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LOWER BOUND FOR THE NUMBER OF ITERATIONS IN SEMIDEFINITE HIERARCHIES FOR THE CUT POLYTOPE

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Hierarchies of semidefinite relaxations for 0/1 polytopes have been constructed by Lasserre (2001a) and by Lovász and Schrijver (1991). The cut polytope of a graph on n nodes can be expressed as a projection of such a semidefinite relaxation after at most n steps. We show that $\lceil n/2 \rceil$ iterations are needed for finding the cut polytope of the complete graph K_n.

1. Introduction.

1.1. Preamble. Given a graph $G = (V, E)$ and a subset $A \subseteq V$, the *cut* determined by A is the vector $\delta(A) \in \{\pm 1\}^E$ with ijth entry -1 if and only if $|A \cap \{i, j\}| = 1$. The *cut polytope* $\mathrm{CUT}(G)$ is the polytope in $\mathbb{R}^E$ defined as the convex hull of all cuts $\delta(A)$ ($A \subseteq V$). Given edge weights $w \in \mathbb{Q}^E$, the *max-cut problem* is the problem of finding a cut $\delta(A)$ whose weight $\sum_{ij \in E \mid i \in A,\, j \notin A} w_{ij}$ is maximum. Hence, it can be formulated as the linear programming problem:

$$\max \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_{ij}) \qquad \text{subject to } x \in \mathrm{CUT}(G)$$

over the cut polytope.

Because the max-cut problem is NP-hard, extensive research has been done for finding good and efficient relaxations of the cut polytope. Research has focused, in particular, on finding large classes of valid inequalities and facets for the cut polytope by exploiting the combinatorial structure of the specific max-cut problem (see Deza and Laurent 1997 for a detailed account). Research has also focused on developing general purpose methods applied to arbitrary 0/1 (or ± 1) problems. Let us mention, in particular, the lift-and-project method of Balas et al. (1993), the matrix-cut method of Lovász and Schrijver (1991), the linearization-reformulation technique of Sherali and Adams (1990, 1999), and the more recent real-algebraic method of Lasserre (2001a, b) (and related work of Nesterov 2000; Parrilo 2000, 2003, Shor 1987). A common feature of these methods is the construction of a hierarchy of linear or semidefinite relaxations for a given 0/1 (or ± 1) polytope P, converging to P in d steps (if P lies in the d-space). The various constructions have been compared in Laurent (2003a) and it is shown there that the Lasserre hierarchy of semidefinite relaxations refines all the other hierarchies.

Let us define the *rank* of a given method (with respect to some initial relaxation of P) as the smallest number of iterations needed for finding the integer polytope P. The dimension of the ambient space is a common upper bound for the rank of the various procedures. Lower bounds have been established for several examples; for instance, for the Sherali-Adams procedure (Laurent 2003a), and for the Lovász-Schrijver procedure applied to the matching polytope (Stephen and Tunçel 1999), to the knapsack polytope and to the "pigeon

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hole principle" polytope (Grigoriev et al. 2001), to the traveling salesman polytope and to some other examples (Cook and Dash 2001, Dash 2000, Goemans and Tunçel 2001). As pointed out to us by D. Pasechnik, it follows from arguments in Grigoriev (2001) (given in the context of Positivstellensatz calculus) that $\lfloor n/2 \rfloor$ iterations of the Lasserre procedure are needed to prove that the polytope $\{x \in [0,1]^n \mid \sum_{i=1}^n x_i = \lfloor n/2 \rfloor + \frac{1}{2}\}$ contains no integer point (see also Grigoriev et al. 2001). The same holds for the (semidefinite) Lovász-Schrijver procedure, thus improving the lower bound $n/4$ given in Grigoriev et al. (2001, Theorem 7.1). (This follows from the fact, shown in Laurent 2003a, that the tth iterate in the semidefinite Lovász-Schrijver hierarchy contains the tth iterate in the Lasserre hierarchy for any integer $t \geq 1$.)

When $P = \mathrm{CUT}(G)$ is the cut polytope of a graph G, let $Q_t(G)$ (resp., $N^t(G)$, $N_+^t(G)$) denote the hierarchy of relaxations for $\mathrm{CUT}(G)$ obtained using the Lasserre construction (resp., using the N and N_+ operators in the Lovász-Schrijver construction). The sets $N^t(G)$ are *linear* relaxations while $N_+^t(G)$ and $Q_t(G)$ are *semidefinite* relaxations of $\mathrm{CUT}(G)$; these hierarchies of relaxations are studied in detail in Laurent (2001, 2003b). The sets $Q_t(G)$ will be defined in §1.2, but we do not need here a precise definition of the sets $N^t(G)$ and $N_+^t(G)$; let us simply mention that they are obtained by applying the Lovász-Schrijver N and N_+ operators to the relaxation of the cut polytope defined by the triangle inequalities. It is shown in Laurent (2003b) that

$$Q_{t+2}(G) \subseteq N_+^t(G) \subseteq N^t(G) \tag{1}$$

(for $t \geq 0$) and in Laurent (2001) that $\mathrm{CUT}(G) = N^t(G)$ if G has t edges whose contraction produces a graph with no K_5-minor. Define the parameter $\rho_L(G)$ as the smallest integer t for which $\mathrm{CUT}(G) = Q_t(G)$, called the *rank* of the graph G with respect to the Lasserre procedure; analogously, define $\rho_N(G)$ (resp., $\rho_{N_+}(G)$) as the smallest t for which $\mathrm{CUT}(G) = N^t(G)$ (resp., $N_+^t(G)$). Thus,

$$\rho_L(G) - 2 \leq \rho_{N_+}(G) \leq \rho_N(G) \leq \max(0, n - \alpha(G) - 3),$$

and in particular, $\rho_N(K_n) \leq n-4$ for $n \geq 4$. It is conjectured in Laurent (2003a) that $\rho_N(K_n) = n-4$, i.e., that $n-4$ iterations are, in fact, needed for finding $\mathrm{CUT}(K_n)$ using the N operator. Equality $\rho_N(K_n) = n-4$ is known to hold for $n \leq 7$.

The main contribution of this paper is to show the lower bound

$$\rho_L(K_n) \geq \left\lceil \frac{n}{2} \right\rceil$$

for the rank of the complete graph K_n with respect to the Lasserre procedure. As a consequence, this implies the lower bound

$$\rho_N(K_n) \geq \rho_{N_+}(K_n) \geq \left\lceil \frac{n}{2} \right\rceil - 2$$

for the rank of the Lovász-Schrijver procedure. We conjecture that equality $\rho_L(K_n) = \lceil n/2 \rceil$ holds.

The paper is organized as follows. In §§1.2 and 1.3, we introduce the semidefinite relaxations $Q_t(G)$ for the cut polytope $\mathrm{CUT}(G)$ and sketch the method of proof utilized for showing the strict inclusion: $\mathrm{CUT}(K_n) \subset Q_t(K_n)$ when $t \leq \lceil n/2 \rceil - 1$ (which constitutes the main result of the paper). This can be reduced to establishing positive semidefiniteness of some matrix Y_n (indexed by all subsets of $\{1, \ldots, n\}$ of size $\leq \lceil n/2 \rceil - 1$) (see Theorem 6). Section 2 is devoted to proving that matrix Y_n is indeed positive semidefinite; the proof relies on algebraic tools like the Johnson association scheme and hypergeometric series.

Section 3 recalls the algebraic background underlying the Lasserre procedure, about representations of nonnegative polynomials as sums of squares of polynomials. It contains, in particular, an algebraic reformulation of our bound on the rank of the Lasserre procedure for the cut polytope. Finally, in the Appendix, we group a number of remarkable features displayed by the matrix Y_n, which plays a central role in the paper.

1.2. Semidefinite relaxations for the cut polytope. Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and let $t \geq 1$ be an integer; $\mathcal{U}_t(V)$ denotes the collection of subsets of V whose cardinality is $\leq t$ and has the same parity as t, and $\mathcal{E}(V)$ the collection of even subsets of V. Given a vector $y = (y_I)_{I \in \mathcal{E}(V), |I| \leq 2t}$, its *moment matrix of order t* is the matrix $M_t(y)$ indexed by $\mathcal{U}_t(V)$ with (I, J)th entry $y(I \Delta J)$.

It is useful to observe that $M_t(y)$ can alternatively be indexed by the set $\{I \Delta O \mid I \in \mathcal{U}_t(V)\}$, for any given set $O \subseteq V$. From this follows, in particular, that

$$M_n(y) = M_{n-1}(y).$$

Indeed, say n is even. Then $M_n(y)$ is indexed by all even subsets while $M_{n-1}(y)$ is indexed by all odd subsets. As the collection of odd subsets of V is equal to $\{I \Delta O \mid I \in \mathcal{E}(V)\}$ where O is any given odd set, $M_{n-1}(y)$ can be assumed to be indexed by all even sets, and therefore $M_{n-1}(y) = M_n(y)$.

Let $\mathcal{F}_t(n)$ denote the set of moment matrices of order t that are positive semidefinite and have an all-ones main diagonal, and let $Q_t(G)$ denote the projection of $\mathcal{F}_t(n)$ on the subspace $\mathbb{R}^E$ indexed by the edge set of G. That is,

$$\mathcal{F}_t(n) = \left\{ Y \succeq 0 \mid Y = M_t(y) \text{ for some } y = (y_I)_{I \in \mathcal{E}(V), |I| \leq 2t} \text{ with } y_\varnothing = 1 \right\},$$

and

$$Q_t(G) = \left\{ x \in \mathbb{R}^E \mid \exists\, y = (y_I)_{I \in \mathcal{E}(V), |I| \leq 2t} \quad \text{such that } y_\varnothing = \varnothing,\ y_{ij} = x_{ij}\ (ij \in E), \text{ and } M_t(y) \succeq 0 \right\}.$$

Then the set $Q_t(G)$ is a semidefinite relaxation of the cut polytope, i.e.,

$$\text{(2)} \qquad \mathrm{CUT}(G) \subseteq Q_t(G),$$

and

$$\text{(3)} \qquad \mathrm{CUT}(G) = Q_{n-1}(G).$$

The inclusion (2) follows from the fact that each cut $\delta(A)$ is equal to the projection on $\mathbb{R}^E$ of the moment matrix of the vector $y^A \in \mathbb{R}^{\mathcal{E}(V)}$ defined by

$$\text{(4)} \qquad y^A(I) := (-1)^{|I \cap A|} \qquad \text{for } I \in \mathcal{E}(V),$$

and from the fact that

$$\text{(5)} \qquad M_{n-1}(y^A) = y^A (y^A)^T$$

is therefore positive semidefinite. Equality (3) is proved by Lasserre (2001b) using results about moment sequences. The following elementary proof was given by Laurent (2003a, b).

LEMMA 1. *Given $y \in \mathbb{R}^{\mathcal{E}(V)}$, the eigenvectors of the matrix $M_{n-1}(y)$ are the 2^{n-1} distinct vectors y^A $(A \subseteq V)$ with respective eigenvalues $y^T y^A$.*

PROOF. Direct verification shows that each vector y^A is an eigenvector of $M_{n-1}(y)$ with corresponding eigenvalue $y^T y^A$. Moreover, two distinct vectors y^A and y^B are orthogonal, since $(y^A)^T y^B = \sum_{I\in\mathscr{E}(V)}(-1)^{|I\cap A|}(-1)^{|I\cap B|} = \sum_{I\in\mathscr{E}(V)}(-1)^{|I\cap(A\Delta B)|}$, which is equal to 0 if $A\Delta B \neq \varnothing, V$. Therefore, the 2^{n-1} distinct vectors y^A form a basis of eigenvectors for $M_{n-1}(y)$. □

Hence any matrix $Y = M_{n-1}(y) \in \mathscr{F}_{n-1}(n)$ can be written as

$$Y = \sum_{A\subseteq V\setminus\{n\}} \lambda_A y^A (y^A)^T,$$

where $\lambda_A := (1/(2^{n-1}))y^T y^A \geq 0$ and $\sum_A \lambda_A = 1$. Therefore, $\mathscr{F}_{n-1}(n)$ is a 2^{n-1}-dimensional simplex with the cut matrices $M_{n-1}(y^A) = y^A(y^A)^T$ as vertices and its projection $Q_{n-1}(G)$ on the edge subspace $\mathbb{R}^E$ is the cut polytope $\text{CUT}(G)$.

REMARK 2. We have introduced here only the Lasserre relaxations $Q_t(G)$ of the cut polytope that are considered in the paper. More generally, given a polytope $K \subseteq [0,1]^n$, the Lasserre construction produces a hierarchy of semidefinite relaxations $Q_t(K)$ of the polytope $P := \text{conv}(K \cap \{0,1\}^n)$, with the property that $P = Q_n(K)$ (Lasserre 2001b). In the case of max-cut, one can apply the Lasserre construction to the linear relaxation K of the cut polytope, defined by the triangle inequalities, or proceed as indicated above in this section and obtain the relaxations $Q_t(G)$; these two possibilities are referred to as the "edge model" and the "node model" in Laurent (2003a, 2003b) and they are described in detail there. We focus on the relaxations $Q_t(G)$ since they have a much simpler description.

The following properties of the parameter $\rho_L(G)$, defined as the smallest t for which $Q_t(G) = \text{CUT}(G)$, are shown in Laurent (2003b).

PROPOSITION 3. (i) *If G has an edge e whose contraction produces a graph G/e with $\rho_L(G/e) \leq t$, then $\rho_L(G) \leq t+1$.*
(ii) *The class of graphs G with $\rho_L(G) \leq t$ is closed under taking minors.*

In view of Proposition 3(ii) (by the results of Robertson and Seymour 1988), the class of graphs with bounded rank t can be characterized by a *finite* list of minimal forbidden minors. Such a list is known only for $t \leq 2$; namely,

$$\rho_L(G) \leq 1 \quad \Leftrightarrow \quad G \text{ has no } K_3\text{-minor};$$
$$\rho_L(G) \leq 2 \quad \Leftrightarrow \quad G \text{ has no } K_5\text{-minor}.$$

The exact value of the rank of K_n is known for $n \leq 7$:

$$\rho_L(K_2) = 1, \qquad \rho_L(K_3) = \rho_L(K_4) = 2, \qquad \rho_L(K_5) = \rho_L(K_6) = 3, \qquad \rho_L(K_7) = 4.$$

Therefore, by applying Proposition 3(i), we find that $\rho_L(K_n) \leq n-3$ for $n \geq 6$. Generally, we conjecture:

CONJECTURE 4. $\rho_L(K_n) = \lceil n/2 \rceil$ *for all* $n \geq 2$.

We show in this paper that $\lceil n/2 \rceil$ is a lower bound for the Lasserre rank of K_n.

THEOREM 5. $\rho_L(K_n) \geq \lceil n/2 \rceil$ *for all* $n \geq 2$.

Observe that, in view of Proposition 3, it suffices to show Theorem 5 for all odd values of n and Conjecture 4 for all even values of n.

1.3. Method of proof for Theorem 5. We indicate here the proof method for Theorem 5. As observed earlier we can assume that n is odd, $n \geq 3$. Our objective is to show that the inclusion $\mathrm{CUT}(K_n) \subseteq Q_{\lceil n/2\rceil-1}(K_n)$ is strict. For this, we consider the inequality:

$$\sum_{1\leq i<j\leq n} x_{ij} \geq \frac{1-n}{2}. \tag{6}$$

This inequality is valid for the cut polytope $\mathrm{CUT}(K_n)$, since $e^T xx^T e = (e^T x)^2 \geq 1$ for all $x \in \{\pm 1\}^n$ (with e denoting the all ones vector). Define

$$p_t^* := \min \sum_{1\leq i<j\leq n} x_{ij} \text{ s.t. } x \in Q_t(K_n) = \min \sum_{1\leq i<j\leq n} y_{ij} \text{ s.t. } M_t(y) \succeq 0,\ y_\varnothing = 1. \tag{7}$$

Then, for $1 \leq t \leq n$,

$$-\frac{n}{2} \leq p_t^* \leq \frac{1-n}{2}. \tag{8}$$

The lower bound $-n/2$ follows from the fact that $M_t(y) \succeq 0$ implies that $M_1(y) \succeq 0$ and, thus, $e^T M_1(y) e = n + 2\sum_{1\leq i<j\leq n} y_{ij} \geq 0$. The inequality (6) is valid for $N_+^{\lceil n/2\rceil-2}(K_n)$ (Laurent 2001, Proposition 5.1), and thus, for $Q_{\lceil n/2\rceil}(K_n)$ by (1). That is,

$$p_t^* = \frac{1-n}{2} \qquad \text{for } t \geq \left\lceil \frac{n}{2} \right\rceil.$$

We show here that

$$p_t^* = -\frac{n}{2} \qquad \text{for } t \leq \left\lceil \frac{n}{2} \right\rceil - 1. \tag{9}$$

For this we have to construct a positive semidefinite moment matrix $M_t(y)$ with $y_\varnothing = 1$ and $\sum_{ij} y_{ij} = -n/2$. By symmetry, we can assume that all y_{ij} are equal; that is, $y_{ij} = -1/(n-1)$ for all $1 \leq i < j \leq n$. Direct inspection tells us what the remaining coordinates of y should be. (Indeed, as $M_1(y)e = 0$ by the choice of y_{ij}, the vector $(1, \ldots, 1, 0, \ldots, 0)^T$—with exactly n ones—belongs to the kernel of $M_t(y)$. From this, one can deduce what the value of y_I should be for $|I| \leq t+2$, and so on.) Namely, define the scalars $a_0, a_2, \ldots, a_{n-1}$ by

$$a_0 := 1, \quad a_{2r+2} := -a_{2r} \frac{2r+1}{n-2r-1} \qquad \text{for } r = 0, 1, \ldots, \frac{n-3}{2}; \tag{10}$$

in other words,

$$a_{2r} = (-1)^r \prod_{\substack{i \text{ odd} \\ 1\leq i\leq 2r-1}} \frac{i}{n-i} = \left(-\frac{1}{4}\right)^r \frac{\binom{2r}{r}}{\binom{(n-1)/2}{r}}.$$

Moreover, define the vector $y \in \mathbb{R}^{\mathscr{E}(V)}$ by

$$y_I := a_{|I|} \qquad \text{for all even subsets } I \subseteq V. \tag{11}$$

Then Theorem 5 will follow from the following result, whose proof is given in the next section.

THEOREM 6. *The matrix $Y_n := M_{\lceil n/2\rceil-1}(y)$ is positive semidefinite, where y is given by* (10), (11).

2. Proof of Theorem 6. Set $k := \lceil n/2 \rceil - 1 = (n-1)/2$ and let $Y_n = M_k(y)$, where y is given by (10), (11). Thus, Y_n is indexed by the set $\mathcal{U}_k(V)$ and has entries $Y_n(I, J) = y(I\Delta J) = a_{|I\Delta J|}$. The order of Y_n is

$$N := \sum_{\substack{i \leq k \\ i \equiv k \bmod 2}} \binom{n}{i}. \tag{12}$$

We proceed as follows for proving that Y_n is positive semidefinite. First, we show that Y_n has a large kernel, namely, that

$$\dim\ker Y_n \geq D := \sum_{\substack{i \leq k \\ i \not\equiv k \bmod 2}} \binom{n}{i} \tag{13}$$

(see Corollary 9). Second, we identify a principal submatrix Z of Y_n of order $N - D$ which is positive definite (see Proposition 11). By the interlacing property of eigenvalues (see, e.g., Horn and Johnson 1990, Theorem 4.3.15), this implies that Y_n has at least $N - D$ positive eigenvalues. Therefore, Y_n has exactly D zero eigenvalues and $N - D$ positive eigenvalues which shows that $Y_n \succeq 0$, thus proving Theorem 6.

2.1. Identifying a large kernel of Y_n. For $H \subseteq V$, define the vector $z^H \in \{0, 1\}^{\mathcal{U}_k(V)}$ by

$$z^H(I) := 1 \qquad \text{if and only if } |I\Delta H| = 1, \text{ i.e., } \quad \text{if } I = H\Delta\{i\} \text{ for some } i \in V,$$

and set $\mathcal{U}'_k(V) := \{I \subseteq V \mid |I| \leq k \text{ and } |I| \not\equiv k \bmod 2\}$.

LEMMA 7. *We have* $Y_n z^H = 0$ *for all* $H \in \mathcal{U}'_k(V)$.

PROOF. For $H \in \mathcal{U}'_k(V)$ and $I \in \mathcal{U}_k(V)$, we have

$$Y_n z^H(I) = \sum_{J \in \mathcal{U}_k(V)} a_{|I\Delta J|} z^H(J) = \sum_{i=1}^{n} a_{|I\Delta H\Delta\{i\}|}.$$

As the cardinalities of the sets I and H have distinct parities, $|I\Delta H| =: 2r+1$ for some $0 \leq r < k$. Then,

$$Y_n z^H(I) = (2r+1)a_{2r} + (n-2r-1)a_{2r+2}$$

which is equal to 0 by the definition (10) of a. □

LEMMA 8. *The vectors* z^H $(H \in \mathcal{U}'_k(V))$ *are linearly independent.*

PROOF. For $T \subseteq V$, define the vector $e_T \in \{0, 1\}^{\mathcal{U}_k(V)}$ with entries $e_T(I) = 1$ if and only if $T \subseteq I$. Then we have

$$(e_T)^T(z^H) = \begin{cases} 0 & \text{if } |H \cap T| \leq |T| - 2, \\ 1 & \text{if } |H \cap T| = |T| - 1, \\ n - |T| & \text{if } H \supseteq T. \end{cases}$$

Consider a linear dependency $\sum_{H \in \mathcal{U}'_k(V)} \alpha_H z^H = 0$. We show that all α_H are equal to 0. For this, we first show by induction on $t = 0, 1, \ldots, k$ that

$$\sum_{H \in \mathcal{U}'_k(V) | H \supseteq T} \alpha_H = 0 \qquad \text{for any set } T \subseteq V \text{ with } |T| = t. \tag{14}$$

Taking the scalar product of both sides of $\sum_H \alpha_H z^H = 0$ with the vector $e_\varnothing = (1, \ldots, 1)^T$, we find that $\sum_H \alpha_H = 0$; that is, (14) holds for the base of induction $t = 0$. Suppose that

(14) holds for $t-1$; we show that it also holds for t. Given a set $T \subseteq V$ of size t, take the inner product of $\sum_H \alpha_H z^H = 0$ with the vector e_T. This yields

$$\sum_{H \mid |H \cap T| = |T|-1} \alpha_H + (n-t) \sum_{H \mid H \supseteq T} \alpha_H = 0. \tag{15}$$

We have

$$\sum_{H \mid |H \cap T| = t-1} \alpha_H = \sum_{i \in T} \sum_{H \mid H \cap T = T \setminus \{i\}} \alpha_H = \sum_{i \in T} \left(\sum_{H \mid H \supseteq T \setminus \{i\}} \alpha_H - \sum_{H \mid H \supseteq T} \alpha_H \right),$$

which is equal to $-t \sum_{H \mid H \supseteq T} \alpha_H$ since $\sum_{H \mid H \supseteq T \setminus \{i\}} \alpha_H = 0$ by the induction assumption. Combining with (15), this implies that $0 = (n-2t) \sum_{H \mid H \supseteq T} \alpha_H$, and thus, (14) holds.

One can now easily derive that $\alpha_H = 0$ for all $H \in \mathcal{U}'_k(V)$ by induction on $|H| \le k-1$. □

COROLLARY 9. *The dimension of the kernel of Y_n is at least the value D given in* (13).

2.2. Identifying a large positive definite principal submatrix of Y_n. Let Z denote the principal submatrix of Y_n indexed by the collection of all subsets $I \subseteq \{1, \dots, n-1\}$ with cardinality $|I| = k$.

LEMMA 10. *The order of Z is equal to $N - D$, where N and D are given by* (12) *and* (13), *respectively.*

PROOF. The order of Z being equal to $\binom{n-1}{k}$, we have to show the identity

$$\sum_{\substack{i \le k \\ i \equiv k \bmod 2}} \binom{n}{i} = \binom{n-1}{k} + \sum_{\substack{i \le k \\ i \not\equiv k \bmod 2}} \binom{n}{i}. \tag{16}$$

The collection of sets $I \subseteq V$ with $|I| \le k$ and $|I| \equiv k \bmod 2$ can be partitioned into the following two classes:

- The sets $I \subseteq V \setminus \{n\}$ with $|I| \le k$ and $|I| \equiv k \bmod 2$.
- The sets $I = J \cup \{n\}$ for $J \subseteq V \setminus \{n\}$ with $|J| \le k-1$ and $|J| \not\equiv k \bmod 2$.

Therefore, the left-hand side of (16) is equal to

$$\sum_{\substack{i \le k \\ i \equiv k \bmod 2}} \binom{n-1}{i} + \sum_{\substack{i \le k-1 \\ i \not\equiv k \bmod 2}} \binom{n-1}{i},$$

which is equal to $\binom{n-1}{k} + \sum_{i \le k-1} \binom{n-1}{i}$. Using the identity: $\binom{n-1}{i} + \binom{n-1}{i+1} = \binom{n}{i+1}$, we can now conclude that

$$\sum_{i \le k-1} \binom{n-1}{i} = \sum_{\substack{j \le k-1 \\ j \not\equiv k \bmod 2}} \binom{n}{j}$$

and, thus, (16) holds. □

It remains now to prove that the matrix Z is positive definite. For this, we use the fact that Z belongs to the Bose-Mesner algebra of the Johnson scheme $J(n-1, k)$, which implies that its eigenvalues can be explicitly expressed in terms of the Eberlein polynomials.

We briefly recall some definitions and refer to Bannai and Ito (1984) or van Lint and Wilson (1992) for details about the Johnson scheme. Given integers k, v with $k \le v/2$, the Johnson scheme $J(v, k)$ is defined on the set $\mathscr{X}$ of all k-subsets of $\{1, \dots, v\}$. Its adjacency matrices are the 0/1 matrices $A_0, A_1, \dots, A_k$ indexed by $\mathscr{X}$ with the (I, J)th entry of A_i being equal to 1 if and only if $|I \Delta J| = 2i$; thus $A_0 = I$ and $A_0 + \cdots + A_k = J$. A fundamental property is that the linear span of $A_0, \dots, A_k$ is closed under matrix multiplication and forms a commutative algebra, called the *Bose-Mesner algebra* of the scheme. Another fundamental property is the existence of a common orthonormal basis of eigenvectors for all matrices in the Bose-Mesner algebra. For $l = 0, 1, \ldots, k$, the distinct eigenvalues of A_l are given by the Eberlein polynomial,

(17) $$P_l(u) = \sum_{j=0}^{l} (-1)^j \binom{u}{j}\binom{k-u}{l-j}\binom{v-k-u}{l-j},$$

for $u = 0, 1, \ldots, k$. Therefore, the distinct eigenvalues of a matrix $X := \sum_l x_l A_l$ in the Bose-Mesner algebra of $J(v, k)$ are the quantities $\sum_l x_l P_l(u)$ for $u = 0, \ldots, k$.

Here and below, for two integers $m, n \geq 0$, $\binom{m}{n}$ is the binomial coefficient defined by

$$\binom{m}{n} = \begin{cases} m!/n!(m-n)! & \text{if } n \leq m, \\ 0 & \text{if } n > m. \end{cases}$$

In our case, we have $v = n - 1 = 2k$; the matrix Z has the form $\sum_{l=0}^{k} a_{2l} A_l$, and therefore, belongs to the Bose-Mesner algebra of the Johnson scheme $J(n-1, k)$. Hence, the distinct eigenvalues of Z are

(18) $$\lambda_u := \sum_{l=0}^{k} a_{2l} P_l(u) = \sum_{l=0}^{k} a_{2l} \sum_{j=0}^{l} (-1)^j \binom{u}{j}\binom{k-u}{l-j}^2$$

for $u = 0, \ldots, k$.

PROPOSITION 11. *We have $\lambda_u > 0$ for all $u = 0, \ldots, k$; that is, the matrix Z is positive definite.*

The rest of the section gives the proof of Proposition 11. In order to evaluate λ_u, we use the tool of hypergeometric series. A detailed account on hypergeometric series and on how they can be used for explicitly computing rational series is given in the book by Petkovsek et al. (1996). We thank Dima Pasechnik for bringing this book to our attention and a result in Grigoriev et al. (2001, §8) about the degree of Positivstellensatz calculus refutations for knapsack, whose proof technique uses hypergeometric series and has inspired our proof below.

Let us briefly recall some facts we need. A *hypergeometric series* is a series $\sum_{i \geq 0} \alpha_i$ where $\alpha_0 = 1$ and the ratio $(\alpha_{i+1})/\alpha_i$ is the quotient of two polynomials in i, say

$$\frac{\alpha_{i+1}}{\alpha_i} = \frac{(i+a_1)\cdots(i+a_p)}{(i+b_1)\cdots(i+b_q)} \frac{x}{i+1},$$

where $x, a_1, \ldots, a_p, b_1, \ldots, b_q$ are complex numbers such that $b_1, \ldots, b_q$ are not nonpositive integers. Equivalently, the series $\sum_{i \geq 0} \alpha_i$ has the form

(19) $$\sum_{i \geq 0} \frac{(a_1)_i \cdots (a_p)_i}{(b_1)_i \cdots (b_q)_i} \frac{x^i}{i!},$$

which is commonly abbreviated as

$${}_pF_q\left[\begin{matrix} a_1 \cdots a_p \\ b_1 \cdots b_q \end{matrix}; x\right].$$

Here, $(a)_i$ denotes the *rising factorial function* defined by

$$(a)_i := \begin{cases} a(a+1)\cdots(a+i-1) & \text{if } i \geq 1, \\ 1 & \text{if } i = 0. \end{cases}$$

The monograph by Petkovsek et al. (1996) contains a number of identities permitting us to compute explicitly certain hypergeometric series; in particular, the *Gauss identity*:

$$ {}_2F_1\left[\begin{matrix} a & b \\ c & \end{matrix} ; 1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \tag{20} $$

if b is a nonpositive integer or if $c-a-b$ has a positive real part. Here $\Gamma(\cdot)$ is the Gamma function which is defined on the complex plane except on the nonpositive integers. Recall that

$$ \Gamma(n+1) = n! = (1)_n, \qquad \frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n \tag{21} $$

for an integer $n \geq 0$.

We now proceed to compute the value of λ_u in (18). Let us first rewrite λ_u by changing the summation in the following way. Set $i = l - j$. Then,

$$ \lambda_u = \sum_{i=0}^{k-u} \binom{k-u}{i}^2 \left[\sum_{j=0}^{k-i} a_{2i+2j} (-1)^j \binom{u}{j} \right]. $$

In a first step, we compute the inner sum

$$ s_i := \sum_{j=0}^{k-i} a_{2i+2j} (-1)^j \binom{u}{j} =: \sum_{j \geq 0} \alpha_j. $$

We have that $\alpha_0 = a_{2i}$ and

$$ \frac{\alpha_{j+1}}{\alpha_j} = \frac{(j+i+1/2)}{(j+i-k)} \frac{(j-u)}{(j+1)}. $$

Therefore,

$$ s_i = a_{2i} \cdot {}_2F_1\left[\begin{matrix} -u & i+1/2 \\ i-k & \end{matrix} ; 1\right]. $$

(This hypergeometric series is well defined, although $i-k$ is a negative integer; indeed, $(-u)_h = 0$ for all $h \geq u+1$ and, thus, the summation obtained by expanding the above hypergeometric series as in (19) can be restricted to $0 \leq h \leq u$, in which case $(i-k)_h \neq 0$.) Using Gauss formula (20) and relation (21), we find that

$$ {}_2F_1\left[\begin{matrix} -u & i+1/2 \\ i-k & \end{matrix} ; 1\right] = \frac{\Gamma(u-k-1/2)\Gamma(i-k)}{\Gamma(-k-1/2)\Gamma(i-k+u)} = \frac{(-k-1/2)_u}{(i-k)_u}. $$

Therefore,

$$ \lambda_u = \sum_{i=0}^{k-u} a_{2i} \binom{k-u}{i}^2 \frac{(-k-1/2)_u}{(i-k)_u} =: \sum_{i \geq 0} \beta_i. $$

We compute this sum again using hypergeometric series. We have that the starting term is $\beta_0 = (-k-1/2)_u/(-k)_u$ and

$$ \frac{\beta_{i+1}}{\beta_i} = \frac{i+1/2}{i-k} \left(\frac{i+u-k}{i+1}\right)^2 \frac{i-k}{i-k+u} = \frac{(i+1/2)(i+u-k)}{(i+1)^2}. $$

Therefore,

$$ \lambda_u = \beta_0 \cdot {}_2F_1\left[\begin{matrix} 1/2 & u-k \\ 1 & \end{matrix} ; 1\right]. $$

Using Gauss formula (20) and (21) again, we find that λ_u is equal to

$$\beta_0 \frac{\Gamma(k-u+1/2)\Gamma(1)}{\Gamma(1/2)\Gamma(k-u+1)} = \frac{(-k-1/2)_u}{(-k)_u} \frac{(1/2)_{k-u}}{(k-u)!} = \frac{1 \cdot 3 \cdots (2k+1)}{2^k \cdot k! \cdot (2k-2u+1)}.$$

Therefore, $\lambda_u > 0$ for all $u = 0, \ldots, k$. This concludes the proof of Proposition 11 and, thus, of Theorem 6.

3. Algebraic interpretation. The Lasserre hierarchy of semidefinite relaxations was originally motivated by results about moment sequences and the dual theory of nonnegative polynomials and their representation as sums of squares of polynomials. It applies to general polynomial programming problems of the form

$$(22) \qquad p^* := \min p_0(x) \qquad \text{subject to } x \in K := \{x \in \mathbb{R}^n \mid p_l(x) \geq 0 \ \forall l = 1, \ldots, m\},$$

where $p_0, p_1, \ldots, p_m$ are polynomials in $x = (x_1, \ldots, x_n)$. Given $x \in \mathbb{R}^n$, define the vector $y := (x^\alpha)_{\alpha \in \mathbb{Z}^n_+}$ whose components are the monomials $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and, given an integer $t \geq 1$, define $M_t^{\mathbb{Z}}(y)$ as the matrix indexed by the sequences $\alpha \in \mathbb{Z}^n_+$ with $\sum_i \alpha_i \leq t$ and whose (α, β)-entry is equal to $y_{\alpha+\beta}$. Observe that the matrix $M_t^{\mathbb{Z}}(y)$ is positive semidefinite as well as each matrix $M_t^{\mathbb{Z}}(p_l * y)$, setting $p_l * y := (\sum_\beta (p_l)_\beta y_{\alpha+\beta})_\alpha$. This observation leads to defining the following semidefinite relaxations of problem (22):

$$(23) \qquad \begin{aligned} p_t^* := \min \quad & \sum_\alpha (p_0)_\alpha y_\alpha \\ \text{s.t.} \quad & M_t^{\mathbb{Z}}(y) \succeq 0, \\ & M_{t-v_l}^{\mathbb{Z}}(p_l * y) \succeq 0 \qquad (l = 1, \ldots, m), \\ & y_0 = 1, \end{aligned}$$

for any $t \geq \max(v_l)$, where $v_l := \lceil \deg(p_l)/2 \rceil$. The dual semidefinite program to (23) takes the form:

$$(24) \qquad \begin{aligned} \rho_t^* := \max \quad & \rho \\ \text{s.t.} \quad & p_0(x) - \rho = q_0(x) + \sum_l q_l(x) p_l(x), \\ & \text{where } q_0, q_l \text{ are sums of squares of polynomials and} \\ & \deg(q_0) \leq 2t, \ \deg(q_l) \leq 2(t - v_l). \end{aligned}$$

(See Lasserre 2001a, 2002 for details.) We have $\rho_t^* \leq p_t^* \leq p^*$ by weak semidefinite duality. Lasserre (2001a) shows asymptotic convergence of the bound ρ_t^* to p^* as t goes to infinity. His proof is based on a result of Putinar (1993) asserting that, if K is compact and satisfies some additional technical condition, then any polynomial *positive* on K has a decomposition of the form $q_0(x) + \sum_l q_l(x) p_l(x)$, where q_0, q_l are sums of squares of polynomials. Putinar's (1993) result holds, in particular, in the ± 1 case when the polynomials $x_i^2 = 1$ are present in the description of K. Moreover, in this case, there is finite convergence in n steps. This was proved by Lasserre (2001b) using a result about rank extensions of moment matrices from Curto and Fialkow (2000); this can also be proved in an elementary way using Lemma 1 (see Laurent 2003a). Indeed, the presence of the constraints $g_i(x) := x_i^2 - 1 = 0$ in the description of K and, thus, of the constraints $M_{t-1}^{\mathbb{Z}}(g_i * y) = 0$ in the program (23), enables us to replace the moment matrix $M_t^{\mathbb{Z}}(y)$ (indexed by integer sequences) by the "combinatorial" moment matrix $M_t(y)$ (indexed by subsets of V) considered earlier in this paper.

In the case of max-cut considered in this paper, we have the problem of finding the minimum value p^* of a quadratic polynomial $p_0(x) := \sum_{1 \leq i < j \leq n} c_{ij} x_i x_j$ over the set

$K := \{x \in \mathbb{R}^n \mid x_i^2 - 1 = 0 \text{ for } i = 1, \ldots, n\}$. Thus, the semidefinite program (23) gives the lower bound,

$$p_t^* := \min \sum_{1 \le i < j \le n} c_{ij} y_{ij} \qquad \text{subject to } M_t(y) \succeq 0, \ y_\varnothing = 1$$

(as in (7)), and its dual (24) gives the bound ρ_t^* defined as the maximum value of ρ for which the polynomial $\sum_{1 \le i < j \le n} c_{ij} x_i x_j - \rho$ has a decomposition

$$q_0(x) + \sum_{i=1}^{n} q_i(x)(x_i^2 - 1), \tag{25}$$

where q_0 is a sum of squares of polynomials, $\deg(q_0) \le 2t$, and $\deg(q_i) \le 2(t-1)$ for $i = 1, \ldots, n$. The latter condition about the degree of $q_1, \ldots, q_n$ can, in fact, be omitted, since it follows from the degree bound on q_0. (Indeed, the polynomial $p(x) := \sum c_{ij} x_i x_j - \rho - q_0(x)$ belongs to the ideal I generated by the polynomials $g_i(x) := x_i^2 - 1$ $(i = 1, \ldots, n)$, which form a Groebner basis of I. Thus, $p = \sum_{i=1}^n q_i g_i$ with $\deg(q_i g_i) \le \deg(p)$ if $q_i \ne 0$.)

In fact, there is no duality gap, i.e., $\rho_t^* = p_t^*$ (since the Slater condition holds). As

$$\sum_{1 \le i < j \le n} x_i x_j + \frac{n}{2} = \frac{1}{2}\left(\sum_{i=1}^n x_i\right)^2 + \frac{1}{2}\sum_{i=1}^n (1 - x_i^2),$$

we find that the value $\rho = -n/2$ is dual feasible, and thus, $p_t^* \ge -n/2$ for any $t \ge 1$ (as in (8)).

The result from Relation (9) (or equivalently from Theorem 6) shown earlier can be reformulated in the following way: If the polynomial $\sum_{1 \le i < j \le n} x_i x_j - \rho$ has a decomposition (25), where q_0 is a sum of squares and $\deg(q_0) \le 2(\lceil n/2 \rceil - 1)$, then $\rho \le -n/2$.

Similarly, Conjecture 4 amounts to proving the following result: Assume that the inequality $\sum_{1 \le i < j \le n} c_{ij} x_{ij} \ge p^*$ is valid for $\mathrm{CUT}(K_n)$, i.e., that $p(x) := \sum_{1 \le i < j \le n} c_{ij} x_i x_j - p^* \ge 0$ for all $x \in \{\pm 1\}^n$. Then, the polynomial p has a decomposition (25), where q_0 is a sum of squares and $\deg(q_0) \le 2\lceil n/2 \rceil$. This conjecture is known to be true for $n \le 7$. For $n \ge 7$, one only knows that such a decomposition exists having $\deg(q_0) \le 2(n-3)$ (since $\rho_L(K_n) \le n - 3$).

Appendix: More on the moment matrix Y_n. The matrix $Y_n = M_{(n-1)/2}(y)$ (for n odd), whose positive semidefiniteness was proven in Theorem 6, seems to have quite remarkable structural properties. Let us mention a few properties, observed in some earlier attempts for proving Theorem 6.

About the spectrum of Y_n. We have computed the eigenvalues of Y_n for odd $n = 3, \ldots, 11$. The distinct eigenvalues are displayed in Figure 1. All eigenvalues have multiplicity greater than 1, except the eigenvalues λ_3, λ_8 marked with a star (*) which have multiplicity

$n=3$	$n=5$	$n=7$	$n=9$	$n=11$
0	0	0	0	0
$\lambda_1 := \frac{3}{2}$	$\lambda_2 := \frac{5}{4}\lambda_1$	$\lambda_4 := \frac{7}{6}\lambda_2$	$\lambda_6 := \frac{9}{8}\lambda_4$	$\lambda_9 := \frac{11}{10}\lambda_6$
	$\lambda_3 := \frac{13}{8}$ (*)	$\lambda_5 := \frac{7}{6}\lambda_3$	$\lambda_7 := \frac{9}{8}\lambda_5$	$\lambda_{10} := \frac{11}{10}\lambda_7$
			$\lambda_8 := \frac{263}{128}$ (*)	$\lambda_{11} := \frac{11}{10}\lambda_8$

FIGURE 1. Distinct eigenvalues of Y_n.

1. Interestingly, the distinct eigenvalues of Y_n for $(n-1)/2$ odd appear to be of the form $(n/(n-1))\lambda$, where the λs are the distinct eigenvalues of Y_{n-2}. The same holds when $(n-1)/2$ is even except that a new eigenvalue with multiplicity one appears (namely, λ_3 for $n=5$ and λ_8 for $n=9$). Although we could not prove it, it seems quite likely that this behaviour of the spectrum of Y_n holds for any odd n. If true, this could lead to an alternative proof of positive semidefiniteness of Y_n.

A tentative iterative proof. Matrix Y_n has a natural block decomposition into the blocks Y_{rs} $(r, s \leq k,\ r, s \equiv k \bmod 2)$, where Y_{rs} is the submatrix of Y_n whose rows are indexed by all r-subsets of V and columns by all s-subsets.

Let $V_0^{(r)} \cup V_1^{(r)} \cup \cdots \cup V_r^{(r)}$ denote the orthogonal decomposition of the space $\mathbb{R}^{\mathscr{X}}$ $(\mathscr{X} = \binom{n}{r})$ corresponding to the distinct eigenvalues of matrices in the Bose-Mesner algebra of the Johnson scheme $J(n, r)$. Then the orthogonal projections $E_i^{(r)}$ of $\mathbb{R}^{\mathscr{X}}$ onto $V_i^{(r)}$ form another basis of the Bose-Mesner algebra of $J(n, r)$.

Suppose (to fix ideas) that k is odd. Then, Y_n has the form

$$Y_n = \begin{pmatrix} Y_{11} & Y_{13} & Y_{15} & \cdots \\ Y_{31} & Y_{33} & Y_{35} & \cdots \\ Y_{51} & Y_{53} & Y_{55} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The upper left corner Y_{11} of Y_n belongs to the Bose-Mesner algebra of $J(n, 1)$. Moreover,

$$Y_{11} = \frac{n}{n-1} E_1^{(1)}, \qquad Y_{i1} E_0^{(1)} = 0 \qquad \text{for odd } i \geq 3. \tag{26}$$

Therefore, $Y_n \succeq 0$ if and only if the matrix

$$Y_n^{(1)} := \begin{pmatrix} Y_{33} & Y_{35} & \cdots \\ Y_{53} & Y_{55} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} - \frac{n-1}{n} \begin{pmatrix} Y_{31} \\ Y_{51} \\ \vdots \end{pmatrix} \begin{pmatrix} Y_{31} \\ Y_{51} \\ \vdots \end{pmatrix}^T =: \begin{pmatrix} Y_{33}^{(1)} & Y_{35}^{(1)} & \cdots \\ Y_{53}^{(1)} & Y_{55}^{(1)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is positive semidefinite. We verified that the upper left corner $Y_{33}^{(1)}$ of $Y_n^{(1)}$ belongs to the Bose-Mesner algebra of $J(n, 3)$ and

$$Y_{33}^{(1)} = \beta E_3^{(3)}, \qquad Y_{i3}^{(1)} E_u^{(3)} = 0 \qquad \text{for odd } i \geq 5, u = 0, 1, 2 \tag{27}$$

(with $\beta := n(n-2)(n-4)/(n-1)(n-3)(n-5)$). Analogously, $Y_n^{(1)} \succeq 0$ if and only if the matrix

$$Y_n^{(2)} := \begin{pmatrix} Y_{55}^{(1)} & Y_{57}^{(1)} & \cdots \\ Y_{75}^{(1)} & Y_{77}^{(1)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} - \frac{1}{\beta} \begin{pmatrix} Y_{53}^{(1)} \\ Y_{73}^{(1)} \\ \vdots \end{pmatrix} \begin{pmatrix} Y_{53}^{(1)} \\ Y_{73}^{(1)} \\ \vdots \end{pmatrix}^T$$

is positive semidefinite. Again the upper left corner of $Y_n^{(2)}$ is a positive multiple of the last idempotent $E_5^{(5)}$, etc. Quite probably, this reasoning can be carried out throughout, thus "peeling off" one block layer of Y_n at each iteration until finding $Y_n^{(n-3)/4}$ which should be proven to be a positive multiple of $E_k^{(k)}$. If true, this would show that Y_n is positive semidefinite. We did not succeed in carrying out this type of proof. Checking (27) and its further analogues indeed becomes technically very complicated, a difficulty being that although the entries in all the blocks of the initial matrix Y_n are governed by a single sequence a, at a later iteration $t \geq 1$, the entries in each of the blocks of $Y_n^{(t)}$ are governed by distinct sequences.

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