MATHEMATICS OF OPERATIONS RESEARCH
Vol. 17, No. 4, November 1992
Printed in U.S.A.

CLIQUE-WEB FACETS FOR MULTICUT POLYTOPES*

M. DEZA, M. GRÖTSCHEL AND M. LAURENT

Let $G = (V, E)$ be a graph. An edge set $\{uv \in E | u \in S_i, v \in S_j, i \neq j\}$, where $S_1, \ldots, S_k$ is a partition of V, is called a multicut with k shores. We investigate the polytopes $\mathrm{MC}_k^{\leq}(n)$ and $\mathrm{MC}_k^{\geq}(n)$ that are defined as the convex hulls of the incidence vectors of all multicuts with at most k shores and at least k shores, respectively, of the complete graph K_n. We introduce a large class of inequalities, called clique-web inequalities, valid for these polytopes, and provide a quite complete characterization of those clique-web inequalities that define facets of $\mathrm{MC}_k^{\leq}(n)$ and $\mathrm{MC}_k^{\geq}(n)$. Using general facet manipulation techniques like collapsing and node splitting we construct further new classes of facets for these multicut polytopes. We also exhibit a class of clique-web facets for which the separation problem can be solved in polynomial time.

Introduction and notation. We denote graphs by $G = (V, E)$; V is the node set and E the edge set of G. An edge between nodes i and j is denoted by ij. If this notation leads to ambiguities we write $\{i, j\}$ instead. All graphs we consider are undirected and have neither loops nor multiple edges. The complete graph on n nodes is denoted by K_n. The set $[1, n] := \{1, 2, \ldots, n\}$ will usually be considered as the node set of K_n. The edge set of K_n will be denoted by E_n. An **interval** in the set $[1, n]$ is a subset of $[1, n]$ of the form $\{i, i+1, \ldots, j\}$ if $i \leq j$ or of the form $\{i, i+1, \ldots, n, 1, \ldots, j\}$ if $j < i$.

A **partition** of a set S is a system $S_1, \ldots, S_k$ of subsets of S such that $S_i \neq \emptyset$ $(i = 1, \ldots, k)$, $S_i \cap S_j = \emptyset$ $(1 \leq i < j \leq k)$, and $S = \bigcup_{i=1}^k S_i$. If S is a set and $b_i \in \mathbb{R}$ are weights for all $i \in S$ then, for any $T \subseteq S$, $b(T)$ denotes the sum $\sum_{i \in T} b_i$.

If $G = (V, E)$ is a graph and $S_1, \ldots, S_k$ a partition of V then $\delta(S_1, \ldots, S_k) := \{uv \in E | \; \exists i \neq j \text{ with } u \in S_i, \; v \in S_j\}$ is called the **k-cut** of G associated with $S_1, \ldots, S_k$. The sets $S_1, \ldots, S_k$ are called the **shores** of the k-cut. If we do not want to specify the number k of the shores we will simply speak of a **multicut**.

To make notation easier we will sometimes speak of a **$^{\leq}k$-cut** and a **$^{\geq}k$-cut** if $\delta(S_1, \ldots, S_h)$ is an h-cut with $h \leq k$ and $h \geq k$, respectively. The (standard) cuts usually considered in graph theory are our $^{\leq}2$-cuts. The symbol most frequently used to denote these cuts is $\delta(S)$, i.e., $\delta(S) = \{ij \in E | i \in S, j \notin S\}$. For $S \neq V$, we will use either $\delta(S)$ or $\delta(S, V \setminus S)$ to denote standard cuts depending on which notation is more convenient.

In the remainder of the paper we will only study multicuts of the complete graph K_n. Thus we will frequently drop reference to the graph with respect to which a multicut is considered.

Let $\mathbb{R}^{\binom{n}{2}}$ denote the vector space where each of the $n(n-1)/2$ components is indexed by an edge of the complete graph K_n. The incidence vector of a multicut $\delta(S_1, \ldots, S_k)$ is the vector $\chi^{\delta(S_1, \ldots, S_k)} \in \mathbb{R}^{\binom{n}{2}}$ with $\chi_e^{\delta(S_1, \ldots, S_k)} = 1$ if $e \in \delta(S_1, \ldots, S_k)$

*Received March 26, 1990; revised May 21, 1991.
AMS 1980 subject classification. Primary: 68E10, 68C25.
IAOR 1973 subject classification. Main: Graphs. Cross references: Combinatorial analysis.
OR/MS Index 1978 subject classification. Primary: 481 Networks/Graphs. Secondary: 012 Computational Complexity.
Key words. Polyhedral combinatorics, polytopes, facets, cuts, multicuts.

0364-765X/92/1704/0981/$01.25
Copyright © 1992, The Institute of Management Sciences/Operations Research Society of America

and $\chi_e^{\delta(S_1,\dots,S_k)} = 0$ otherwise. Since this (standard) notation is so clumsy and inconvenient in our case we slightly change the symbol to denote the incidence vector and write

$$\chi(S_1,\dots,S_k) := \chi^{\delta(S_1,\dots,S_k)}.$$

To shorten notation we call $\chi(S_1,\dots,S_k)$ a **multicut** or **k-cut vector** or simply (abusing language) also a multicut. The incidence vector $\chi(S)$ of a 2-cut is called a **cut vector.**

There are a number of interesting combinatorial optimization problems associated with multicuts. Given weights c_e for all edges e of K_n, one can ask to find a maximum weight multicut or a maximum weight multicut with at most, at least, or exactly k shores. We introduce here the polytopes associated with these problems. Let "conv" denote the convex hull operator and let n and k be positive integers with $1 \leqslant k \leqslant n$. Set

$$\mathrm{MC}(n) := \mathrm{conv}\{\chi(S_1,\dots,S_h) | \delta(S_1,\dots,S_h) \text{ a multicut of } K_n\},$$

$$\mathrm{MC}_k^{\leqslant}(n) := \mathrm{conv}\{\chi(S_1,\dots,S_h) | \delta(S_1,\dots,S_h) \text{ a } {}^{\leqslant}k\text{-cut of } K_n\},$$

$$\mathrm{MC}_k^{\geqslant}(n) := \mathrm{conv}\{\chi(S_1,\dots,S_h) | \delta(S_1,\dots,S_h) \text{ a } {}^{\geqslant}k\text{-cut of } K_n\},$$

$$\mathrm{MC}_k^{=}(n) := \mathrm{conv}\{\chi(S_1,\dots,S_k) | \delta(S_1,\dots,S_k) \text{ a } k\text{-cut of } K_n\},$$

$$\mathrm{EMC}(n) := \mathrm{conv}\{\chi(S_1,\dots,S_h) | \delta(S_1,\dots,S_h) \text{ a multicut with } \\ \| S_i| - |S_j\| \leqslant 1,\ 1 \leqslant i < j \leqslant h\}.$$

We call MC(n) **the multicut polytope**, $\mathrm{MC}_k^{\leqslant}(n)$ the **${}^{\leqslant}k$-cut polytope**, $\mathrm{MC}_k^{\geqslant}(n)$ the **${}^{\geqslant}k$-cut polytope**, and EMC(n) the **equi multicut polytope** of K_n. If it is not necessary to be precise about the name we will simply speak of a multicut polytope $\mathrm{MC}_k^{\leqslant}(n)$, $\mathrm{MC}_k^{\geqslant}(n)$, etc. Moreover, to save parameters we will sometimes drop the "n" and write $\mathrm{MC}_k^{\leqslant}$ instead of $\mathrm{MC}_k^{\leqslant}(n)$, etc., if it is clear from the context what the underlying complete graph K_n is.

In this paper we will study the polytopes $\mathrm{MC}_k^{\geqslant}(n)$ and $\mathrm{MC}_k^{\leqslant}(n)$. (Note that the polytopes $\mathrm{MC}_1^{\leqslant}(n)$ and $\mathrm{MC}_n^{\geqslant}(n)$ are trivial. They consist of a single point and, thus, they will not be considered further.) The two extreme cases $\mathrm{MC}(n) = \mathrm{MC}_n^{\leqslant}(n) = \mathrm{MC}_1^{\geqslant}(n)$ and $\mathrm{MC}_2^{\leqslant}(n)$ have been investigated intensively before.

$\mathrm{MC}_2^{\leqslant}(n)$ is nothing but the standard cut polytope studied, e.g., in [BM], [DL1], [DL2], [DDL] and other papers mentioned in these references. MC(n) is the "complement" of the clique partitioning polytope. A **clique partitioning** is an edge set $E(S_1,\dots,S_k) := E_n \setminus \delta(S_1,\dots,S_k)$, where E_n is the edge set of the complete graph K_n, and the **clique partitioning polytope** $\mathscr{P}_n$ is nothing but $\mathscr{P}_n = \{\mathbf{1} - x | x \in \mathrm{MC}(n)\}$. ($\mathbf{1}$ denotes the vector all of whose components are 1.) The polyhedron $\mathscr{P}_n$ has been studied in [GW1], [GW2], and [W]. Due to the simple relationship between MC(n) and $\mathscr{P}_n$ it is easy to transform a result about $\mathscr{P}_n$ into a result about MC(n) and vice versa. Multicut polytopes (for general graphs) have been studied in [CR1] and the polytope $\mathrm{MC}_k^{\leqslant}(n)$ in [CR2]; in particular, see [CR1] for integer programming formulations of multicut polytopes.

The standard cut problem has many real world applications, see, for instance, [BGJR], and so does the clique partitioning problem, see [GW3]. Equicut problems come up in physics and VLSI-design, see [BGJR], and multicut problems with upper or lower bounds on the number of shores appear frequently, e.g., in cluster and quantitative data analysis. It is not the purpose of this paper to describe these applications and their (sometimes complicated) modeling. We provide theoretical results on the associated polytopes that will eventually form the basis of a cutting plane algorithm to solve these problems. The present paper is a follow-up work of the papers [DL2] and [GW1] that study $\mathrm{MC}_k^{\leqslant}(n)$ for $k = 2$ and $k = n$. We investigate here the "intermediate" cases.

We use the standard notation of polyhedral theory. We say that an inequality $a^T x \leqslant \alpha$ is **valid** for a polyhedron P if $P \subseteq \{x | a^T x \leqslant \alpha\}$; a valid inequality $a^T x \leqslant \alpha$ defines (or induces) a **facet** of P if the dimension of $\{x \in P | a^T x = \alpha\}$ is one less than the dimension of P. An inequality $a^T x \leqslant \alpha$ is called **pure** if all coefficients of a are elements of $\{0, +1, -1\}$. If $a^T x \leqslant \alpha$ is valid for some multicut polytope and if $\delta(S_1, \ldots, S_k)$ is a multicut such that its incidence vector satisfies this inequality with equality, then we call the multicut vector $\chi(S_1, \ldots, S_k)$ a **root** of $a^T x \leqslant \alpha$.

The main aim of our paper is to produce large classes of inequalities that define facets of $\mathrm{MC}_k^{\leqslant}(n)$ and $\mathrm{MC}_k^{\geqslant}(n)$.

Let us begin with some easy but important facts that we will use in the sequel.

If an inequality is valid for $\mathrm{MC}(n)$, then it is also valid for all polytopes $\mathrm{MC}_k^{\leqslant}(n)$, $\mathrm{MC}_k^{\geqslant}(n)$ and $\mathrm{MC}_k^{=}(n)$ since $\mathrm{MC}(n)$ contains all polytopes $\mathrm{MC}_k^{\leqslant}(n), \mathrm{MC}_k^{\geqslant}(n)$, $\mathrm{MC}_k^{=}(n)$.

Every multicut vector $\chi(S_1, \ldots, S_k)$, $k \geqslant 2$, can be written as a nonnegative linear combination of usual cut vectors, namely:

$$(*) \qquad \chi(S_1, \ldots, S_k) = \frac{1}{2} \sum_{i=1}^{k} \chi(S_i, [1, n] \setminus S_i) = \frac{1}{2} \sum_{i=1}^{k} \chi(S_i),$$

and, similarly, every incidence vector of a k-cut, $k \geqslant 3$, can be expressed as a nonnegative linear combination of incidence vectors of $(k-1)$-cuts

$$(**) \qquad \chi(S_1, \ldots, S_k) = \frac{1}{\binom{k}{2} - 1} \times \sum_{1 \leqslant i < j \leqslant k} \chi(S_i \cup S_j, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_{j-1}, S_{j+1}, \ldots, S_k),$$

and hence as a nonnegative linear combination of incidence vectors of h-cuts for all $2 \leqslant h < k$. Clearly, such an expression is not unique.

An immediate consequence of $(*)$ is that, for each fixed k, $2 \leqslant k \leqslant n$, the cone generated by all ${}^{\leqslant}k$-cuts coincides with the **cut cone** C_n (the cone generated by all ${}^{\leqslant}2$-cuts). In other words, all multicut polytopes $\mathrm{MC}_k^{\leqslant}(n)$ have the same set of homogeneous facets, i.e., facets of the form $a^T x \leqslant 0$, as the cut cone C_n. Similarly, $(**)$ implies that the cone generated by the k-cuts is the same as the cone generated by the ${}^{\geqslant}(k-1)$-cuts.

A remarkable property of the cut polytope $\mathrm{MC}_2^{\leqslant}(n)$ is that all its nonhomogeneous facets can be obtained from its homogeneous ones via the so-called switching operation [BM]. It would be nice to have such a tool for multicuts. However, we could not find a natural extension of this property for the case $k \geqslant 3$. Indeed, it is shown in [DGL] that the only symmetries of the multicut polytope $\mathrm{MC}_k^{\leqslant}(n)$, $3 \leqslant k \leqslant n$, are those induced by permutations of the n nodes.

From $(*)$ and $(**)$ also follows that if a k-cut vector $\chi(S_1, \ldots, S_k)$ is a root of a valid homogeneous inequality $a^T x \leqslant 0$ then the 2-cut $\chi(S_i, [1, n] \setminus S_i)$ and the $(k-1)$-cut vectors $\chi(S_i \cup S_j, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_j, S_{j+1}, \ldots, S_k)$ are also roots of $a^T x \leqslant 0$.

It is well known that the cut polytope $\mathrm{MC}_2^{\leqslant}(n)$ is full dimensional, see [BM]. Since $\mathrm{MC}_k^{\leqslant}(n) \supseteq \mathrm{MC}_2^{\leqslant}(n)$ for $2 \leqslant k \leqslant n$ we see that the dimension of $\mathrm{MC}_k^{\leqslant}(n)$ is also equal to $\binom{n}{2}$. The polytope $\mathrm{MC}_{n-1}^{\geqslant}(n)$ is nothing but the simplex in $\mathbb{R}^{\binom{n}{2}}$ generated by the $\binom{n}{2}$ vectors $\mathbf{1} - e_i$, where e_i is the ith unit vector, and by the n-cut vector $\mathbf{1} = \chi(\{1\}, \ldots, \{n\})$. Thus $\mathrm{MC}_{n-1}^{\geqslant}(n)$ is of full dimension. And since $\mathrm{MC}_k^{\geqslant}(n) \supseteq \mathrm{MC}_{n-1}^{\geqslant}(n)$, $1 \leqslant k \leqslant n-1$, the multicut polytopes $\mathrm{MC}_k^{\geqslant}(n)$, $1 \leqslant k \leqslant n-1$, are full dimensional as well.

After these preliminaries let us survey the contents of our paper. In the first section, we introduce clique-web inequalities (CW-inequalities for short); they are a generalization of those homogeneous clique-web inequalities introduced in [DL1] and proved to be facet-inducing for the cut polytope $\mathrm{MC}_2^{\leqslant}(n)$ in [DL2]. We prove that clique-web inequalities are valid for the multicut polytope $\mathrm{MC}(n)$ (cf., Proposition (1.5)) and, hence, for all polytopes $\mathrm{MC}_k^{\leqslant}(n), \mathrm{MC}_k^{\geqslant}(n)$ for $2 \leqslant k \leqslant n$. Then we study whether CW-inequalities induce facets of $\mathrm{MC}_k^{\leqslant}(n)$ and $\mathrm{MC}_k^{\geqslant}(n)$ and we group the results in our main Theorem (1.20). For example, concerning facethood for $\mathrm{MC}_k^{\leqslant}(n)$, there are only two values of k for which facethood is undecided. We also exhibit a class of CW-facets, the so-called odd wheel facets (cf. Proposition (1.23)), for which the separation problem can be solved in polynomial time.

In the second section we extend to multicut polytopes the operation of collapsing valid inequalities considered in [DL2], [DDL] for the cut polytope $\mathrm{MC}_2^{\leqslant}(n)$; it enables us to construct general (collapsed) clique-web inequalities. The inverse operation to collapsing is a special case of lifting. Using this special lifting procedure, we prove facethood for some classes of CW-inequalities.

1. Clique-web facets. Clique-web inequalities, introduced in [DL1], were proved to be facet-inducing for the cut cone C_n in [DL2]. Because of $(*)$ and since they are homogeneous, they are valid and thus they also induce facets of the multicut polytopes $\mathrm{MC}_k^{\leqslant}(n)$ for all $k \leqslant n$ and hence of $\mathrm{MC}(n)$. By relaxing conditions on their parameters, we construct a nonhomogeneous version of these inequalities and prove that they induce facets of $\mathrm{MC}(n)$ and $\mathrm{MC}_k^{\leqslant}(n), \mathrm{MC}_k^{\geqslant}(n)$ for suitable k.

1.1. *Clique-web inequalities.* Given nonnegative integers p, r such that $p \geqslant 2r+1$, let $([1,p], \mathrm{AW}_p^r)$ denote the **antiweb** with parameters p, r, i.e., the circular graph on node set $[1,p]$ with edges $\{i, i+1\}, \{i, i+2\}, \ldots, \{i, i+r\}$ for $i \in [1,p]$ (setting $p+1=1$). The complement of $([1,p], \mathrm{AW}_p^r)$ is the **web** $([1,p], W_p^r)$. To shorten notation we will simply use W_p^r or AW_p^r to denote a web or antiweb since the edge set implicitly defines the node set. It will be clear from the context whether we mean the graph or its edge set. Antiwebs are Cayley graphs on the additive group $\mathbf{Z}_p$.

(1.1) DEFINITION. Let $n, p, q \geqslant 1$, $r \geqslant 0$ be integers such that

$$(1.2)\qquad n = p + q, \quad p - q \geqslant 2r + 1.$$

The **clique-web inequality** (**CW-inequality**, for short) with parameters p, q, r satisfying (1.2) is the inequality

$$(1.3)\quad \mathrm{CW}_{p,q}^r \cdot x := \sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} - \sum_{ij \in \mathrm{AW}_p^r} x_{ij} \leqslant \frac{(p-q)(p-q-2r-1)}{2}$$

with $b = (1, \ldots, 1, -1, \ldots, -1)^T$, where the first p coefficients of b are $+1$ and the last q coefficients are -1.

To make technical arguments more transparent we shall use the following notation concerning CW-inequalities. In the sequel we denote by $[1, p]$ the set of nodes i with $b_i = +1$ (the "positive" nodes) and by $[1', q']$ (or $[p+1, n]$) the set of nodes with $b_i = -1$ (the "negative" nodes). Thus the coefficients of the edges of E_n in (1.3) are as follows. All edges of the web W_p^r on $[1, p]$ have value $+1$; the edges of its antiweb AW_p^r have value zero; the edges with one endnode in $[1, p]$ and one in $[p+1, n] = [1', q']$ have value -1, the edges in the clique $[p+1, n]$ have value $+1$.

(1.4) REMARK. (i) In case the parameters p, q, r of the CW-inequality (1.3) satisfy $p - q = 2r + 1$, then the right-hand side of inequality (1.3) is equal to 0 and inequality (1.3) coincides with the clique-web inequality $\mathrm{CW}_{p,q}^r \cdot x \leqslant 0$ proved to be facet-inducing for $\mathrm{MC}_2^{\leqslant}(n)$ in [DL2] for $q \geqslant 2$. In the following we shall therefore restrict ourselves to the case $p - q > 2r + 1$, i.e., to the nonhomogeneous CW-inequalities for multicut polytopes.

(ii) Inequality (1.3) for the case $r = 0$ coincides (after transformation $x \to \mathbf{1} - x$) with the $[S, T]$-inequality (with $S \cup T = [1, n]$, $|S| \neq |T|$) introduced in [GW1] and shown there to be facet-inducing for $\mathrm{MC}(n)$. Also, inequality (1.3) for the case $r = 0$ is a subcase of the generalized hypermetric inequality introduced in [CR2] and proved there to be facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$ for k satisfying $p - q + 1 \leqslant k \leqslant p$.

(iii) In case the parameters satisfy $p - q = kr + 1$ and $r \geqslant 1$ then inequality (1.3) coincides with the antiweb inequality introduced in [CR2] and proved there to be valid for $\mathrm{MC}_k^{\leqslant}(n)$ and facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$ if the additional condition $p \geqslant 2kr$ and $r \leqslant k - 2$ holds. □

To prove validity and characterize the roots of CW-inequalities for $\mathrm{MC}(n)$ we use the following two facts about homogeneous CW-inequalities for the cut polytope $\mathrm{MC}_2^{\leqslant}(n)$. As before, AW_p^r denotes the edge set of the antiweb on the node set $[1, p]$ with parameters p, r.

(1.5) PROPOSITION (See [A]). *Let S be a subset of $[1, p]$ of size s. The following assertions hold*:

(i) *If $s \leqslant r$, then $|\delta(S) \cap \mathrm{AW}_p^r| \geqslant s(2r + 1 - s)$.*

(ii) *If $r \leqslant s \leqslant p - r$ then $|\delta(S) \cap \mathrm{AW}_p^r| \geqslant r(r + 1)$.*

(1.6) PROPOSITION (See [DL2, Theorem 1.8]). *Given integers $p, q, r, n \geqslant 1$ with $p - q = 2r + 1$ and $n = p + q$, consider the CW-inequality (1.3) $\mathrm{CW}_{p,q}^r \cdot x \leqslant 0$ defined on the nodes $[1, p] \cup [1', q']$. The roots in $\mathrm{MC}_2^{\leqslant}(n)$ of this inequality are the cut vectors $\chi(S)$ for which the node set S is of one of the following two types*:

(R1) *S or $[1, n] \setminus S$ induces a clique of the antiweb AW_p^r (i.e., any two nodes are adjacent in AW_p^r).*

(R2) *$S = S^+ \cup S^-$ where S^+ is an interval of $[1, p]$ of size s^+ with $r + 1 \leqslant s^+ \leqslant p - r - 1$ and S^- is a subset of $[1', q']$ of size s^- with $s^+ - s^- \in \{r, r + 1\}$.*

Note that any subset of $[1, p]$ of type (R1) has size less than or equal to $r + 1$; also any subset of an interval of size $r + 1$ of $[1, p]$ is of type (R1).

It follows from (*) that for homogeneous CW-inequalities $\mathrm{CW}_{p,q}^r \cdot x \leqslant 0$ with $p - q = 2r + 1$, a multicut vector $\chi(S_1, \ldots, S_k)$ is a root if and only if, for all $i \in \{1, \ldots, k\}$, the cut vector $\chi(S_i)$ is a root of $\mathrm{CW}_{p,q}^r \cdot x \leqslant 0$, i.e., S_i is of type (R1) or of type (R2).

(1.7) PROPOSITION. *The clique-web inequality (1.3) is valid for the multicut polytope* $\mathrm{MC}(n)$.

(1.8) PROPOSITION. *Given integers $p, q, r \geq 1$ with $p - q \geq 2r + 2$, the multicut vector $\chi(S_1, \ldots, S_k)$ is a root of the CW-inequality* (1.3), *i.e., satisfies*

$$\mathrm{CW}_{p,q}^r \cdot \chi(S_1, \ldots, S_k) = \frac{(p-q)(p-q-2r-1)}{2},$$

if and only if, for all $i \in [1, k]$, one of the following assertions holds:

(R1) *S_i induces a clique of the antiweb AW_p^r.*

(R2) *$S_i = S_i^+ \cup S_i^-$ where S_i^+ is an interval of size s_i^+ of $[1, p]$ with $r + 1 \leq s_i^+ \leq p - r - 1$ and S_i^- is a subset of size s_i^- of $[1', q']$ with $s_i^+ - s_i^- \in \{r, r+1\}$.*

PROOF OF PROPOSITIONS (1.7) AND (1.8). The proof relies upon $(*)$, i.e., the fact that every multicut vector $\chi(S_1, \ldots, S_k)$ can be expressed as a linear combination of cut vectors, namely

$$(1.9) \qquad \chi(S_1, \ldots, S_k) = \frac{1}{2} \sum_{i=1}^{k} \chi(S_i).$$

Take a multicut vector $\chi(S_1, \ldots, S_k)$. Using (1.9) and $\sum_{i=1}^k b(S_i) = p - q$ we obtain

$$\begin{aligned} \mathrm{CW}_{p,q}^r \cdot \chi(S_1, \ldots, S_k) &= \frac{1}{2} \sum_{i=1}^{k} \mathrm{CW}_{p,q}^r \cdot \chi(S_i) \\ &= \frac{1}{2} \sum_{i=1}^{k} b(S_i)(p - q - b(S_i)) - \frac{1}{2} \sum_{i=1}^{k} |\delta(S_i) \cap \mathrm{AW}_p^r| \\ &= \frac{(p-q)^2}{2} - \frac{1}{2} \sum_{i=1}^{k} b(S_i)^2 - \frac{1}{2} \sum_{i=1}^{k} |\delta(S_i) \cap \mathrm{AW}_p^r|. \end{aligned}$$

Hence, proving validity of inequality (1.3) amounts to verifying that the following inequality holds:

$$\sum_{i=1}^{k} |\mathrm{AW}_p^r \cap \delta(S_i)| \geq \sum_{i=1}^{k} b(S_i)(2r + 1 - b(S_i)).$$

For this, it is enough to show that for all $i \in [1, k]$ the following inequality holds:

$$(1.10) \qquad |\mathrm{AW}_p^r \cap \delta(S_i)| \geq b(S_i)(2r + 1 - b(S_i)).$$

For each $i \in [1, k]$ set $S_i^+ := S_i \cap [1, p]$ and $S_i^- := S_i \cap [1', q']$ with respective cardinalities s_i^+, s_i^-; hence $b(S_i) = s_i^+ - s_i^-$. Define the following index sets: $I := \{i \in [1, k]: s_i^+ \leq r\}$, $J := \{i \in [1, k]: s_i^+ \geq p - r + 1\}$, and $K := \{i \in [1, k]: r + 1 \leq s_i^+ \leq p - r\}$. We show that relation (1.10) holds by distinguishing the cases whether i is in I, J or K.

First, if $i \in I$, i.e., $s_i^+ \leq r$, then $s_i^+ - s_i^- \leq s_i^+ \leq r$, implying that

$$|\delta(S_i) \cap \mathrm{AW}_p^r| \geq s_i^+(2r + 1 - s_i^+) \geq (s_i^+ - s_i^-)(2r + 1 - (s_i^+ - s_i^-)),$$

the first inequality following from Proposition (1.5)(i) and the second inequality from the fact that the function $x(2r + 1 - x)$ is monotone nondecreasing for $x \leq r$.

Then, if $i \in J$, i.e., $s_i^+ \geq p - r + 1$, then, since $\delta(S_i) = \delta([1, n] \setminus S_i)$, we obtain that

$$|\delta(S_i) \cap \mathrm{AW}_p^r| \geq (p - s_i^+)(2r + 1 - p + s_i^+) \geq (s_i^+ - s_i^-)(2r + 1 - (s_i^+ - s_i^-)),$$

the first inequality following from Proposition (1.5)(i) and the second one from the fact that $x(2r + 1 - x)$ is monotone nonincreasing for $x \geq r + 1$ and relation $r + 1 \leq 2r + 1 - p + s_i^+ \leq s_i^+ - s_i^-$.

Finally, if $i \in K$, i.e., $r + 1 \leq s_i^+ \leq p - r$, then

$$|\delta(S_i) \cap \mathrm{AW}_p^r| \geq r(r + 1) \geq b(S_i)(2r + 1 - b(S_i)),$$

the first inequality following from Proposition (1.5)(ii) and the second one from the fact that $x(2r + 1 - x) \leq r(r + 1)$ for all integers x.

This concludes the proof of validity of inequality (1.3) for MC(n).

We now identify the multicut vectors $\chi(S_1, \dots, S_k)$ which are roots of inequality (1.3) in the case $p - q \geq 2r + 2$. From the above observations, $\chi(S_1, \dots, S_k)$ is a root of inequality (1.3) if and only if $|\delta(S_i) \cap \mathrm{AW}_p^r| = b(S_i)(2r + 1 - b(S_i))$ for all $i \in [1, k]$, i.e., equality holds in (1.10) for all $i \in [1, k]$. We again distinguish the cases whether $i \in I$, J or K.

— If $i \in I$, equality holds in (1.10) if and only if $s_i^- = 0$ and $|\delta(S_i) \cap \mathrm{AW}_p^r| = s_i^+(2r + 1 - s_i^+)$, i.e., S_i defines a root of inequality $\mathrm{CW}_{p, p-2r-1}^r \cdot x \leq 0$. It follows from Proposition (1.6) that S_i induces a clique of AW_p^r.

— If $i \in J$, equality in (1.10) implies that $s_i^+ - s_i^- = 2r + 1 - p + s_i^+$ and thus $s_i^- = p - 2r - 1 \geq q + 1$, yielding a contradiction; therefore, J must be empty.

— If $i \in K$, equality holds in (1.10) if and only if $s_i^+ - s_i^- = r$, $r + 1$ and $|\delta(S_i) \cap \mathrm{AW}_p^r| = r(r + 1)$, i.e., S_i defines a root of inequality $\mathrm{CW}_{p, p-2r-1}^r \cdot x \leq 0$ and, thus, from Proposition (1.6)(R2), S_i is as in Proposition (1.8)(R2). This concludes the proof. □

1.2. *Clique-web facets.*

(1.11) THEOREM. *For any integers $p, q \geq 1$ and $r \geq 0$ with $n = p + q$, $p - q \geq 2r + 1$ and $q \geq 2$ if $p - q = 2r + 1$, the clique-web inequality* (1.3) *defines a facet of* MC(n).

PROOF. In view of Remark (1.4)(i), (ii), we can suppose that $p - q \geq 2r + 2$ and $r \geq 1$. Take an inequality $b^T x \leq b_0$ valid for MC(n) such that

$$\left\{x \in \mathrm{MC}(n) : \mathrm{CW}_{p,q}^r \cdot x = \frac{(p - q)(p - q - 2r - 1)}{2}\right\}$$

is contained in $\{x \in \mathrm{MC}(n): b^T x = b_0\}$; we prove through the following claims that, for some positive scalar α, $b^T x = \alpha \mathrm{CW}_{p,q}^r \cdot x$ holds, henceforth inequality (1.3) defines a facet of MC(n).

(1.12) *Claim.* $b_{ij} = 0$ for all $ij \in \mathrm{AW}_p^r$.

PROOF. Let χ_1 denote the root of inequality (1.3) (and hence of $b^T x \leq b_0$) defined by the partition of $[1, p] \cup [1', q']$ with classes: $[p - q - r + 1, p] \cup [1', q'], \{i\}$, for $1 \leq i \leq p - q - r$ and, given some u with $2 \leq u \leq r + 1$, let χ_2 denote the root defined by the partition with classes $[p - q - r + 1, p] \cup [1', q'], \{1, u\}, \{i\}$, for $2 \leq i \leq p - q - r$ and $i \neq u$. Hence, $0 = b^T \chi_1 - b^T \chi_2$, implying that $b_{1u} = 0$. Then Claim (1.12) follows by symmetry. □

(1.13) *Claim.* For some scalar α, $b_{ii'} = \alpha$ for all $i \in [1, p]$, $i' \in [1', q']$.

PROOF. Set $A := [p - q - r + 1, p] \cup [1', q'] \setminus \{i'\}$, where i' is a given element of $[1', q']$. Consider the roots χ_3, χ_4, χ_5 defined, respectively, by the following partitions:

— with classes $A, [1, r + 1] \cup \{i'\}, \{r + 2\}, \{i\}$, with $r + 3 \leqslant i \leqslant p - q - r$,

— with classes $A, [2, r + 2] \cup \{i'\}, \{1\}, \{i\}$, with $r + 3 \leqslant i \leqslant p - q - r$,

— with classes $A, [1, r + 2] \cup \{i'\}, \{i\}$, with $r + 3 \leqslant i \leqslant p - q - r$.

Using Claim (1.12) and relations $0 = b^T\chi_3 - b^T\chi_4$ and $0 = b^T\chi_4 - b^T\chi_5$, we obtain, respectively, that $b_{1i'} = b_{r+2i'}$ and $b_{1i'} = -b_{1r+2}$, hence implying that $b_{1i'} = \alpha_1$ is a constant that does not depend on the choice of i' in $[1', q']$ and $\alpha_1 = \alpha_{r+2} = -b_{1r+2}$; similarly, $b_{2i'} = \alpha_2 = \alpha_{r+3} = -b_{2r+3}$ for $i' \in [1', q']$.

Set $B := [p - q - r + 3, p] \cup [3', q']$ and consider the roots χ_6, χ_7, χ_8 defined respectively by the following partitions:

— with classes $B, [1, r + 3] \cup \{1', 2'\}, \{i\}$, for $r + 4 \leqslant i \leqslant p - q - r + 2$,

— with classes $B, [2, r + 3] \cup \{1', 2'\}, \{1\}, \{i\}$, for $r + 4 \leqslant i \leqslant p - q - r + 2$,

— with classes $B, [1, r + 2] \cup \{1', 2'\}, \{r + 3\}, \{i\}$, for $r + 4 \leqslant i \leqslant p - q - r + 2$.

From relations $0 = b^T\chi_6 - b^T\chi_7$ and $0 = b^T\chi_6 - b^T\chi_8$, we obtain that $\alpha_1 = -b_{1r+3}$, $\alpha_{r+3} = -b_{1r+3}$ and thus $\alpha_1 = \alpha_{r+3} = \alpha_2$. By symmetry, we can conclude that all α_i are equal. This proves Claim (1.13). □

(1.14) *Claim.* For some scalar β, $b_{i'j'} = \beta$ for all $1' \leqslant i' < j' \leqslant q'$.

PROOF. Take i', j' with $1' \leqslant i' < j' \leqslant q'$, set $C = [p - q - r + 2, p] \cup [1', q'] \setminus \{i', j'\}$ and consider the roots χ_9, χ_{10} defined by the following partitions:

— with classes $C \cup \{j'\}, [1, r + 2] \cup \{i'\}, \{i\}$, for $r + 3 \leqslant i \leqslant p - q - r + 1$,

— with classes $C, [1, r + 2] \cup \{i', j'\}, \{i\}$, for $r + 3 \leqslant i \leqslant p - q - r + 1$.

Then $0 = b^T\chi_9 - b^T\chi_{10}$, yielding relation

$$b_{i'j'} = (q - 3)\alpha + \sum_{h' \in [1', q'] \setminus \{i', j'\}} b_{j'h'} = (q - 3)\alpha + b_{j'k'} + \sum_{h' \in [1', q'] \setminus \{i', j', k'\}} b_{j'h'}$$

where k' is an element of $[1', q'] \setminus \{i', j'\}$. Similarly, one has $b_{k'j'} = (q - 3)\alpha + b_{i'j'} + \sum_{h' \in [1', q'] \setminus \{i', j', k'\}} b_{j'h'}$, henceforth yielding that $b_{i'j'} - b_{k'j'} = b_{j'k'} - b_{i'j'}$ and thus $b_{i'j'} = b_{j'k'}$. Hence Claim (1.14) is proved. □

(1.15) *Claim.* $\alpha = -\beta$.

PROOF. Obviously, we can suppose that $q \geqslant 2$. With $A = [p - q - r + 1, p] \cup [1', q']$, χ_{11}, χ_{12} are the roots defined by the partitions:

— with classes $A, [1, r + 1], \{i\}$, for $r + 2 \leqslant i \leqslant p - q - r$,

— with classes $A - \{1'\}, [1, r + 1] \cup \{1'\}, \{i\}$, for $r + 2 \leqslant i \leqslant p - q - r$.

From relation $0 = b^T\chi_{11} - b^T\chi_{12}$, we deduce that $\alpha = -\beta$. □

(1.16) *Claim.* $b_{ij} = -\alpha$ for all $ij \in W_p^r = E_p \setminus \mathrm{AW}_p^r$.

PROOF. We already know that $b_{1,r+2} = -\alpha$ (cf. proof of Claim (1.13)). We show by induction on u, $r + 2 \leqslant u \leqslant (p + 1)/2$, the following assertion:

$$(H_u) \qquad b_{ij} = -\alpha \quad \text{for all } 1 \leqslant i < j \leqslant u \text{ such that } ij \notin \mathrm{AW}_p^r.$$

Assertion (H_{r+2}) holds. Assume (H_{u-1}) holds; we prove that (H_u) holds, i.e., that $b_{iu} = -\alpha$ for all $1 \leqslant i \leqslant u - r - 1$. For this, given $i \in [1, u - r - 1]$ set $D := [u + 1, q + i + 2r]$ (interval of $[1, p]$ of size $q + i + 2r - u$ starting at point $u + 1$) and let χ_{13}, χ_{14} denote the roots defined by the following partitions:

— with classes $D \cup [1', (q + r + i - u)'], [i + 1, u] \cup [(q + r + i - u + 1)', q']$, $\{j\}$, for $j \in [1, p] \setminus (D \cup [i + 1, u])$,

— with classes $D \cup [1', (q+r+i-u)'], [i, u] \cup [(q+r+i-u+1)', q'], \{j\}$, for $j \in [1, p] \setminus (D \cup [i, u])$.

Using the induction assumption and relation $0 = b^T\chi_{13} - b^T\chi_{14}$, we deduce that $b_{iu} = -\alpha$, hence showing assertion (H_u). □

This finishes the proof of Theorem (1.11). □

Theorem (1.11) states that the CW-inequalities (1.3) define facets of the multicut polytope MC(n). In fact, by looking closely at the proof, we can refine this result and prove that inequality (1.3) is, in fact, facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$ and $\mathrm{MC}_k^{\geqslant}(n)$ for suitable k. More precisely, we can show the following.

(1.17) THEOREM. *Given integers $p, q, r, n \geqslant 1$ with $p - q \geqslant 2r + 2$ and $n = p + q$, the following assertions hold*:

(i) *If $2 \leqslant k \leqslant p - q - 2r$, then inequality* (1.3) *is facet-inducing for* $\mathrm{MC}_k^{\geqslant}(n)$.

(ii) *If $\lceil (p-q)/(r+1) \rceil + 2 \leqslant k \leqslant n$, then inequality* (1.3) *is facet-inducing for* $\mathrm{MC}_k^{\leqslant}(n)$.

(iii) *If $p - q - r + 2 \leqslant k \leqslant n$, then inequality* (1.3) *is not facet-inducing for* $\mathrm{MC}_k^{\geqslant}(n)$.

(iv) *If $2 \leqslant k \leqslant \lceil (p-q)/(r+1) \rceil - 1$, then inequality* (1.3) *is not facet-inducing for* $\mathrm{MC}_k^{\leqslant}(n)$.

(v) *If $p - q = k(r+1)$ and $k \geqslant 2$, then inequality* (1.3) *is not facet-inducing for* $\mathrm{MC}_k^{\leqslant}(n)$.

PROOF. (i) By simply checking the roots χ_i ($i = 1, 2, \ldots, 14$) used in the proof of Theorem (1.11), one can notice that all of them are incidence vectors of h-cuts for h taking one of the following values: $p-q-2r$, $p-q-2r+1$, $p-q-2r+2$, $p-q-r$, $p-q-r+1$. This simple observation implies (i).

(ii) We can reformulate the proof of Theorem (1.11), such that all arguments remain correct and each root used in the newly formulated proof has at most $\lceil (p-q)/(r+1) \rceil + 2$ shores. Thus we obtain a proof for (ii). The reformulation is simple. Each root used in the proof of Theorem (1.11) has to be changed by just grouping singleton-shores into larger shores satisfying property (R1) of Proposition (1.8). We leave the easy-to-find details to the reader.

(iii) and (iv) We show that under the conditions of (iii) and (iv) inequality (1.3) does not have a root at all. Namely, suppose the k-cut $\chi(S_1, \ldots, S_k)$ is a root of inequality (1.3). We use the description of roots from Proposition (1.8). We can suppose that $S_1, \ldots, S_h$ are of type (R1) while $S_{h+1}, \ldots, S_k$ are of type (R2) with $0 \leqslant h \leqslant k-1$. Then $S_i = S_i^+ \cup S_i^-$ for all i, and $s_i^- = 0$, $1 \leqslant s_i^+ \leqslant r+1$ for $i \in [1, h]$, while $s_i^+ - s_i^- \in \{r, r+1\}$ for $i \in [h+1, k]$. Then we have that

$$p = \sum_{i=1}^{k} s_i^+ \leqslant \sum_{i=1}^{h} s_i^+ + \sum_{i=h+1}^{k} s_i^- + (k-h)(r+1)$$

$$\leqslant h(r+1) + q + (k-h)(r+1) = q + k(r+1),$$

implying that $k \geqslant (p-q)/(r+1)$. We also have that

$$p = \sum_{i=1}^{k} s_i^+ \geqslant h + \sum_{i=h+1}^{k} s_i^- + (k-h)r = q + h + (k-h)r$$

implying that $p - q \geqslant h + (k-h)r \geqslant k + r - 1$ and thus $k \leqslant p - q - r + 1$.

(v) Take a multicut vector $\chi(S_1,\dots,S_t)$ which is a root of inequality (1.3), suppose that $S_1,\dots,S_h$ are of type (R1) and $S_{h+1},\dots,S_t$ are of type (R2) with $0 \leqslant h \leqslant t-1$ (cf. Proposition (1.8)); then $p = \sum_{i=1}^{t} s_i^+ \leqslant h(r+1) + q - (t-h)(r+1)$, yielding that $p - q = k(r+1) \leqslant t(r+1)$ and thus $t \geqslant k$. Therefore, the multicuts of $\mathrm{MC}_k^{\leqslant}(n)$ that define roots of inequality (1.3) are k-cuts $\delta(S_1,\dots,S_k)$ satisfying $s_i^+ - s_i^- = r+1$ for all $i \in [1,k]$ and, henceforth, every root of inequality (1.3) for $\mathrm{MC}_k^{\leqslant}$ is, in fact, root of the following inequality:

$$\sum_{1 \leqslant i<j \leqslant n} b_i b_j x_{ij} \leqslant \frac{(p-q)(p-q-r-1)}{2}$$

where $b = (1,\dots,1,-1,\dots,-1)^T$ consists of p coefficients $+1$ and q coefficients -1. To see this, observe that

$$\begin{aligned}\sum_{1 \leqslant i<j \leqslant n} b_i b_j \chi(S_1,\dots,S_k)_{ij} &= \frac{1}{2}\sum_{i=1}^{k} b(S_i)(p-q-b(S_i)) \\ &= \frac{k}{2}(r+1)(p-q-r-1) \\ &= \frac{(p-q)(p-q-r-1)}{2}\end{aligned}$$

at all roots $\chi(S_1,\dots,S_k)$ of inequality (1.3) for $\mathrm{MC}_k^{\leqslant}(n)$. Since $r \geqslant 1$, the linear forms $\mathrm{CW}_{p,q}^r \cdot x$ and $\sum_{1 \leqslant i \leqslant j \leqslant n} b_i b_j x_{ij}$ are distinct, hence implying that inequality (1.3) is not facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$. □

There remain only two values of k for which facethood for $\mathrm{MC}_k^{\leqslant}(n)$ is undecided, namely $k = \lceil (p-q)/(r+1) \rceil$ and $k = \lceil (p-q)/(r+1) \rceil + 1$. In both cases, some roots indeed exist. For the first case $k = \lceil (p-q)/(r+1) \rceil$, we believe that inequality (1.3) is not facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$.

Concerning facethood of inequality (1.3) for $\mathrm{MC}_k^{\geqslant}(n)$, there is the interval $[p-q-2r+1, p-q-r+1]$ of undecided values of k. Probably, one could improve Theorem (1.17)(i) by suitably modifying the roots used in the proof of Theorem (1.11) (but, this time, trying to increase the number of classes in the partitions defining the roots) in a similar manner as we did for Theorem (1.17)(ii).

(1.18) REMARK. We would like to mention again that, in the case $p - q = hr + 1$, $p \geqslant 2hr$ and $1 \leqslant r \leqslant h-2$, the CW-inequality (1.3) was shown to be facet-inducing for $\mathrm{MC}_h^{\leqslant}(n)$ in Theorem 6.3 of [CR2]. Actually, if $p - q = hr + 1$, then

$$\left\lceil \frac{p-q}{r+1} \right\rceil = h - \left\lfloor \frac{h-1}{r+1} \right\rfloor$$

and Theorem (1.17)(ii) implies that inequality (1.3) is facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$ for all $k \geqslant h + 2 - \lfloor (h-1)/(r+1) \rfloor$. Therefore, if $\lfloor (h-1)/(r+1) \rfloor = 1$, then Chopra and Rao's result mentioned above solves one of the two undecided cases, namely facethood for $\mathrm{MC}_h^{\leqslant}(n)$. The case $k = h-1$ remains still undecided, and, probably, (1.3) does not define a facet (recall (1.17) (v)). If $\lfloor (h-1)/(r+1) \rfloor = 2$ and $p \geqslant 2hr$, then Theorem (1.17) (ii) coincides with Chopra and Rao's result for this case. But, if $\lfloor (h-1)/(r+1) \rfloor \geqslant 3$, then Theorem (1.17) (ii) is more general than Chopra and Rao's result.

(1.19) REMARK. In all preceding results we considered all inequalities with $r \geqslant 1$. For the case $r = 0$, which corresponds to the class of $[S,T]$-inequalities studied in

[GW1], we can do a similar work of counting the number of partitions forming the roots used for the proof of facethood of inequality $CW^0_{p,q} \cdot x \leqslant (p-q)(p-q-1)/2$. By simply looking at the roots used in the proof of Theorem (4.2) ([GW1]), one can check that they are all h-cuts with h taking values $p, p-1$, hence implying that inequality $CW^0_{p,q} \cdot x \leqslant (p-q)(p-q-1)/2$ is facet-inducing for $MC^{\geqslant}_k(n)$ for all $k \leqslant p-1$. On the other hand, by suitably modifying the roots in such a way that they are h-cuts with h small, it is not hard to see that the proof of Theorem (4.2) ([GW1]) can be modified so that it uses only h-cuts as roots with h taking values $p-q+1$, $p-q+2$ and, therefore, inequality $CW^0_{p,q} \cdot x \leqslant (p-q)(p-q-1)/2$ is facet-inducing over $MC^{\leqslant}_k(n)$ for all $k \geqslant p-q+2$. Furthermore, in a similar way as we did for CW-inequalities with $r \geqslant 1$, one can give an alternative proof for the validity for $MC(n)$ of inequality $CW^0_{p,q} \cdot x \leqslant (p-q)(p-q-1)/2$ and derive the full description of its roots (the details are left to the reader); so a multicut vector $\chi(S_1, \ldots, S_k)$ is a root of inequality $CW^0_{p,q} \cdot x \leqslant (p-q)(p-q-1)/2$ if and only if $b(S_i) = s_i^+ - s_i^- \in \{0,1\}$ for all $i \in [1,k]$, where $b = (1, \ldots, 1, -1, \ldots, -1)$. An immediate consequence is that the roots are k-cuts with $p-q \leqslant k \leqslant p$. We summarize all above observations in the next theorem.

(1.20) THEOREM. *Given integers $n = p+q$, $p, q \geqslant 1$ with $p-q \geqslant 2$, the roots of inequality $CW^0_{p,q} \cdot x \leqslant (p-q)(p-q-1)/2$ are the multicut vectors $\chi(S_1, \ldots, S_k)$ for which $s_i^+ - s_i^- \in \{0,1\}$, for all $i \in [1,k]$ and, hence, they satisfy $p-q \leqslant k \leqslant p$. Furthermore, inequality $CW^0_{p,q} \cdot x \leqslant (p-q)(p-q-1)/2$ is facet-inducing for $MC^{\leqslant}_k(n)$ for all $k \geqslant p-q+2$, is not facet-inducing for $MC^{\leqslant}_k(n)$ for $k \leqslant p-q-1$, and is facet-inducing for $MC^{\geqslant}_k(n)$ for $k \leqslant p-1$.*

(1.21) REMARK. It follows from (1.20) that—by our proof—there remain only two values of k for which facethood of the inequality $CW^0_{p,q} \cdot x \leqslant \sigma(\sigma-1)/2$ for $MC^{\leqslant}_k(n)$ is undecided, namely $k \in \{\sigma, \sigma+1\}$, where $\sigma = p-q$. In fact (recall Remark 1.4(ii)) Chopra and Rao [CR2] proved facethood of the above inequality for $MC^{\leqslant}_{\sigma+1}(n)$, hence solving one of the two undecided cases.

1.3. *A class of CW facets that is polynomially separable.* Given an integer $r \geqslant 0$, let C be a cycle of length $p = 2r+3$ with nodeset $V(C) \subset [1,n]$ and i_0 be a node of $[1,n] \setminus V(C)$. The following inequality

$$(1.22) \qquad \sum_{ij \in C} x_{ij} - \sum_{i \in V(C)} x_{i_0 i} \leqslant \frac{|C|-1}{2}$$

is called **p-wheel inequality** or just, if we do not want to specify the cycle length p, **odd wheel inequality.**

Since the web $W^r_p = E_p \setminus AW^r_p$ is just a cycle of length $2r+3$, the CW-inequality $CW^r_{2r+3,1} \cdot x \leqslant r+1$, with parameters $p = 2r+3$, $q = 1, r$, coincides (up to permutation of the p positive nodes) with the p-wheel inequality (1.22). In fact, the only cases where the web W^r_p is a cycle are given by the parameter relation $p = 2r+3$; and thus, since $p-q \geqslant 2r+1$, the only CW-inequalities for which W^r_p is a cycle are determined by the parameters $p = 2r+3$ and $q \in \{1,2\}$. The case $q = 1$ is the above odd wheel inequality while the case $q = 2$ is (up to switching) the known bicycle odd wheel inequality introduced in [BGM], [BM] for the bipartite subgraph polytope and the cut polytope $MC^{\leqslant}_2(n)$ (and shown to define a facet). The bicycle odd wheel inequality, being a homogeneous facet for $MC^{\leqslant}_2(n)$, is therefore facet-inducing for $MC^{\leqslant}_k(n)$ for all $2 \leqslant k \leqslant n$. On the other hand, the odd wheel inequality (1.22) is facet-inducing for $MC^{\leqslant}_k(n)$ for $4 \leqslant k \leqslant n$ (see [CR1] and also Theorem

(1.17)(ii)), and is not facet-inducing for $MC_2^{\leqslant}(n)$ (cf. Remark 1.6 in [DL2] and Theorem (1.17)(v)).

Both the odd wheel and the bicycle odd wheel inequalities share the remarkable property that they can be tested for separation in polynomial time. This fact was proved for bicycle odd wheel inequalities by Gerards ([G]). By the same type of argument (see also [GLS] for the related odd wheel inequalities for the stable set polytope) we can prove

(1.23) PROPOSITION. *The separation problem for the following set of inequalities*
(i) $x_{ij} \leqslant 1$ *for all* $1 \leqslant i < j \leqslant n$,
(ii) $x_{ij} - x_{ik} - x_{jk} \leqslant 0$ *for all distinct* i, j, k *in* $[1, n]$,
(iii) $\sum_{ij \in C} x_{ij} - \sum_{i \in V(C)} x_{i_0 i} \leqslant (|C| - 1)/2$ *for every node* $i_0 \in [1, n]$ *and for every odd cycle* C *with node set* $V(C) \subseteq [1, n] \setminus \{i_0\}$,
can be solved in polynomial time.

PROOF. Given a vector $y \in \mathbb{Q}^{\binom{n}{2}}$, one can obviously test in polynomial time whether y satisfies inequalities (i), (ii). If some inequality (i) or (ii) is violated, then we are finished. We now suppose that y satisfies all inequalities (i), (ii).

For each node $i_0 \in [1, n]$, we define a weighting of the edges of $K_n - \{i_0\}$ by

$$w_{ij}^{i_0} := \tfrac{1}{2} - y_{ij} + \tfrac{1}{2}(y_{i_0 i} + y_{i_0 j}) \quad \text{for all } ij \in E_n \setminus \delta(i_0).$$

By assumption, y satisfies (i) and (ii), thus $w_{ij}^{i_0} \geqslant 0$. The construction of the weights $w_{ij}^{i_0}$ implies that y violates an inequality of type (iii) if and only if there exist a node $i_0 \in [1, n]$ and an odd cycle $C \subseteq E_n \setminus \delta(i_0)$ such that

$$w^{i_0}(C) = \frac{|C|}{2} - y(C) + \sum_{i \in V(C)} y_{i_0 i} < \frac{1}{2}.$$

Using the algorithm described in [GP] one can decide in polynomial time whether or not such a node i_0 and an odd cycle C exist. If so, any odd cycle C^* with $w^{i_0}(C^*) < \frac{1}{2}$ defines an odd wheel inequality

$$\sum_{ij \in C^*} x_{ij} - \sum_{i \in V(C^*)} x_{i_0 i} \leqslant \frac{|C^*| - 1}{2}$$

that is violated by y. □

Combining polynomial time separation algorithms for various classes of inequalities for multicut polytopes (e.g., the algorithms described in [GP], [BM], and [G]) with the algorithm of Proposition (1.23) and the ellipsoid method yields (see [GLS]) polynomial time algorithms that optimize linear objective functions over certain LP-relaxations of multicut optimization problems. Since this paper is not intended to treat computational aspects, we do not go into further detail here.

2. Collapsing and lifting. There are various techniques known with which valid or facet-defining inequalities can be manipulated such that new valid or facet-defining inequalities are obtained. The operations "collapsing" and "lifting" have been studied in [DL1], [DL2], and [DDL], for the cut polytope $MC_2^{\leqslant}(n)$. We extend these operations in this section to the case of multicuts.

2.1. *Collapsing*. We first recall the definition of collapsing of a vector or a graph from [DL2], [DDL]. Given integers $N \geqslant n$, let π be a partition of the set $[1, N]$ into n parts $I_1, \ldots, I_n$. Given a vector $v \in \mathbb{R}^{\binom{N}{2}}$, its **$\pi$-collapsing** is the vector v_π of $\mathbb{R}^{\binom{n}{2}}$ defined by

$$(2.1) \qquad (v_\pi)_{ij} = \sum_{h \in I_i, k \in I_j} v_{hk} \quad \text{for } 1 \leqslant i < j \leqslant n.$$

In particular, if $G = (V, E)$ is an edge weighted graph on node set $[1, N]$ and v is its edge weight vector, where, for $1 \leqslant i < j \leqslant N$, v_{ij} denotes the weight of edge $ij \in E$ and $v_{ij} = 0$ for $ij \notin E$, then the π-collapsing of G is the weighted graph G_π on node set $[1, n]$ whose edge weight vector is v_π.

For a subset S of $[1, N]$ we set $S_\pi := \{i \in [1, n]: S \cap I_i \neq \varnothing\}$ and, for a subset S of $[1, n]$, we set $S^\pi := \bigcup_{i \in S} I_i$.

The next definitions are extensions to the multicut case of corresponding definitions for the case of ordinary cuts. A partition $S_1, \ldots, S_k$ of $[1, N]$ is called **π-admissible** if for all $i \in [1, n]$ and all $j \in [1, k]$, $I_i \cap S_j \neq \varnothing$ implies $I_i \subseteq S_j$. If $S_1, \ldots, S_k$ is a π-admissible partition of $[1, N]$, then the family $(S_1)_\pi, \ldots, (S_k)_\pi$ defines a partition of $[1, n]$. The following relations can be easily checked.

$$(2.2) \qquad v_\pi^T \chi(S_1, \ldots, S_k) = v^T \chi\big((S_1)^\pi, \ldots, (S_k)^\pi\big)$$

for every partition $S_1, \ldots, S_k$ of $[1, n]$ and every $v \in \mathbb{R}^{\binom{N}{2}}$.

An immediate consequence of relation (2.2) is that, if $v^T x \leqslant \alpha$ is a valid inequality for $\mathrm{MC}_k^{\leqslant}(N)$ (respectively, $\mathrm{MC}_k^{\geqslant}(N)$, $\mathrm{MC}_k^{=}(N)$), then its π-collapsing $v_\pi^T x \leqslant \alpha$ is a valid inequality for $\mathrm{MC}_k^{\leqslant}(n)$ (respectively, $\mathrm{MC}_k^{\geqslant}(n)$, $\mathrm{MC}_k^{=}(n)$); furthermore, the roots of inequality $v_\pi^T x \leqslant \alpha$ are the multicut vectors $\chi((S_1)_\pi, \ldots, (S_k)_\pi)$ for which $S_1, \ldots, S_k$ is a π-admissible partition of $[1, N]$ and $\chi(S_1, \ldots, S_k)$ is a root of inequality $v^T x \leqslant \alpha$.

The collapsing operation enables us to define the general CW-inequalities $\mathrm{CW}_{p,q}^r(b) \cdot x \leqslant \sigma(\sigma - 2r - 1)/2$ where $b = (b_1, \ldots, b_n)^T$ are $n = p + q$ integers satisfying $\sigma := \sum_{i=1}^n b_i \geqslant 2r + 1$, $b_1, \ldots, b_p > 0$ and $b_{p+1}, \ldots, b_n < 0$. For this we use the notion of a **collapsed antiweb** $\mathrm{AW}_p^r(b_1, \ldots, b_p)$ (see [DL2]). Setting $P := \sum_{i=1}^p b_i$ and letting $\pi_+(b)$ denote the partition of $[1, P]$ into the p intervals $I_0 := [1, b_1]$, $I_i := [b_1 + \cdots + b_i + 1, b_1 + \cdots + b_i + b_{i+1}]$ for $i = 1, 2, \ldots, p - 1$, the antiweb $\mathrm{AW}_p^r(b_1, \ldots, b_p)$ is the weighted graph on node set $[1, p]$ obtained by $\pi_+(b)$-collapsing of the antiweb AW_P^r on node set $[1, P]$. We set $Q := \sum_{i=p+1}^n |b_i|$, $N := P + Q = \sum_{i=1}^n |b_i|$ and $\sigma := P - Q = \sum_{i=1}^n b_i$.

(2.3) PROPOSITION. *With the above notation, the* **general** ***CW*-inequality**

$$(2.4) \qquad \mathrm{CW}_{p,q}^r(b) \cdot x = \sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} - \sum_{(i,j) \in \mathrm{AW}_p^r(b)} x_{ij} \leqslant \frac{\sigma(\sigma - 2r - 1)}{2}$$

is valid for the multicut polytope $\mathrm{MC}(n)$, *and, hence, for* $\mathrm{MC}_k^{\leqslant}(n)$, $\mathrm{MC}_k^{=}(n)$, *and* $\mathrm{MC}_k^{\geqslant}(n)$ *for all* $k \leqslant n$, *where* $b = (b_1, \ldots, b_n)^T$, $\sigma := \sum_{i=1}^n b_i \geqslant 2r + 1$.

Observe that inequality (2.4) is, in fact, the $\pi(b)$-collapsing of the valid inequality (1.3)

$$\mathrm{CW}^r_{P,Q} \cdot x \leqslant \frac{\sigma(\sigma - 2r - 1)}{2} = \frac{(P-Q)(P-Q-2r-1)}{2} \quad \text{for } \mathrm{MC}(N),$$

where $\pi(b)$ is any partition of $[1, N] = [1, P] \cup [1', Q']$ into the p intervals $I_0, \ldots, I_{p-1}$ (forming the previously considered partition $\pi_+(b)$ of $[1, P]$) and q arbitrary parts of sizes $|b_{p+1}|, \ldots, |b_n|$ partitioning $[1', Q']$. Therefore, validity of inequality (2.4) follows from Proposition (1.7) and the observations described after relation (2.2) above. Observe that the CW-inequalities defined in (1.3) are exactly the pure (general) CW-inequalities defined in (2.4).

Another immediate consequence (using the bounds given in Proposition (1.17)(iii) and (iv), Theorem (1.20) on the number k of parts in partitions defining roots of pure CW-inequalities) is that, for any multicut $\delta(S_1, \ldots, S_k)$ that defines a root of inequality (2.4), the following relation holds

$$(2.5) \qquad \frac{\sigma}{r+1} \leqslant k \leqslant \sigma - r + 1 \quad \text{if } r \geqslant 1, \quad \text{and} \quad \sigma \leqslant k \leqslant p \quad \text{if } r = 0.$$

Consequently, inequality (2.4) does neither induce a facet of $\mathrm{MC}^{\leqslant}_{\lceil \sigma/(r+1) \rceil - 1}(n)$ nor of $\mathrm{MC}^{\geqslant}_{\sigma - r + 2}(n)$ if $r \geqslant 1$, and it is not facet-inducing for $\mathrm{MC}^{\leqslant}_{\sigma - 1}(n)$ nor for $\mathrm{MC}^{\geqslant}_{p+1}(n)$ if $r = 0$.

Also, using Proposition (1.8) and Theorem (1.20) we deduce that every multicut $\delta(S_1, \ldots, S_k)$ that defines a root of inequality (2.4) with $r \geqslant 1$ (resp., $r = 0$) satisfies $b(S_i) \in [1, r+1]$ (resp., $b(S_i) \in \{0, 1\}$) for all $i \in [1, k]$. This simple observation enables us to derive the following necessary condition for a general CW-inequality (2.4) to define a facet.

(2.6) Proposition. *Take an integer $r \geqslant 0$ and an integral vector $b = (b_1, \ldots, b_n)^T$ with $b_1, \ldots, b_p > 0$, $b_{p+1}, \ldots, b_n < 0$, and assume that $\sigma := \sum_{i=1}^n b_i \geqslant 2r + 2$ holds. Suppose that b_1, b_2 are the largest two integers among $b_1, \ldots, b_p$ and that b_{n-1}, b_n the smallest two among $b_{p+1}, \ldots, b_n$. If inequality* (2.4) $\mathrm{CW}^r_{p,q}(b) \cdot x \leqslant \sigma(\sigma - 2r - 1)/2$ *defines a facet for $\mathrm{MC}^{\leqslant}_k(n)$ or $\mathrm{MC}^{\geqslant}_k(n)$ for some $2 \leqslant k \leqslant n$, then the following assertions hold*:

(i) $b_1 + b_2 + \sum_{i=p+1}^n b_i \leqslant r + 1$,

(ii) $b_{n-1} + b_n + \sum_{i=1}^p b_i \geqslant \min(r, 1)$.

Proof. We use the fact that, if $\chi(S_1, \ldots, S_k)$ is a root of inequality (2.4), then $b(S_i) \in [1, r+1]$ if $r \geqslant 1$ and $b(S_i) \in \{0, 1\}$ if $r = 0$, for all $i \in [1, k]$. Assume condition (i) is violated; then, if for some root $\chi(S_1, \ldots, S_k)$, nodes 1, 2 belong to the same part S_i, we have that $b(S_i) \geqslant b_1 + b_2 + \sum_{i=p+1}^n b_i > r + 1$, yielding a contradiction. Hence, if (i) is violated, then inequality (2.4) is dominated by the valid inequality $x_{12} \leqslant 1$ and thus is not facet-inducing. Similarly, if condition (ii) is violated, then nodes $n-1, n$ belong to distinct parts of every partition defining a root of (2.4), implying that (2.4) is dominated by inequality $x_{n-1,n} \leqslant 1$. □

In §5 of [CR2] the following inequality is considered (as a special case of the class of general cycle inequalities)

$$(2.7) \qquad \sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} - \frac{r(r+1)}{2} \sum_{ij \in C} x_{ij} \leqslant 0,$$

where, in our notation, $b = (b_1, \ldots, b_n)^T$ are n integers satisfying $\sum_{i=1}^n b_i = 2r + 1$,

$b_1, \ldots, b_p > 0$, $b_{p+1}, \ldots, b_n < 0$ and C is a cycle on $[1, p]$. In the case $b_i \geq r$ for all $i \in [1, p]$, the collapsed antiweb $AW_p^r(b_1, \ldots, b_p)$ is just the cycle $C = (1, 2, \ldots, p)$ with weight $r(r+1)/2$ on its edges (see Example 1.5 in [DL2] and, for the roots, Proposition 1.19, [DL2]) and, therefore, the general cycle inequality (2.7) of [CR2] coincides with the CW-inequality $CW_{p,q}^r(b) \cdot x \leq 0$. Otherwise, if $b_i < r$ for some $i \in [1, p]$, the general cycle inequality (2.7) is not facet-inducing for $MC_2^{\leq}(n)$ in view of the following fact.

(2.8) LEMMA. *Given* $b_1, \ldots, b_p > 0$ *with* $\sum_{i=1}^p b_i \geq 2r + 1$ *and the cycle* $C = (1, 2, \ldots, p)$, *then*

$$|\delta(S) \cap AW_p^r(b_1, \ldots, b_p)| \leq \frac{r(r+1)}{2} |\delta(S) \cap C|$$

holds for all subsets $S \subseteq [1, p]$.

PROOF. We assume first that $b_1 = \cdots = b_p = 1$. If S is an interval of $[1, p]$, then

$$\frac{r(r+1)}{2} |\delta(S) \cap C| = r(r+1),$$

$$|\delta(S) \cap AW_p^r| = r(r+1) \quad \text{if } r \leq |S| \leq p - r \quad \text{and}$$

$$|\delta(S) \cap AW_p^r| = s(2r + 1 - s) \leq r(r+1) \quad \text{if } |S| \leq r.$$

Therefore, $|\delta(S) \cap AW_p^r| \leq r(r+1)$ holds for any interval S of $[1, p]$. Take now an arbitrary subset S of $[1, p]$. Then S can be viewed as a union $S = S_1 \cup \cdots \cup S_u$ of $u \geq 2$ intervals S_i of $[1, p]$. Hence,

$$|\delta(S) \cap C| \frac{r(r+1)}{2} = r(r+1)u,$$

while

$$|\delta(S) \cap AW_p^r| = \left| AW_p^r \cap \left(\bigcup_{i=1}^u \delta(S_i) \setminus \delta(S_1, \ldots, S_u) \right) \right|$$

$$\leq \sum_{i=1}^u |AW_p^r \cap \delta(S_i)| \leq r(r+1)u,$$

since each S_i is an interval. Therefore, we have that

$$|\delta(S) \cap AW_p^r| \leq \frac{r(r+1)}{2} |\delta(S) \cap C| \quad \text{for all } S \subseteq [1, p].$$

We now consider the general case $b_1, \ldots, b_p \geq 1$. Set $P := \sum_{i=1}^p b_i$ and let π denote a partition of $[1, P]$ into p consecutive intervals of sizes $b_1, \ldots, b_p$. Then the π-collapsing of the antiweb AW_P^r and of the cycle $C' = (1, 2, \ldots, P)$ are, respectively, the weighted antiweb $AW_p^r(b_1, \ldots, b_p)$ and the cycle $C = (1, 2, \ldots, p)$. Then, using the properties of collapsing, we have that for $S \subseteq [1, p]$,

$$|\mathrm{AW}_p^r(b_1, \ldots, b_p) \cap \delta(S)| = |\mathrm{AW}_p^r \cap \delta(S^\pi)|$$

$$\leqslant \frac{r(r+1)}{2} |\delta(S^\pi) \cap C'| = \frac{r(r+1)}{2} |\delta(S) \cap C|.$$

This proves the claim. □

We deduce from Lemma (2.8) that the general cycle inequality (2.7) is dominated by the CW-inequality $\mathrm{CW}_{p,q}^r(b) \cdot x \leqslant 0$. Therefore, the general cycle inequality (2.7) is not facet-inducing unless it coincides with the CW-inequality $\mathrm{CW}_{p,q}^r(b) \cdot x \leqslant 0$, i.e., unless $\mathrm{AW}_p^r(b_1, \ldots, b_p)$ coincides with the weighted cycle $C = (1, 2, \ldots, p)$ with weight $r(r+1)/2$ on all edges which occurs if and only if $b_1, \ldots, b_p \geqslant r$.

2.2. *Construction of facets by collapsing-lifting*. In this section we give a general method for characterizing facets of multicut polytopes which is based on collapsing (cf. Theorem (2.9)). We derive from it a lifting procedure for constructing clique-web facets (cf. Theorem (2.11)). In general, lifting means any procedure permitting us to construct facets for $\mathrm{MC}_k^{\leqslant}(n+1)$ (or $\mathrm{MC}_k^{\geqslant}(n+1)$) from given facets for $\mathrm{MC}_k^{\leqslant}(n)$ (or $\mathrm{MC}_k^{\geqslant}(n)$). The lifting we consider is based on the following node-splitting operation which is, in fact, the converse operation to the collapsing operation from the preceding section. Given vectors $v \in \mathbb{R}^{\binom{n}{2}}$, $v' \in \mathbb{R}^{\binom{n+1}{2}}$, we say that v' is obtained from v by **splitting node** 1 **into nodes** $1, n+1$ if $v_{1i} = v'_{1i} + v'_{in+1}$ holds for all $i \in [2, n]$, or, in other words, if v is obtained from v' by $\{\{1, n+1\}, \{2\}, \{3\}, \ldots, \{n\}\}$-collapsing. The easiest example of such a lifting is **zero-lifting**. In this case v, v' satisfy $v'_{ij} = v_{ij}$ for $1 \leqslant i < j \leqslant n$ and $v'_{in+1} = 0$ for $i \in [1, n]$. If an inequality $v^T x \leqslant \alpha$ is facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$ (resp., $\mathrm{MC}_k^{\geqslant}(n)$) and v' obtained from v by zero-lifting (and, for the case $\mathrm{MC}_k^{\geqslant}(n+1)$, $(v')^T x \leqslant \alpha$ is valid) then $(v')^T x \leqslant \alpha$ is facet-inducing for $\mathrm{MC}_k^{\leqslant}(n+1)$ (resp., $\mathrm{MC}_k^{\geqslant}(n+1)$). In other words, zero-lifting preserves facethood (this result was proved for polytopes $\mathrm{MC}_k^{\leqslant}$, for $k = 2$ in Theorem 2.2 ([DL1]), for $k = n$ in Theorem 3.9 ([GW], with some additional condition), for arbitrary k in Theorem 4.1 ([CR2]); the general result follows immediately from Theorem (2.9) below, cf. Remark (2.10)).

In general, collapsing does not preserve facethood. However, under some conditions, collapsing can be used as a tool for proving facethood, as the following result shows, which is, in some sense, a companion result to Theorem 4 ([DDL]).

(2.9) THEOREM. *Let $v^T x \leqslant \alpha$ be a valid inequality for $\mathrm{MC}_k^{\leqslant}(n)$ (resp., $\mathrm{MC}_k^{\geqslant}(n)$) and j_1, j_2, j_3, j_4 be distinct elements of $[1, n]$. Assume that the following conditions hold:*

(i) *$\alpha \neq 0$ or, for some $i, j \notin \{j_1, j_2, j_3, j_4\}$, $v_{ij} \neq 0$.*

(ii) *The three inequalities obtained from inequality $v^T x \leqslant \alpha$ by collapsing nodes $\{j_1, j_2\}$, nodes $\{j_1, j_3\}$, or nodes $\{j_1, j_4\}$, respectively, are facet-inducing for $\mathrm{MC}_k^{\leqslant}(n-1)$ (resp. $\mathrm{MC}_k^{\geqslant}(n-1)$).*

Then inequality $v^T x \leqslant \alpha$ is facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$ (resp., $\mathrm{MC}_k^{\geqslant}(n)$).

PROOF. We give the proof for $\mathrm{MC}_k^{\leqslant}(n)$; the proof for the case $\mathrm{MC}_k^{\geqslant}(n)$ is identical. Take an inequality $w^T x \leqslant \beta$ which is valid for $\mathrm{MC}_k^{\leqslant}(n)$ and such that

$$\{x \in \mathrm{MC}_k^{\leqslant}(n) : v^T x = \alpha\} \subseteq \{x \in \mathrm{MC}_k^{\leqslant}(n) : w^T x = \beta\}.$$

We show the existence of a scalar $\lambda > 0$ for which $w = \lambda v$, $\beta = \lambda \alpha$. We can suppose without loss of generality that $j_4 = 1$, $j_1, j_2, j_3 \in [2, n]$. For $u = 1, 2, 3$ denote by π_u

the partition of $[1, n]$ with classes $\{1, j_u\}$ and singletons $\{i\}$ for $i \in [2, n] \setminus \{j_u\}$. Then inequality $w_{\pi_1}^T x \leqslant \beta$ obtained from $w^T x \leqslant \beta$ by π_1-collapsing is valid for $\mathrm{MC}_k^{\leqslant}(n-1)$. Furthermore, every root of inequality $v_{\pi_1}^T x \leqslant \alpha$ is, in fact, a root of inequality $w_{\pi_1}^T x \leqslant \beta$. Therefore, since by assumption inequality $v_{\pi_1}^T x \leqslant \alpha$ is facet-inducing for $\mathrm{MC}_k^{\leqslant}(n-1)$, we deduce the existence of a scalar $\lambda_1 > 0$ for which $\beta = \lambda_1 \alpha$ and

(a) $w_{\pi_1} = \lambda_1 v_{\pi_1}$.

Similarly, there exist scalars $\lambda_2, \lambda_3 > 0$ such that $\beta = \lambda_2 \alpha = \lambda_3 \alpha$ and

(b) $w_{\pi_2} = \lambda_2 v_{\pi_2}$ and

(c) $w_{\pi_3} = \lambda_3 v_{\pi_3}$.

Using assumption (i) one deduces easily that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. From (a) we obtain that

(d) $w_{ij} = \lambda v_{ij}$ for all ij with $i, j \notin \{1, j_1, j_2, j_3\}$, and from (b), that

(e) $w_{ij_1} = \lambda v_{ij_1}$ for all $i \notin \{1, j_1, j_2\}$,

and, in particular, $w_{j_1 j_3} = \lambda v_{j_1 j_3}$. From (a) we have that $w_{1j_2} + w_{j_1 j_2} = \lambda(v_{1j_2} + v_{j_1 j_2})$, implying that $w_{1j_2} = \lambda v_{1j_2}$. By symmetry we can conclude that $w = \lambda v$. □

(2.10) Remark. A trivial consequence of Theorem (2.9) is that zero-lifting preserves facethood for multicut polytopes, namely, if an inequality $v^T x \leqslant \alpha$ is facet-inducing for $\mathrm{MC}_k^{\leqslant}(n)$ (resp., $\mathrm{MC}_k^{\geqslant}(n)$), then it is also facet-inducing for $\mathrm{MC}_k^{\leqslant}(n+1)$ (resp., $\mathrm{MC}_k^{\geqslant}(n+1)$) provided $\alpha \neq 0$ or $n \geqslant 5$. (These conditions ensure that assertion (i) from Theorem (2.9) holds.) □

As an application of Theorem (2.9) we obtain the following lifting result for CW-inequalities.

(2.11) Theorem. *Take an integer* $r \geqslant 0$, *and an integral vector* $b = (b_1, \ldots, b_n)^T$ *with* $b_1, \ldots, b_p > 0$, $b_{p+1}, \ldots, b_n < 0$ *and assume that* $\sigma := \sum_{i=1}^n b_i \geqslant 2r + 1$ *holds. Let* j_1, j_2, j_3 *be distinct elements of* $[1, n]$ *and* d *be an integer such that* $b_{j_2} = b_{j_3} = d$. *We define the integer vector* $b' \in \mathbb{R}^{n+1}$ *by* $b'_{j_1} := b_{j_1} - d$, $b'_{n+1} := d$, *and* $b'_i := b_i$ *for* $i \in [1, n] \setminus \{j_1\}$ *and we denote by* p' (*resp.*, q') *the number of positive* (*resp.*, *nonpositive*) *coefficients* b'_i. *Assume that inequality* $\mathrm{CW}_{p,q}^r(b) \cdot x \leqslant \sigma(\sigma - 2r - 1)/2$ *is facet-inducing for* $\mathrm{MC}_k^{\leqslant}(n)$ (*resp.*, $\mathrm{MC}_k^{\geqslant}(n)$) *and that one of the following conditions holds*:

(i) $r = 0$,

(ii) $r \geqslant 1$, $d < 0$ *and* $b_{j_1} - d \leqslant 0$, *or*

(iii) $r \geqslant 1$, $d < 0$ *and* $b_{j_1} \geqslant r$.

Then inequality $\mathrm{CW}_{p',q'}^r(b') \cdot x \leqslant \sigma(\sigma - 2r - 1)/2$ *is facet-inducing for* $\mathrm{MC}_k^{\leqslant}(n+1)$ (*resp.*, $\mathrm{MC}_k^{\geqslant}(n+1)$).

Before showing (2.11) we first state and prove an observation on collapsed antiwebs.

(2.12) Lemma. *Given integers* $b_1, b'_1, b_2, \ldots, b_p \geqslant 1$ *with* $b_1 \neq b'_1$, *the following assertions are equivalent*:

(i) $\mathrm{AW}_p^r(b_1, b_2, \ldots, b_p) = \mathrm{AW}_p^r(b'_1, b_2, \ldots, b_p)$,

(ii) $b_1 \geqslant r$ *and* $b'_1 \geqslant r$.

Proof. It is clear that the antiwebs $\mathrm{AW}_p^r(b_1, b_2, \ldots, b_p)$ and $\mathrm{AW}_p^r(r, b_2, \ldots, b_p)$ coincide, if $b_1 \geqslant r$. Hence, (ii) implies (i).

If $b_1 \leqslant r$, then from Proposition (1.5) (i) $|\mathrm{AW}_p^r(b_1, \ldots, b_p) \cap \delta(\{1\})| = b_1(2r + 1 - b_1)$, since in the original pure antiweb AW_P^r, $P = \sum_{i=1}^p b_i$, from which $\mathrm{AW}_p^r(b_1, \ldots, b_p)$ comes by collapsing, the singleton $\{1\}$ corresponds to an interval of size $b_1 \leqslant r$. On the other hand, if $b_1 \geqslant r$, then from Proposition (1.5) (ii) $|\mathrm{AW}_p^r(b_1, b_2, \ldots, b_p) \cap \delta(\{1\})| \geqslant r(r+1)$. Therefore, if $\mathrm{AW}_p^r(b_1, b_2, \ldots, b_p) = \mathrm{AW}_p^r(b'_1, b_2, \ldots, b_p)$, then $b_1, b'_1 \geqslant r$, i.e., (i) implies (ii). □

PROOF OF THEOREM (2.11). Set $v^T x = \mathrm{CW}^r_{p,q}(b)$, $w^T x = \mathrm{CW}^r_{p',q'}(b')$, and $\rho := \sigma(\sigma - 2r - 1)/2$. The proof is based on Theorem (2.9) and hence consists of showing that the inequalities $w^T_{\pi_1} x \leqslant \rho$, $w^T_{\pi_2} x \leqslant \rho$, $w^T_{\pi_3} x \leqslant \rho$, obtained from $w^T x \leqslant \rho$ by collapsing, respectively, nodes $\{j_1, n+1\}$, nodes $\{j_1, j_2\}$, nodes $\{j_1, j_3\}$, are facet-defining.

In case (i), i.e., $r = 0$, inequality $w^T_{\pi_1} x \leqslant \rho$ coincides with inequality $v^T x \leqslant \rho$. Inequalities $w^T_{\pi_2} x \leqslant \rho$ and $w^T_{\pi_3} x \leqslant \rho$ are permutation equivalent to $v^T x \leqslant \rho$ and all three are henceforth facet-inducing.

In case (ii) we are in the same situation as in case (i) because, since nodes j_1, j_2, j_3 all belong to $[p+1, n]$, all three collapsings leave the antiweb $\mathrm{AW}^r_p(b_1, \ldots, b_p)$ (occurring in both inequalities $v^T x \leqslant \rho$, $w^T x \leqslant \rho$) unaffected.

In case (iii) we can argue similarly for the following reason. We have

$$\begin{aligned} w^T_{\pi_1} x &= \sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} - \sum_{ij \in \mathrm{AW}^r_p(b_1, \ldots, b_{j_1 - 1}, b_{j_1} - d, \ldots, b_p)} x_{ij} \\ &= \sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} - \sum_{ij \in \mathrm{AW}^r_p(b_1, \ldots, b_{j_1}, \ldots, b_p)} x_{ij} \\ &= v^T x, \end{aligned}$$

since from Lemma (2.12) the antiwebs $\mathrm{AW}^r_p(b_1, \ldots, b_{j_1} - d, \ldots, b_p)$ and $\mathrm{AW}^r_p(b_1, \ldots, b_{j_1}, \ldots, b_p)$ coincide because $b_{j_1} - d > b_{j_1} \geqslant r$. Therefore, all three collapsings are facet-inducing. This concludes the proof. □

(2.13) REMARK. (1) Observe that Theorem (2.11)(iii) remains valid under the condition that the antiwebs $\mathrm{AW}^r_p(b_1, \ldots, b_p)$ and $\mathrm{AW}^r_p(b'_1, \ldots, b'_r)$ coincide, but this condition is not more general than condition $b_{j_1} \geqslant r$ since, from Lemma (2.12) both antiwebs coincide if and only if $b_{j_1} \geqslant r$.

(2) In the case $r = 0$, Theorem (2.11)(i) also follows from Theorem 4.2 of [CR]. □

We now give some examples of applications of the lifting Theorem (2.11). It is not difficult to construct other classes of facets using Theorem (2.11).

(2.14) COROLLARY. *Take integers $b_1, \ldots, b_n$ with $\sigma := \sum_{i=1}^n b_i \geqslant 1$ such that $b_{j_1} = b_{j_2} = d$ for two distinct elements j_1, j_2 of $[1, n]$. If inequality $\mathrm{CW}^0_{p,q}(b_1, \ldots, b_n) \cdot x \leqslant \sigma(\sigma - 1)/2$ is facet-inducing for $\mathrm{MC}^{\leqslant}_k(n)$ (resp., $\mathrm{MC}^{\geqslant}_k(n)$), then inequality $\mathrm{CW}^0_{p+1,q+1}(b_1, \ldots, b_n, d, -d) \cdot x \leqslant \sigma(\sigma - 1)/2$ is facet-inducing for $\mathrm{MC}^{\leqslant}_k(n+2)$ (resp., $\mathrm{MC}^{\geqslant}_k(n+2)$).*

The proof follows directly from Theorem (2.11) and the fact that zero-lifting preserves facethood.

(2.15) COROLLARY. (i) *Given integers $b_1, b_2, \ldots, b_p > 0$, $q \geqslant 1$, $n = p + q$, and $\sigma := \sum_{i=1}^p b_i - q$ such that $2 \leqslant \sigma \leqslant p - 2$, inequality $\mathrm{CW}^0_{p,q}(b_1, \ldots, b_p, -1, \ldots, -1) \cdot x \leqslant \sigma(\sigma - 1)/2$ is facet-inducing for $\mathrm{MC}^{\leqslant}_{\sigma+2}(n)$ and for $\mathrm{MC}^{\geqslant}_{p-1}(n)$.*

(ii) *Given integers $b_1, \ldots, b_q < 0$, $q \geqslant 1$, $n = p + q$, and $\sigma := p + \sum_{i=1}^q b_i \geqslant 2$, inequality $\mathrm{CW}^0_{p,q}(1, \ldots, 1, b_1, \ldots, b_q) \cdot x \leqslant \sigma(\sigma - 1)/2$ is facet-inducing for $\mathrm{MC}^{\leqslant}_{\sigma+2}(n)$ and for $\mathrm{MC}^{\geqslant}_{q+\sigma-1}(n)$.*

PROOF. For assertion (i) we apply Theorem (2.11)(i) with $d = -1$ starting with inequality $\mathrm{CW}^0_{p,p-\sigma}(1, \ldots, 1, -1, \ldots, -1) \cdot x \leqslant \sigma(\sigma - 1)/2$ which from Theorem (1.20) is facet-inducing for $\mathrm{MC}^{\leqslant}_{\sigma+2}$ and $\mathrm{MC}^{\geqslant}_{p-1}$.

For showing assertion (ii) we apply Theorem (2.11)(i) with $d = 1$ starting with inequality $\mathrm{CW}^0_{q+\sigma,q}(1, \ldots, 1, -1, \ldots, -1) \cdot x \leqslant \sigma(\sigma - 1)/2$ which from Theorem (1.20) is facet-inducing for $\mathrm{MC}^{\leqslant}_{\sigma+2}$ and $\mathrm{MC}^{\geqslant}_{q+\sigma-1}$. □

Observe that, using relation (2.5), the only undecided values of k for facethood over $MC_k^{\leqslant}(n)$ of both inequalities from Corollary (2.15)(i) and (ii) are $k = \sigma, \sigma + 1$, while the undecided values of k for facethood for $MC_k^{\geqslant}(n)$ are $k = p$ for the inequality from Corollary (2.15)(i) and $k \in [q + \sigma, p]$ for the inequality from Corollary (2.15)(ii). In fact, using the fact that $CW_{p,q}^0 \cdot x \leqslant (p-q)(p-q-1)/2$ is facet-inducing for $MC_{p-q+1}^{\leqslant}(n)$ (recall Remarks (1.4) (ii) and (1.21)), one can improve the result and state facethood of both inequalities from Corollary 2.15 (i) and (ii) for $MC_{\sigma+1}^{\leqslant}(n)$, as proved in [CR2].

(2.16) COROLLARY. *Given integers* $b_1, b_2, \ldots, b_p \geqslant r \geqslant 1$, $n = p + q$, *and* $q = \sum_{1 \leqslant i \leqslant p} b_i - 2r - 1$, *inequality*

$$CW_{p,q}^r(b_1, \ldots, b_p, -1, \ldots, -1) \cdot x \leqslant 0$$

is facet-inducing for $MC_2^{\leqslant}(n)$ *for all* $p \geqslant 5$.

PROOF. Apply recursively Theorem (2.11)(iii) (with $d = -1$) starting with inequality $CW_{p,q}^r(r, \ldots, r, -1, \ldots, -1)$ which is facet-inducing for $MC_2^{\leqslant}$ for $p \geqslant 5$, $r \geqslant 1$, $pr - q = 2r + 1$ (Theorem 2.2, [DL2]). □

(2.17) COROLLARY. *Inequality* $CW_{p,p+a-6}^2(a, 1, \ldots, 1, -1, \ldots, -1) \cdot x \leqslant 0$ *is facet-inducing for* $MC_2^{\leqslant}(n)$ *for all* $p \geqslant 7$, $a \geqslant 2$, $n = 2p + a - 6$.

PROOF. We apply Theorem (2.11)(iii) with $d = -1$, $b_{j_1} = 2$, i.e., $j_1 = 1$, starting with inequality $CW_{p,p-4}^2(2, 1, \ldots, 1, -1, \ldots, -1) \cdot x \leqslant 0$ which is facet-inducing for $MC_2^{\leqslant}$ for all $p \geqslant 7$ (cf. Theorem 2.3, [DL2]). □

References

[A] Alon, N. (1990). The CW-Inequalities for Vectors in l_1. *European J. Combinatorics* **11** 1–5.

[BGJR] Barahona, F., Grötschel, M., Jünger, M. and Reinelt, G. (1988). An Application of Combinatorial Optimization to Statistical Physics and Circuit Layout Design. *Oper. Res.* **36** 493–513.

[BGM] ______, ______ and Mahjoub, A. R. (1985). Facets of the Bipartite Subgraph Polytope. *Math. Oper. Res.* **10** 340–358.

[BM] ______ and Mahjoub, A. R. (1986). On the Cut Polytope. *Math. Programming* **36** 157–173.

[CR1] Chopra, S. and Rao, M. R. (1989). The Partition Problem. Submitted manuscript.

[CR2] ______ and ______ (1989). Facets of the k-Partition Polytope. Submitted manuscript.

[DGL] Deza, M., Grishukhin, V. P. and Laurent, M. (1991). The Symmetries of the Cut Polytope and of Some Relatives. P. Gritzmann and B. Sturmfels (Eds.), *Applied Geometry and Discrete Mathematics—The Victor Klee Festschrift*. DIMACS Series in Discrete Mathematics and Theoretical Computer Science 4 205–220.

[DL1] ______ and Laurent, M. (1992). Facets for the Cut Cone I. *Math. Programming*, to appear.

[DL2] ______ and ______ (1992). Facets for the Cut Cone II. *Math. Programming*, to appear.

[DDL] ______, De Simone, C. and Laurent, M. (forthcoming). Collapsing and Lifting for the Cut Cone. IASI Report no. 265, Rome (July). *Proceedings of the Second Japan Conference on Graph Theory and Combinatorics* (Hakone 1990).

[C] Gerards, A. M. H. (1985). Testing the Odd Bicycle Wheel Inequalities for the Bipartite Subgraph Polytope. *Math. Oper. Res.* **10** 359–360.

[GLS] Grötschel, M., Lovász, L. and Schrijver, A. (1988). *Geometric Algorithms and Combinatorial Optimization*. Springer, Heidelberg.

[GP] ______ and Pulleyblank, W. R. (1981). Weakly Bipartite Graphs and the Max-Cut Problem. *Oper. Res. Lett.* **1** 23–27.

[GW1] ______ and Wakabayashi, Y. (1990). Facets of the Clique Partitioning Polytope *Math. Programming* **47** 367–387.

[GW2] Grötschel, M. and Wakabayashi, Y. (1990). Composition of Facets of the Clique Partitioning Polytope. In: *Topics in Combinatorics and Graph Theory. Essays in Honor of Gerhard Ringel.* R. Bodendiek, R. Henn (Eds.), Physica, Heidelberg, 277–284.

[GW3] ______ and ______ (1989). A Cutting Plane Algorithm for a Clustering Problem. *Math. Programming* **B45** 59–96.

[W] Wakabayashi, Y. (1986). Aggregation of Binary Relations: Algorithmic and Polyhedral Investigations. Ph.D. Thesis, Universität Augsburg, Germany.

DEZA: 17 PASSAGE DE L'INDUSTRIE, 75010 PARIS, FRANCE

GRÖTSCHEL: KONRAD-ZUSE-ZENTRUM, HEILBRONNER STR. 10, D-1000 BERLIN 31, GERMANY

LAURENT: UNIVERSITÉ PARIS IX, LAMSADE, PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY, 75775 PARIS CEDEX 16, FRANCE