqslant i + t - 1. \tag{4.6}$$

By using the fact that $\lfloor x+y \rfloor \geqslant \lfloor x \rfloor + \lfloor y \rfloor$ holds for all reals x, y and the inequality $\lfloor n(q-1)/\alpha \rfloor \geqslant t$, we deduce from (4.5) and (4.6) that $i+t-1 \geqslant u_{k+q-1} \geqslant u_k + \lfloor n(q-1)/\alpha \rfloor \geqslant i+t$ holds, yielding a contradiction. □

Theorem 4.7. *The valid inequality*:

$$\sum_{e \in E} x_e \leqslant \left\lfloor \frac{n(q-1)}{t} \right\rfloor \tag{4.8}$$

induces a facet of the polytope $P = P(\mathscr{AW}(n, t, q))$ *if and only if* $n = t$ *or* t *does not divide* $n(q-1)$.

Proof. When $n = t$, then, as we already mentioned, the (n, n, q)-generalized antiweb corresponds to a generalized clique for which it is known that (4.8) induces a facet of P (see [9, 11, 15]). Hence, we can assume that $t \lneqq n$.

Let us first assume that t divides $n(q-1)$. Then (4.8) can be obtained as a linear combination of the inequalities: $\sum_{e \in E^i} x_e \leqslant q-1$ that are valid for P for all $i \in N$ and therefore (4.8) does not induce a facet of P.

We now assume that t does not divide $n(q-1)$. We prove that (4.8) induces a facet of P by using Theorem 3.2, i.e. we show that the critical graph $G=G_E(\mathscr{I})$ associated with the antiweb is connected. For this, take the independent set I defined by relation (4.4) that we considered in the proof of Proposition 4.2. By construction, we have that $e_1 \in I$ and also $e_n \notin I$. Else, if $e_n \in I$, then this implies that $n=u_{\alpha-1}=1+\lfloor(\alpha-1)n/\alpha\rfloor=1+n+\lfloor -n/\alpha\rfloor$ and thus $\lfloor -n/\alpha\rfloor=-1$, yielding the inequality $n\leqslant\alpha$; using relation (4.3), we deduce that $\alpha t+\rho=n(q-1)\leqslant\alpha(q-1)\lneqq\alpha t$ yielding a contradiction. We define the set $I'=I-e_1+e_n$; then, in order to show that e_1, e_n are adjacent in G, it is enough to check that I' also is an independent set. For this, it can be easily seen that it suffices to verify that $|I'\cap E^{n-t+1}|\leqslant q-1$, i.e. $|I\cap E^{n-t+1}|\leqslant q-2$. Suppose for contradiction that $|I\cap E^{n-t+1}|\geqslant q-1$ holds; this implies that $e_{u_{\alpha-q+1}}\in E^{n-t+1}$, i.e. $n-t+1\leqslant u_{\alpha-q+1}=1+\lfloor(\alpha-q+1)n/\alpha\rfloor$ from which we deduce that $n(q-1)/\alpha\leqslant t$ and thus $n(q-1)/\alpha=t$, i.e. $n(q-1)/t=\alpha$. We therefore obtain a contradiction with the assumption that t does not divide $n(q-1)$. □

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References

[1] C. Berge, *Graphs and Hypergraphs* (Dunod, Paris, 1970).
[2] E. Balas, "Facets of the knapsack polytope," *Mathematical Programming* 8 (1975) 146–164.
[3a] E. Balas and E. Zemel, "Critical cutsets of graphs and canonical facets of set packing polytopes," *Mathematics of Operations Research* (1977) 15–19.
[3b] E. Balas and E. Zemel, "Facets of the knapsack polytope from minimal covers," *SIAM Journal of Applied Mathematics* 34 (1978) 119–148.
[4] V. Chvátal, "On certain polytopes associated with graphs," *Journal of Combinatorial Theory* (*B*) 18 (1975) 138–154.
[5] M. Conforti and M. Laurent, "On the facial structure of independence system polyhedra," *Mathematics of Operations Research*, to appear.
[6] G. Cornuéjols and A. Sassano, "On the 0, 1 facets of the set covering polytope," *Mathematical Programming* 43 (1989) 45–55.
[7] J. Edmonds, "Matroids and the greedy algorithm," *Mathematical Programming* 1 (1971) 127–136.
[8] J. Edmonds, "Matroid intersection," *Annals of Discrete Mathematics* 4 (1979) 39–49.
[9] R. Euler, M. Junger and G. Reinelt, "Generalizations of cliques, odd cycles and anticycles and their relation to independence system polyhedra," preprint, University of Augsburg.
[10] R. Giles, "Submodular functions, graphs and integer polyhedra," Ph.D. thesis, University of Waterloo (1975).
[11] G.L. Nemhauser and L.E. Trotter, "Properties of vertex packing and independence system polyhedra," *Mathematical Programming* 6 (1974) 48–61.
[12] M.W. Padberg, "On the facial structure of set packing polyhedra," *Mathematical Programming* 5 (1973) 199–215.
[13] M.W. Padberg, "A note on zero-one programming," *Operations Research* 23 (1975) 833–837.

[14] A. Sassano, "On the facial structure of the set covering polytope," *Mathematical Programming*, to appear.

[15] Y. Sekiguchi, "A note on node packing problems on hypergraphs," *Operations Research Letters* 2(5) (1983) 243–247.

[16] L.E. Trotter, "A class of facet producing graphs for vertex packing polyhedra," *Discrete Mathematics* 12 (1975) 373–388.

[17] D. Welsh, *Matroid Theory* (Academic Press, New York, 1976).