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Collapsing and lifting for the cut cone

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Abstract

The cut polytope $P_C(G)$ of a graph G is the convex hull of the incidence vectors of all cuts of G; the cut cone $C(G)$ of G is the cone generated by the incidence vectors of all cuts of G. We introduce the operation of collapsing an inequality valid over the cut cone $C(K_n)$ of the complete graph with n vertices: it consists of identifying vertices and adding the weights of the corresponding incident edges. Using collapsing and its inverse operation (lifting), we give several methods to find facets of $C(K_n)$. We also show how to construct facets of $C(K_n)$ from the difference of inequalities valid over $C(K_n)$. When G is an induced subgraph of a graph H, we give sufficient conditions to derive inequalities defining facets of $P_C(H)$ from inequalities defining facets of $P_C(G)$. Finally, the description (up to permutation) of the cut cone $C(K_7)$ is given.

1. Introduction and preliminaries

We use the standard graph-theoretical terminology as in [9, 10]. An edge with endpoints i and j in an undirected graph will be denoted by ij (or ji). The complete graph on n vertices is denoted by K_n. Let $G=(V,E)$ be a graph, and let S be a (possibly empty) subset of V. The *cut* corresponding to S is the set $\delta(S)$ of edges with exactly one endpoint in S. (In particular, we allow $S=\emptyset$, in which case $\delta(S)$ is a zero vector.) Throughout this paper, we shall let $\delta(S)$ stand for both a cut and its incidence vector. The *cut cone* $C(G)$ of a graph G is the cone generated by the incidence vectors of all

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edge sets of cuts of G; the *cut polytope* $P_C(G)$ of a graph G is the convex hull of the incidence vectors of all edge sets of cuts of G. For every graph G, the cone $C(G)$ and the polytope $P_C(G)$ are full dimensional. As usual, we let B^A denote the set of all mappings from A to B; elements of B^A can be thought of as vectors whose components are subscripted by elements of A and take values in B.

Let $G=(V,E)$ be a graph, and let v be a vector in $\mathbb{R}^E$. If the inequality $v^{\mathrm{T}}x \leqslant 0$ is satisfied by all points in $C(G)$ or, equivalently, by all cut vectors $\delta(S)$, we say that the inequality $v^{\mathrm{T}}x \leqslant 0$ is valid over $C(G)$. The face defined by the inequality $v^{\mathrm{T}}x \leqslant 0$ is the set $F_v = \{x \in C(G)\colon v^{\mathrm{T}}x = 0\}$. A *root* of the vector v is a nonzero cut vector which belongs to F_v. The dimension of a face F_v, denoted by $\dim(F_v)$, is the largest number of affinely independent points in F_v minus one or, equivalently, the largest number of linearly independent roots of v (since F_v contains the zero vector). The codimension of a face F_v is equal to $\binom{n}{2} - \dim(v)$. A facet of $C(G)$ is a face of dimension $|E|-1$.

For every graph $G=(V,E)$, and for every vector v in $\mathbb{R}^E$, we define a graph $G(v)$ as follows: its edges are all the edges ij in G for which $v_{ij} \neq 0$, and its vertices are all the endpoints of these edges; to every edge ij, the weight v_{ij} is assigned. The graph $G(v)$ is called the *supporting graph* of v. Let $v^{\mathrm{T}}x \leqslant 0$ be an inequality valid over $C(G)$. If all nonzero components of v are ± 1, then we say that the inequality $v^{\mathrm{T}}x \leqslant 0$ is *pure*. As usual, a vector with components all equal to zero will be denoted by $\underline{0}$.

When G is the complete graph K_n with n vertices, the corresponding cut cone will be denoted by C_n. Points in C_n can be interpreted as semi-metrics on n points; in fact, C_n coincides with the family of all the semi-metrics on n points which are isometrically embeddable into L^1; in this context, the study of the cut cone C_n was started in 1960 by Deza [12]. (For more informations, see for instance [2, 6, 13, 14, 24].)

We now describe two classes of inequalities valid over the cone C_n. The first class is the class of *hypermetric inequalities* which were introduced by Deza [12] and later, independently, by Kelly [22]. For every integer row vector, $b=(b_1, \ldots, b_n)$ such that $b_1 + \cdots + b_n = 1$, the hypermetric inequality specified by the vector b is the inequality

$$\sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} \leqslant 0. \tag{1}$$

We refer to each inequality (1) as Hyp(b). Write $b(S) = \sum_{i \in S} b_i$. To see that (1) is valid, observe that

$$\sum_{ij \in \delta(S)} b_i b_j = \sum_{i \in S} b_i \sum_{j \notin S} b_j = b(S)(1 - b(S))$$

and that $t(1-t) \leqslant 0$ for all integers t. Let v be the vector defined by $v_{ij} = b_i b_j$ for all ij. Note that every root of the vector v is a cut for which $b(S)$ is equal to zero or one.

An hypermetric inequality that will play a special role in our paper is the inequality

$$x_{ij} - x_{ik} - x_{jk} \leqslant 0;$$

we shall refer to such inequality as *triangle inequality*. It is easy to verify that, for $n \geqslant 3$, every triangle inequality defines a facet of C_n.

The second class of inequalities valid over C_n is the class of *cycle inequalities* which were introduced by Deza and Laurent [17]. To specify these inequalities, we need one more definition. Let f be an integer greater than or equal to three; a cycle $C=(1,2,\ldots,f)$ is the graph with vertices $1,2,\ldots,f$ and edges $12,23,\ldots,f1$. For every cycle C, $E(C)$ denotes the set of its edges. Let $b=(b_1,\ldots,b_n)$ be an integer row vector such that $b_1+\cdots+b_n=3$; order the components of b in such a way that $b_1,b_2,\ldots,b_f>0\geqslant b_{f+1},\ldots,b_n$. Then the cycle inequality specified by the vector b is the inequality

$$\sum_{1\leqslant i<j\leqslant n} b_i b_j x_{ij} - \sum_{ij\in E(C)} x_{ij}\leqslant 0; \tag{2}$$

we shall refer to each inequality (2) as Cyc(b).

For every nonnegative integer n, we let $[1,n]$ denote the set $\{1,\ldots,n\}$, and we let N stand for $\binom{n}{2}$. In the following, we describe two operations on an inequality valid over the cone C_n: permutation and switching.

Let v be a vector in $\mathbb{R}^N$. For every permutation σ of the set $[1,n]$, we define a vector v^σ in $\mathbb{R}^N$ by

$$v^\sigma_{ij}=v_{\sigma(i)\sigma(j)} \quad \text{for every } 1\leqslant i<j\leqslant n;$$

we shall say that v^σ has been obtained from v via the *permutation* σ or that v^σ is permutation equivalent to v. Clearly, the inequality $v^{\mathrm{T}}x\leqslant 0$ is valid over C_n if and only if the inequality $(v^\sigma)^{\mathrm{T}}x\leqslant 0$ is valid over C_n. It is easy to verify that if $\mathrm{Hyp}(b_1,\ldots,b_n)$ is an hypermetric facet-defining inequality of C_n, then for every permutation σ of the set $[1,n]$, the inequality $\mathrm{Hyp}(b_{\sigma(1)},\ldots,b_{\sigma(n)})$ defines a facet of C_n. However, if $\mathrm{Cyc}(b_1,\ldots,b_n)$ is a cycle facet-defining inequality of C_n, then the inequality $\mathrm{Cyc}(b_{\sigma(1)},\ldots,b_{\sigma(n)})$ does not define a facet of C_n for every permutation σ of the set $[1,n]$ [17].

The second operation, called *switching*, relates the cut polytope of a graph G with the cut cone of G in the following sense. Since $P_C(G)\subset C(G)$, every inequality valid over $C(G)$ is also valid over $P_C(G)$. Moreover, every facet-defining inequality of $C(G)$ is facet-defining inequality of $P_C(G)$. In fact, the switching operation will show that looking for all facets of $P_C(G)$ is equivalent to looking for all facets of $C(G)$. To describe this operation, consider a graph $G=(V,E)$ and let v be a vector in $\mathbb{R}^E$. For every subset S of V, we define a vector v^S in $\mathbb{R}^E$ by

$$v^S_{ij}=\begin{cases} v_{ij} & \text{if } ij\notin\delta(S),\\ -v_{ij} & \text{otherwise,}\end{cases}$$

we shall say that the vector v^S has been obtained from v by switching the cut $\delta(S)$. Write

$$d=-\sum_{ij\in\delta(S)} v_{ij}.$$

For the case $G=K_n$, Deza [12] (see also [17]) showed that for every vector v in $\mathbb{R}^N$ and for every root $\delta(S)$ of v, the inequality $v^T x \leqslant 0$ defines a facet of C_n if and only if the inequality $(v^S)^T x \leqslant 0$ defines a facet of C_n. For a general graph $G=(V,E)$, Barahona and Mahjoub [8] showed that for every vector v in $\mathbb{R}^E$ and for every cut $\delta(S)$, $v^T x \leqslant b$ defines a facet of $P_C(G)$ if and only if the inequality $(v^S)^T x \leqslant b-d$ defines a facet of $P_C(G)$. Furthermore, they showed that every inequality defining a facet of $P_C(G)$ can be obtained for some inequality defining a facet of $C(G)$ by switching a cut [8]. In [15], it was shown that switching and permutation are the only symmetries of $P_C(K_n)$.

In Section 2, we introduce two operations on an inequality valid over C_n: collapsing and expansion; collapsing an inequality consists of identifying vertices and adding the weights of the corresponding incident edges; the expansion of an inequality is the inverse operation of collapsing.

In Sections 3–5, we give several results on lifting. Lifting is a commonly used technique in polyhedral combinatorics to derive inequalities defining facets of a polyhedron in $\mathbb{R}^n$ from inequalities defining facets of a polyhedron in $\mathbb{R}^{n'}$ with $n'<n$ (see for instance [23]).

Let $G=(V,E)$ and $H=(W,F)$ be two graphs where the former is an induced subgraph of the latter, and let v be a vector in $\mathbb{R}^E$. *Lifting* the vector v means to find a vector v' in $\mathbb{R}^F$ such that the following two conditions hold:

- if $v^T x \leqslant 0$ is valid over $C(G)$, then $(v')^T x \leqslant 0$ is valid over $C(H)$;
- if $v^T x \leqslant 0$ defines a facet of $C(G)$, then $(v')^T x \leqslant 0$ defines a facet of $C(H)$.

If $v'=(v,\underline{0})$ where $\underline{0}$ is the vector in $\mathbb{R}^{F-E}$ with components all equal to zero, then we shall say that v' was obtained from v by *zero-lifting*.

Finally, in Section 6, we give the complete description of the cut cone C_7.

2. Collapsing and expansion

Let n be an integer greater than or equal to two, and let k be an integer such that $1 \leqslant k \leqslant n-1$. Recall that N stands for $\binom{n}{2}$. For every partition, $\pi=\{V_1,\ldots,V_k\}$ of the set $[1,n]$ into k nonempty subsets, and for every vector v in $\mathbb{R}^N$, we define a vector v^π in $\mathbb{R}^{\binom{k}{2}}$ by

$$v^\pi_{ij} = \sum_{s \in V_i,\, t \in V_j} v_{st} \quad \text{for all } 1 \leqslant i < j \leqslant k.$$

We call the vector v^π the *π-collapsing* of v. If $k=n-1$, then precisely one of the k subsets of $[1,n]$, say V_1, has size two, all the others have size one; in this case, if $V_1=\{i,j\}$ then we denote the vector v^π simply by $v^{i,j}$, and we call the vector $v^{i,j}$ the *(i,j)-collapsing* of v. The π-collapsing of an inequality $v^T x \leqslant 0$ is the inequality $(v^\pi)^T x \leqslant 0$.

The π-collapsing of an hypermetric inequality $\mathrm{Hyp}(b_1,\ldots,b_n)$ can be easily obtained in the following way: define a vector v in $\mathbb{R}^N$ by writing $b_i b_j$ for v_{ij}. Clearly, for distinct

i and j in $[1,n]$, the (i,j)-collapsing of the vector v is the vector $v^{i,j}$ given by

$$v_{hk}^{i,j}=\begin{cases}(b_i+b_j)b_k & \text{if } h=i,\ k\in[1,n]-\{i,j\},\\ b_hb_k & \text{if } h,k\in[1,n]-\{i,j\}.\end{cases}$$

Now define a vector $b^{i,j}$ in $\mathbb{R}^{\binom{n-1}{2}}$ by

$$b_h^{i,j}=\begin{cases}b_i+b_j & \text{if } h=i,\\ b_h & \text{if } h\in[1,n]-\{i,j\}.\end{cases}$$

Since $b_1^{i,j}+\cdots+b_n^{i,j}=1$, the inequality $(v^{i,j})^{\mathrm{T}}x\leqslant 0$ is an hypermetric inequality. We call the vector $b^{i,j}$ the (i,j)-collapsing of the vector b. For instance, if $b=(1,1,1,-1,-1)$ then the $(1,2)$-collapsing of b is the vector $(2,1,-1,-1)$.

Proposition 2.1. *Let π be a partition of the set $[1,n]$ into k nonempty subsets $(1\leqslant k\leqslant n-1)$. If the inequality $v^{\mathrm{T}}x\leqslant 0$ is valid over C_n, then the inequality $(v^{\pi})^{\mathrm{T}}x\leqslant 0$ is valid over C_k.*

Proof. Write $\pi=(V_1,\ldots,V_k)$; let S be a subset of $[1,k]$; and set $S'=\bigcup_{i\in S}V_i$. Clearly, S' is a subset of $[1,n]$. Now it is easy to verify that $(v^{\pi})^{\mathrm{T}}\delta(S)=v^{\mathrm{T}}\delta(S')$. □

Let $G=(V,E)$ be a graph; the concept of the π-collapsing of a vector v can be extended to the case when v is a vector in $\mathbb{R}^E$ in the following sense. For every partition $\pi=\{V_1,\ldots,V_k\}$ of V, let E' be the set of edges of the graph G' obtained from G by identifying all the vertices in each V_i into a single vertex (multiple edges are deleted). Define the π-collapsing of v as the vector v^{π} in $\mathbb{R}^{E'}$ given by

$$v_{ij}^{\pi}=\sum_{hk\in E,\,h\in V_i,\,k\in V_j}v_{hk}\quad\text{for all } ij\in E'.$$

Clearly, if $v^{\mathrm{T}}x\leqslant 0$ is valid over $C(G)$, then the inequality $(v^{\pi})^{\mathrm{T}}x\leqslant 0$ is valid over $C(G')$.

Let Σ be the set of all partitions of the set $[1,n]$, and let v be a vector in $\mathbb{R}^N$. For every π in Σ, we denote by $\bar{v}^{\pi}$ the vector in $\mathbb{R}^N$ defined by

$$\bar{v}^{\pi}=(v^{\pi},\underline{0}).$$

The vector $\bar{v}^{\pi}$ is a zero-lifting of v^{π}. Let $L(v)=\{\bar{v}^{\pi}\colon \pi\in\Sigma\}$. It is easy to verify that $L(v)$ is a lattice isomorphic to the set of all the partitions of $[1,n]$; the order of $L(v)$ is the following: for all partitions $\pi=\{V_1,\ldots,V_k\}$ and $\pi'=\{W_1,\ldots,W_h\}$ in Σ, $\bar{v}^{\pi}\geqslant\bar{v}^{\pi'}$ if and only if for every i in $\{1,\ldots,k\}$, $V_i\subseteq W_j$ for some j in $\{1,\ldots,h\}$. Note that the greatest element of the lattice $L(v)$ is v, and that the smallest element of $L(v)$ is $\underline{0}$ (zero vector corresponding to the trivial partition $\pi=\{[1,n]\}$). For every vector v, we call the lattice $L(v)$ the *collapsing lattice* of v.

Let v be a vector in $\mathbb{R}^{\binom{n}{2}}$, and let v' be a vector in $\mathbb{R}^{\binom{n'}{2}}$, with $n'>n$. If v is a π-collapsing of v' for some partition π of $[1,n']$, then we say that v' is an *expansion* of v. Not every expansion of an inequality valid over the cut cone C_n is valid over $C_{n'}$. In fact, every inequality $v^{\mathrm{T}}x\leqslant 0$ valid over C_n admits an expansion which is not valid over some cut cone containing C_n. For instance, let k be an arbitrary element in $[1, n]$; define a vector v' by

$$v'_{ij} = \begin{cases} 1 & \text{if } i=n+1,\ j=k, \\ 0 & \text{if } i \doteq n+1,\ j \neq k, \\ v_{ij} & \text{otherwise.} \end{cases}$$

Note that the vector v is the $(k, n+1)$-collapsing of V', and that the inequality $(v')^T x \leqslant 0$ is not valid over C_{n+1}. On the other hand, every inequality which is valid over C_n admits an expansion which is valid over C_{n+1}: its zero-lifting.

Let v be a vector in $\mathbb{R}^N$, and let v' be an expansion of v. If the inequality $(v')^T x \leqslant 0$ is pure, then we say that $(v')^T x \leqslant 0$ is a *purification* of the inequality $v^T x \leqslant 0$. Every inequality $v^T x \leqslant 0$ valid over C_n admits a purification which is valid over some cut cone containing C_n. If v is not pure then some coefficient v_{hk} is greater than one in modulo. Without loss of generality, we can assume that $v_{hk} > 1$; define a vector v' in $\mathbb{R}^{\binom{n+1}{2}}$ by

$$v'_{ij} = \begin{cases} v_{hk} - 1 & \text{if } i=h,\ j=k, \\ 1 & \text{if } i=n+1,\ j=h, \\ -1 & \text{if } i=n+1,\ j=k, \\ 0 & \text{if } i=n+1,\ j \neq h, k, \\ v_{ij} & \text{otherwise.} \end{cases}$$

Clearly, the vector v is the $(k, n+1)$-collapsing of v'. Now, let S be a subset of $[1, n+1]$; without loss of generalilty, we can assume that $n+1 \notin S$. Clearly, if h and k are both in S or both not in S then $(v')^T \delta(S) \leqslant 0$ (since$(v')^T \delta(S) = v^T \delta(S)$); if $h \in S$ and $k \notin S$ then $(v')^T \delta(S) = v^T \delta(S) - v_{hh} + v_{hk} - 1 + 1$, and so $(v')^T \delta(S) \leqslant 0$; if $h \notin S$ and $k \in S$ then

$$(v')^T \delta(S) = v^T \delta(S) - v_{hk} + v_{hk} - 1 - 1,$$

and so $(v')^T \delta(S) < 0$. Hence, the inequality $(v')^T x \leqslant 0$ is valid over C_{n+1}. Repeating this procedure on v' will yield a purification of the vector v.

Let $\pi = \{V_1, \ldots, V_k\}$ be a partition of the set $[1, n]$, and let S be a subset of $[1, n]$. We say that the cut $\delta(S)$ is *compatible* with the partition π if, for every $i = 1, \ldots, k$, each V_i is a subset of S whenever $S \cap V_i \neq 0$. Recall that, for every vector v in $\mathbb{R}^N$, the vector v^π denotes the π-collapsing of v and v^S denotes the vector obtained from v by switching the cut $\delta(S)$.

Proposition 2.2. *Let* $\pi = \{V_1, \ldots, V_k\}$ *be a partition of the set* $[1, n]$, *and let* S *be a subset of* $[1, n]$. *If the cut* $\delta(S)$ *is compatible with* π *then* $(v^\pi)^S = (v^S)^\pi$.

Proof. Let i and j be two distinct elements in $[1, n]$. Clearly, it is sufficient to show that the vector obtained from the (i, j)-collapsing of v by switching the cut $\delta(S)$ is the

(i,j)-collapsing of the vector obtained from v by switching the cut $\delta(S)$, i.e. $(v^{i,j})^S=(v^S)^{i,j}$. Let π be the corresponding partition of $[1,n]$. Set $x=v^S$, $y=x^{i,j}$, $z=v^{i,j}$, and $w=z^S$. Since the cut $\delta(S)$ is compatible with the partition π, $ij\notin\delta(S)$, and so we may assume that both i and j are in S. Let h and k be distinct elements in $[1,n]-\{i,j\}$; we have

$$y_{ik}=x_{ik}+x_{jk}=\begin{cases}-v_{ik}-v_{jk} & \text{if } k\notin S,\\ v_{ik}+v_{jk} & \text{if } k\in S,\end{cases}$$

$$y_{hk}=x_{hk}=\begin{cases}-v_{hk} & \text{if } hk\in\delta(S),\\ v_{hk} & \text{if } hk\notin\delta(S),\end{cases}$$

$$w_{ik}=\begin{cases}-z_{ik}=-v_{ik}-v_{jk} & \text{if } k\notin S,\\ z_{ik}=v_{ik}+v_{jk} & \text{if } k\in S,\end{cases}$$

$$w_{hk}=\begin{cases}-z_{hk}=-v_{hk} & \text{if } hk\in\delta(S),\\ z_{hk}=v_{hk} & \text{if } hk\notin\delta(S).\end{cases}\qquad\square$$

We refer to [16] for an extension of the notion of collapsing for the multicut polytope.

3. Zero-lifting

In this section, we consider two graphs $G=(V,E)$ and $H=(W,F)$ where the former is an induced subgraph of the latter. Let v be a vector in $\mathbb{R}^E$, and let $\underset{\sim}{0}$ be a vector in $\mathbb{R}^{F-E}$ with components all equal to zero. It is easy to see that if the inequality $v^{\mathrm{T}}x\leqslant d$ is valid over $P_C(G)$, then the inequality $(v,\underset{\sim}{0})^{\mathrm{T}}x\leqslant d$ is valid over $P_C(H)$. Conversely, if the inequality $(v,\underset{\sim}{0})^{\mathrm{T}}x\leqslant d$ is valid over $P_C(H)$, then the inequality $v^{\mathrm{T}}x\leqslant d$ is valid over $P_C(G)$. This is a special case of the following observation.

Theorem 3.1 (De Simone [11]). *If $(v,\underset{\sim}{0})^{\mathrm{T}}x\leqslant d$ defines a facet of $P_C(H)$ then $v^{\mathrm{T}}x\leqslant d$ defines a facet of $P_C(G)$.*

Consider the following problem:
given an inequality $v^{\mathrm{T}}x\leqslant d$ defining a facet of $P_C(G)$,
is the inequality $(v,\underset{\sim}{0})^{\mathrm{T}}x\leqslant d$ defining a facet of $P_C(H)$? (3)

Barahona and Mahjoub [8] showed that for the inequalities

$$x_{ij}\geqslant 0, \qquad x_{ij}\leqslant 1, \tag{4}$$

the answer to (3) is 'yes' if and only if ij does not belong to any triangle of H. In addition, they showed that the answer to (3) is again 'yes' for every other inequality they studied in [8].

We say that an inequality $v^{\mathrm{T}}x \leqslant d$ is nontrivial if the supporting graph $G(v)$ of v has more than two vertices. Note that the supporting graphs of the inequalities (4) have precisely two vertices. De Simone [11] gave a sufficient condition on the graphs G and H under which problem (3) has a positive answer for all the inequalities of the linear description of $P_C(G)$, with the exception of (4).

Theorem 3.2 (De Simone [11]). *Let $G=(V,E)$ be a graph with n vertices; let $n \geqslant 3$, and let $H=(V \cup \{r\}, F)$. If $N(r)-\{v\} \subseteq N(v)$ for some vertex v in G then problem* (3) *has a positive answer for every nontrivial inequality defining a facet of $P_C(G)$.*

Corollary 3.3. (De Simone [11]; Deza and Laurent [17]). *Let G be a complete graph with n vertices and let $n \geqslant 3$. Then $v^{\mathrm{T}}x \leqslant d$ defines a facet of $P_C(G)$ if and only if $[v, \underline{0}]^{\mathrm{T}}x \leqslant d$ defines a facet of the cut polytope of every complete graph with more than n vertices.*

Now consider the graph K_n with $n \geqslant 3$. Write $K_n=(V, E)$ and let v be a vector in $\mathbb{R}^E$. Recall that, for every vector v, $G(v)$ denotes the supporting graph of v. Clearly, if $G(v)=(V', E')$ is a partial subgraph of K_n then the vector v can be written as

$$v=(v', \underline{0}), \quad \text{with } v' \in \mathbb{R}^{E'} \text{ and } \underline{0} \in \mathbb{R}^{E'-E}.$$

Theorem 3.4. *Let $K_n=(V, E)$ with $n \geqslant 3$; let E' be a subset of E, and let v be a vector in $\mathbb{R}^E$ such that $v=(v', \underline{0})$, with v' in $\mathbb{R}^{E'}$, and such that $G(v)=(V', E')$. If $v^{\mathrm{T}}x \leqslant d$ defines a facet of $P_C(K_n)$ then, for every subgraph $H=(W, F)$ of K_n containing $G(v)$, the inequality*

$$(v', \underline{0})^{\mathrm{T}}x \leqslant d,$$

with $v' \in \mathbb{R}^{E'}$ and $\underline{0} \in \mathbb{R}^{F-E'}$, defines a facet of $P_C(H)$.

Proof. Suppose the contrary: there exists a partial subgraph H of K_n containing $G(v)$ such that $(v', \underline{0})^{\mathrm{T}}x \leqslant d$ does not define a facet of $P_C(H)$. Then $(v', \underline{0})^{\mathrm{T}}x \leqslant d$ can be obtained as sum of two other inequalities valid over $P_C(H)$, say $v_1^{\mathrm{T}}x \leqslant d_1$ and $v_2^{\mathrm{T}}x \leqslant d_2$. But the inequalities $[v_1, \underline{0}]^{\mathrm{T}}x \leqslant d_1$ and $[v_2, \underline{0}]^{\mathrm{T}}x \leqslant d_2$, with $\underline{0} \in \mathbb{R}^{E-F}$, are valid over $P_C(K_n)$ and their sum is $[v', \underline{0}]^{\mathrm{T}}x \leqslant d$, contradicting the fact that $[v', \underline{0}]^{\mathrm{T}}x \leqslant d$ defines a facet of $P_C(K_n)$. □

An instant corollary of Theorem 3.4 is the following.

Corollary 3.5. *Let $G=(V, E)$ be the complete graph with n vertices; let $n \geqslant 3$; and let $H=(W, F)$ be a graph containing G as induced subgraph. Then problem* (3) *has a positive answer for every facet-defining inequality of $P_C(G)$.*

Proof. Let $v^{\mathrm{T}}x \leqslant d$ be a facet-defining inequality of $P_C(G)$. Let T denote the set of edges of the complete graph with $|W|$ vertices. Corollary 3.3 implies that the inequality $(v, \underline{0})^{\mathrm{T}}x \leqslant d$, with $\underline{0} \in \mathbb{R}^{T-E}$, defines a facet of $P_C(K_{|W|})$. Since H is a partial subgraph of $K_{|W|}$, the corollary follows from Theorem 3.4. □

We end this section by considering a generalization of problem (3).
Given an inequality $v^{\mathrm{T}}x \leqslant d$ *defining a face of* $P_C(G)$ *of codimension* r,
is the inequality $(v, \underline{0})^{\mathrm{T}}x \leqslant d$ *defining a face of* $P_C(H)$ *of codimension* r?
The answer to the above problem is, in general, 'no'. For instance, consider the vector $b = (1, 1, -1, -1)$ and define a vector v by $v_{ij} = b_i b_j$ $(1 \leqslant i < j \leqslant 4)$. It is easy to verify that while the inequality $v^{\mathrm{T}}x \leqslant 0$ defines a face of $P_C(K_4)$ of codimension four, the inequality $(v, \underline{0})^{\mathrm{T}}x \leqslant 0$, with $\underline{0} \in \mathbb{R}^4$, defines a face of $P_C(K_5)$ of codimension five.

4. Nonzero lifting

In this section, we consider a complete graph with n vertices, $n \geqslant 5$. Recall that N stands for $\binom{n}{2}$. Let v be a vector in $\mathbb{R}^N$, and let $v^{\mathrm{T}}x \leqslant 0$ be a facet-defining inequality of C_n. In Section 3, we have seen that this inequality can be lifted to a facet-defining inequality of $C_{n'}$, with $n' > n$, by just adding zeroes (Corollary 3.3). In this section, we study the more general lifting problem. For this purpose, recall that for distinct i and j in $[1, n]$, the vector $v^{i,j}$ denotes the (i, j)-collapsing of v.

Theorem 4.1. *Let v be a vector in $\mathbb{R}^N$ satisfying the following three conditions:*
(i) *there exists p in $[1, n]$ such that $\sum_{j \in [1,n]-\{p\}} v_{pj} = 0$ ($\delta(\{p\})$ is a root of v);*
(ii) *there exist distinct h and k in $[1, n] - \{p\}$ such that both inequalities $(v^{p,h})^{\mathrm{T}}x \leqslant 0$ and $(v^{p,k})^{\mathrm{T}}x \leqslant 0$ define facets of C_{n-1};*
(iii) *there exist distinct i and j in $[1, n] - \{p, h, k\}$ such that $v_{ij} \neq 0$.*
If the inequality $v^{\mathrm{T}}x \leqslant 0$ is valid over C_n then it defines a facet of C_n.

Proof. Suppose the contrary: $v^{\mathrm{T}}x \leqslant 0$ does not define a facet of C_n. Then $v^{\mathrm{T}}x \leqslant 0$ is the sum of two inequalities, say $u^{\mathrm{T}}x \leqslant 0$ and $w^{\mathrm{T}}x \leqslant 0$ (with $u \neq \underline{0}$ and $w \neq \underline{0}$), valid over C_n. Let $u^{p,h}$ and $u^{p,k}$ be the (p, h)-collapsing and the (p, k)-collapsing of the vector u, respectively; similarly, let $w^{p,h}$ and $w^{p,k}$ be the (p, h)-collapsing and the (p, k)-collapsing of the vector w, respectively. Proposition 2.1 implies that the four inequalities

$$(u^{p,h})^{\mathrm{T}}x \leqslant 0, \qquad (u^{p,k})^{\mathrm{T}}x \leqslant 0,$$

$$(w^{p,h})^{\mathrm{T}}x \leqslant 0, \qquad (w^{p,k})^{\mathrm{T}}x \leqslant 0$$

are valid over C_{n-1}. It is easy to verify that $v^{p,h} = u^{p,h} + w^{p,h}$, and that $v^{p,k} = u^{p,k} + w^{p,k}$. Now, (ii) implies that either $u^{p,h} = \underline{0}$ or $w^{p,h} = \underline{0}$, and that either $u^{p,k} = \underline{0}$ or $w^{p,k} = \underline{0}$. Without loss of generality, we can assume that $u^{p,h} = \underline{0}$, and so

$$u_{pj} + u_{hj} = 0, \quad u_{ij} = 0 \quad \text{for all } i, j \in [1, n] - \{p, h\}. \tag{5}$$

If $w^{p,k}=\underline{0}$ then $v_{ij}=0$, for all i,j in $[1,n]-\{p,h,k\}$, contradicting (iii). Hence, $u^{p,k}=\underline{0}$, and so

$$u_{pj}+u_{kj}=0, \quad u_{ij}=0 \quad \text{for all } i,j\in[1,n]-\{p,k\}. \tag{6}$$

Now, (5) and (6) imply

$$u_{ij}=\begin{cases} u_{ph} & \text{if } i=p,\ j=h \text{ or } k,\\ -u_{ph} & \text{if } i=k,\ j=h,\\ 0 & \text{otherwise.}\end{cases}$$

Since $u^{\mathrm{T}}x\leqslant 0$ is valid over C_n, it follows that $u_{ph}\leqslant 0$ (because $u^{\mathrm{T}}\delta(\{p\})=2u_{ph}$), and so

$$u_{ph}<0 \tag{7}$$

(for otherwise $u=\underline{0}$). Since $w^{\mathrm{T}}x\leqslant 0$ is valid over C_n, it follows that

$$\sum_{j\in[1,n]-\{p\}} w_{pj}\leqslant 0,$$

and so

$$\sum_{j\in[1,n]-\{p\}} v_{pj}\leqslant 2u_{ph}.$$

But, (i) implies that $\sum_{j\in[1,n]-\{p\}} v_{pj}=0$, and so $u_{ph}\geqslant 0$, contradicting (7). □

From the proof of Theorem 4.1, we get the following observation.

Remark 1. Let v be a vector in $\mathbb{R}^N$ satisfying conditions (ii) and (iii) of Theorem 4.1. If the inequality $v^{\mathrm{T}}x\leqslant 0$ is valid over C_n, then either it defines a facet of C_n or it is the sum of two inequalities valid over C_n, one of which is a positive multiple of the triangle facet-defining inequality $x_{hk}-x_{ph}-x_{pk}\leqslant 0$.

In the following, we show some applications of Theorem 4.1 on hypermetric and cycle inequalities.

Corollary 4.2. *Let $b=(b_1,\ldots,b_{n-1})$ be an integer vector satisfying the following conditions:*
- $b_1+\cdots+b_{n-1}=1$;
- *there exist distinct h and k in $[1,n-1]$ such that $b_h=b_k-1$;*
- *there exist distinct i and j in $[1,n-1]-\{h,k\}$ such that $b_ib_j\neq 0$.*

If the hypermetric inequality Hyp(b) *defines a facet of C_{n-1}, then the hypermetric inequality* Hyp(d) *specified by the vector $d=(d_1,\ldots,d_n)$, with*

$$d_i=\begin{cases} b_h & \text{if } i=k,\\ 1 & \text{if } i=n,\\ b_i & \text{otherwise,}\end{cases}$$

defines a facet of C_n.

Proof. Without loss of generality, we may assume that $h=1$ and $k=2$, and so $d=(b_1,b_1,b_3,\dots,b_{n-1},1)$. Let $d^{1,n}$ and $d^{2,n}$ be the $(1,n)$-collapsing and the $(2,n)$-collapsing of the vector d, respectively. We have

$$d^{1,n}=(b_1+1,b_1,b_3,\dots,b_{n-1}),$$

$$d^{2,n}=(b_1,b_1+1,b_3,\dots,b_{n-1}),$$

Since $d^{1,n}$ can be obtained from $d^{2,n}$ by a permutation of the set $[1,n-1]$, it follows that the hypermetric inequality Hyp($d^{1,n}$) is permutation equivalent to the hypermetric inequality Hyp($d^{2,n}$). Note that $d^{2,n}=b$, and so Hyp($d^{2,n}$) defines a facet of C_{n-1}. Define a vector v in $\mathbb{R}^N$ by $v_{ij}=d_id_j$ for all ij. Now the vector v satisfies conditions (i), (ii) and (iii) of Theorem 4.1 with $p=n$. □

For instance, consider the vector $b=(3,2,2,-1,-1,-1,-1,-2)$. Since the hypermetric inequality Hyp(b) defines a facet of C_8, Corollary 4.2 implies that the hypermetric inequality Hyp$(3,2,2,-1,-1,-1,-2,-2,1)$ defines a facet of C_9. (Here, $h=7$ and $k=8$.)

Corollary 4.3. *Let* $c=(c_1,\dots,c_{n-2})$ *be an integer vector satisfying the following three conditions*:

- $c_1+\cdots+c_{n-2}=1$;
- *there exists* h *in* $[1,n-2]$ *such that* $c_h=-1$;
- *there exist distinct* i *and* j *in* $[1,n-2]-\{h\}$ *such that* $c_ic_j\neq 0$.

If the hypermetric inequality Hyp(c) *defines a facet of* C_{n-2}, *then the hypermetric inequality* Hyp(d) *specified by the vector* $d=(d_1,\dots,d_n)$, *with*

$$d_i=\begin{cases} c_i & \text{if } i=1,2,\dots,n-2\\ 1 & \text{if } i=n-1,\\ -1 & \text{if } i=\text{n},\end{cases}$$

defines a facet of C_n.

Proof. Corollary 2.1 guarantees that the hypermetric inequality Hyp(b) specified by the vector $b=(c_1,c_2,\dots,c_{n-2},0)$ defines a facet of C_{n-1}. Now, observe that the vector b satisfies the assumptions of Corollary 4.2 (with $k=n-1$). □

Corollary 4.4. *Let* f *be an integer greater than or equal to three, and let* $b=(b_1,\dots,b_{n-1})$ *be an integer vector satisfying the following four conditions*:

- $b_1+\cdots+b_{n-1}=3$;
- $b_1,\dots,b_f>0\geqslant b_{f+1},\dots,b_n$;

– *there exists distinct h and k in $[1,f]$ such that $b_h = b_k - 1$ with $k = h+1 \pmod f$;*
– *there exists distinct i and j in $[1,n-1]-\{h,k\}$ such that $b_i b_j \neq 0$.*

Let b' be the vector obtained from b by permuting h and k; let d be the vector obtained from b by inserting 1 between b_h and b_k and by replacing b_k with b_h. If the cycle inequality Cyc(b) *defines a facet of C_{n-1}, and if the cycle inequality* Cyc(b') *defines a facet of C_{n-1}, then the cycle inequality* Cyc(d) *defines a facet of C_n.*

Proof. Without loss of generality, we may assume that $h=1$ and $k=2$, and so $d=(b_1,1,b_1,b_3,\ldots,b_{n-1})$. Let $v^{\mathrm{T}}x \leqslant 0$ denote the cycle inequality Cyc(d); write

$$d^{1,2}=(b_1+1,b_1,b_3,\ldots,b_{n-1}),$$
$$d^{2,3}=(b_1,b_1+1,b_3,\ldots,b_{n-1}).$$

It is easy to verify that the (1, 2)-collapsing and (1, 3)-collapsing of the vector v yield the two cycle inequalities Cyc($d^{1,2}$) and Cyc($d^{1,3}$), respectively. Since $d^{1,2}=b'$ and since $d^{2,3}=b$, both inequalities Cyc($d^{1,2}$) and Cyc($d^{2,3}$) define facets of C_{n-1}. Now the vector v satisfies conditions (*i*), (*ii*) and (*iii*) of Theorem 4.1 with $p=2$. □

For instance, consider the vector $b=(3,2,2,-1,-1,-1,-1)$. Since the cycle inequality Cyc(b) defines a facet of C_7, Corollary 4.4 implies that the cycle inequality Cyc$(2,1,2,2,-1,-1,-1,-1)$ defines a facet of C_8. (Here, $h=1$ and $k=2$.)

We ended Section 3 by pointing out that, in general, the zero-lifting of a face does not preserve the codimension. In the following, we show that a similar result holds for the general nonzero-lifting. For this purpose, let n be an integer greater than or equal to eight; let b^n be the vector in $\mathbb{R}^n$ defined by $b^n=(n-6,2,2,1,1,-1,\ldots,-1)$; and let w be the vector in $\mathbb{R}^{\binom{5}{2}}$ with components $w_{12}=w_{23}=3$, $w_{15}=w_{34}=2$, $w_{14}=w_{35}=w_{45}=1$, and $w_{ij}=0$ otherwise. Consider the inequality

$$\sum_{1\leqslant i<j\leqslant n} b_i^n b_j^n x_{ij} - \sum_{1\leqslant i<j\leqslant 5} w_{ij}x_{ij} \leqslant 0. \tag{8}$$

The inequality (8) belongs to the class of *clique-web* inequalities valid over C_n introduced by Deza and Laurent in [18]: (8) is the clique-web inequality $CW_n^2(b^n)$ with corresponding antiweb $AW_5^2(n-6,2,2,1,1)$.

Proposition 4.5. *Let $n \geqslant 8$. Then the inequality* (8) *defines a face of C_n of dimension $\binom{n}{2}-(n-4)$.*

Proof. For every $n \geqslant 8$, define a vector v^n in $\mathbb{R}^N$ by

$$(v^n)_{ij}=\begin{cases} b_i^n b_j^n - w_{ij} & \text{if } 1\leqslant i<j\leqslant 5,\\ b_i^n b_j^n & \text{otherwise.}\end{cases}$$

To prove that the inequality, $(v^n)^{\mathrm{T}}x \leqslant 0$ defines a face of C_n of dimension $\binom{n}{2}-(n-4)$, we use induction on n. A computer check guarantees that $(v^8)^{\mathrm{T}}x \leqslant 0$ defines a face of

C_8 of dimension 24. Now, suppose that the inequality $(v^n)^T x \leqslant 0$ defines a face of C_n of dimension $\binom{n}{2}-(n-4)$. We want to show that the inequality $(v^{n+1})^T x \leqslant 0$ defines a face of C_{n+1} of dimension $\binom{n+1}{2}-(n-3)$. For this purpose, let S be a subset of $[2,n]$. Since $\sum_{i=1}^n b_i^n = 5$, the cut $\delta(S)$ is a root of v^n if and only if

$$b^n(S)(5-b^n(S)) = \sum_{ij \in \delta(S)} w_{ij},$$

and so every root $\delta(S)$ of v^n, with $1 \notin S$, yields a root of v^{n+1}. By the inductive hypothesis, $\dim(v^n) = \binom{n}{2}-(n-4)$, and so we can find a set R_1 that contains $\dim(v^n)$ linearly independent roots of v^{n+1}. Since $\binom{n+1}{2}-(n-3) = \binom{n}{2}-(n-4)+(n-1)$, we only need find $n-1$ additional roots. Consider the following $n-1$ sets:

$$S' = \{2,3,n+1\}, \qquad S' = \{3,4,n+1\}, \qquad S' = \{3,4,5,n+1\},$$
$$S' = \{2,3,4,6,n+1\}, \qquad S' = \{2,3,k,n+1\}, \quad \text{for every } k = 6, \ldots, n.$$

Clearly, every set S' listed above yields a root of v^{n+1}. Let R_2 denote the set of these $n-1$ new roots of v^{n+1}. Now it is easy to verify that all the vectors in $R_1 \cup R_2$ are linearly independent. □

5. Difference of inequalities

In this section, we show how to construct, from a given face of the cut cone C_n, a face of C_n of higher dimension. Recall that N stands for $\binom{n}{2}$. Let v be a vector in $\mathbb{R}^N$ such that $v^T x \leqslant 0$ is valid over C_n; we want to find a vector w in $\mathbb{R}^N$ and two nonzero real numbers α and β such that both inequalities $w^T x \leqslant 0$ and $(\alpha v - \beta w)^T x \leqslant 0$ are valid over C_n. Clearly, if the inequality $(v-w)^T x \leqslant 0$ is valid over C_n, then the face F_v defined by the inequality $v^T x \leqslant 0$ is contained in the face F_w defined by the inequality $w^T x \leqslant 0$. For every vector v in $\mathbb{R}^N$, let m_v and M_v denote the minimum and maximum nonzero value assumed by $|v^T \delta(S)|$ over all subsets S of $[1,n]$, respectively. In this section, we let n stand for an integer greater than or equal to seven.

Proposition 5.1. *Let $v^T x \leqslant 0$ and $w^T x \leqslant 0$ be two inequalities valid over C_n. If $F_v \subseteq F_w$ then the inequality*

$$(M_w v - m_v w)^T x \leqslant 0$$

is valid over C_n.

Proof. Let S be a subset of $[1,n]$. If $\delta(S)$ is a root of w then

$$(M_w v - m_v w)^T \delta(S) = M_w v^T \delta(S),$$

which is $\leqslant 0$. If $\delta(S)$ is not a root of w then it is neither a root of v (since $F_v \subseteq F_w$). Hence

$$(M_w v - m_v w)^{\mathrm{T}} \delta(S) = M_w v^{\mathrm{T}} \delta(S) - m_v w^{\mathrm{T}} \delta(S)$$
$$\leqslant -M_w m_v + m_v M_w = 0. \quad \square$$

Let $v^{\mathrm{T}} x \leqslant 0$ be an hypermetric inequality specified by a vector b in $\mathbb{R}^N$. Clearly, for every cut vector $\delta(S)$,

$$v^{\mathrm{T}} \delta(S) = b(S)(1 - b(S)),$$

which is an even integer. Hence, $m_v \geqslant 2$. Furthermore, if $v^{\mathrm{T}} x \leqslant 0$ is a triangle inequality, then $m_v = M_v = 2$. Similarly, if $v^{\mathrm{T}} x \leqslant 0$ is a cycle inequality specified by a vector b in $\mathbb{R}^N$ and a cycle C, then, for every cut vector $\delta(S)$,

$$v^{\mathrm{T}} \delta(S) = b(S)(3 - b(S)) - |E(C) \cap \delta(S)|,$$

which, again, is an even integer, and so $m_v \geqslant 2$.

Corollary 5.2. *Let $v^{\mathrm{T}} x \leqslant 0$ be an hypermetric or cycle inequality over C_n, and let $w^{\mathrm{T}} x \leqslant 0$ be a triangle inequality. If $F_v \subseteq F_w$ then the inequality*

$$(v - w)^{\mathrm{T}} x \leqslant 0$$

is valid over C_n.

The proof follows directly from Proposition 5.1 and from the fact that $m_v \geqslant M_w = 2$.

Proposition 5.3. *Let $b = (b_1, \ldots, b_n)$ be an integer vector in $\mathbb{R}^n$ such that $b_i \neq 0$ $(i = 1, \ldots, n)$ and such that $b_1 + \cdots + b_n = 1$. Let the components of b be ordered in such a way that $b_1, \ldots, b_p > 0 > b_{p+1}, \ldots, b_n$, for some $p \geqslant 2$. If there exist distinct i and j in $[1, p]$, and if there exists a k in $\{p+1, \ldots, n\}$ such that*

$$b_i + b_j - b_k \geqslant \sum_{i=1}^{p} b_i + 1, \tag{9}$$

then the face of C_n defined by the inequality Hyp(b) *is strictly contained in the face of C_n defined by the triangle inequality $x_{ij} - x_{ik} - x_{jk} \leqslant 0$.*

Proof. Let S be a subset of $[1, n]$ such that $\delta(S)$ is a root of Hyp(b). Without loss of generality, we may assume that $k \notin S$. Note that $b(S)$ is equal to zero or one. If both i and j are in S then

$$b(S) \geqslant b_i + b_j + \sum_{i=p+1}^{n} b_i - b_k,$$

and so (9) implies that $b(S) \geqslant 2$, a contradiction. If at least one of i and j is not in S, then $\delta(S)$ is a root of $x_{ij} - x_{ik} - x_{jk} \leqslant 0$. $\square$

For instance, consider the hypermetric inequality Hyp(b) with $b=(1,3,2,-1,-1,-1,-2)$. Since b satisfies (9) (with $i=2, j=3$ and $k=7$), it follows that every root of Hyp(b) is also a root of the triangle inequality Hyp$(0,1,1,0,0,0,-1)$. Observe that an hypermetric inequality Hyp(b) satisfying the assumptions of Proposition 5.3 does not define a facet of C_n.

Proposition 5.4. *Let $b=(b_1,\dots,b_n)$ be an integer vector in $\mathbb{R}^n$ such that $b_1+\cdots+b_n=3$. Let the components of b be ordered in such a way that $b_1,\dots,b_f>0>b_{f+1},\dots,b_n$ for some $f\geqslant 3$. If there exist distinct i and j in $[1,f]$, and if there exists a k in $\{f+1,\dots,n\}$ such that*

$$b_i+b_j-b_k \geqslant \sum_{i=1}^{f} b_i,$$

then the face of C_n defined by the inequality Cyc(b) *is strictly contained in the face of C_n defined by the triangle inequality $x_{ij}-x_{ik}-x_{jk}\leqslant 0$.*

Proof. The proof is similar to the proof of Proposition 5.3 and relies on the fact that if $\delta(S)$ is a root of Cyc(b) then $b(S)$ is equal to one or two. □

Consider again the hypermetric inequality Hyp(b) with $b=(1,3,2,-1,-1,-1,-2)$, and let Hyp$(d_1,\dots,d_7)$ denote the triangle inequality Hyp$(0,1,1,0,0,0,-1)$. We have seen that every root of Hyp(b) is also a root of Hyp(d). Hence, Corollary 5.2 implies that the inequality

$$\sum_{1\leqslant i<j\leqslant 7} (b_ib_j-d_id_j)x_{ij}\leqslant 0 \tag{10}$$

is valid over C_7. Now, the inequality Hyp(b) has 19 roots, all of which are linearly independent. Since every root of Hyp(b) is a root of (10), to show that (10) defines a facet of C_7, we only need find one root of (10) which is linearly independent from the other 19; our choice for such a root is $\delta(\{1,7\})$. Hence, (10) defines a facet of C_7. Theorem 5.5 will generalize this procedure. Incidentally, inequality (10) can be obtained from the cycle inequality Cyc$(3,2,2,-1,-1,-1,-1)$ by switching the cut $\delta(\{1,7\})$.

Theorem 5.5. *Let $b=(2n-13,3,2,-1,-1,-1,-2,\dots,-2)$ and $d=(n-7,1,1,0,0,0,-1,\dots,-1)$ be two vectors in $\mathbb{R}^n$. Then the inequality*

$$\sum_{1\leqslant i<j\leqslant n} (b_ib_j-d_id_j)x_{ij}\leqslant 0$$

defines a facet of C_n.

Proof. Write

$$v^{\mathrm{T}}x=\sum_{1\leqslant i<j\leqslant n} (b_ib_j-d_id_j)x_{ij}.$$

To prove validity of the inequality $v^{\mathrm{T}}x \leqslant 0$, let S be a subset of the set $[1, n]$. Without loss of generality, we can assume that $1 \notin S$. We have

$$v^{\mathrm{T}}\delta(S) = b(S)(1-b(S)) - d(S)(1-d(S))$$
$$= (b(S)-d(S))(1-b(S)-d(S)).$$

Set

$$\alpha = |S \cap \{2,3\}|, \qquad \beta = |S \cap \{4,5,6\}|, \qquad \gamma = |S \cap ([1,n]-[1,6]).$$

It is easy to verify that

$$b(S) = k - \beta - 2\gamma, \qquad d(S) = \alpha - \gamma,$$

where $k \in \{0, 2, 3, 5\}$, and so

$$v^{\mathrm{T}}\delta(S) = (k-\alpha-\beta-\gamma)(1-k-\alpha+\beta+3\gamma).$$

Now it is a routine but tedious matter to verify that $v^{\mathrm{T}}\delta(S) \leqslant 0$.

We prove that $v^{\mathrm{T}}x \leqslant 0$ defines a facet of C_n, for all $n \geqslant 7$, by induction on n. For this purpose, note that when $n=7$, $v^{\mathrm{T}}x \leqslant 0$ is inequality (10), and so it defines a facet of C_7. Now assume that $v^{\mathrm{T}}x \leqslant 0$ defines a facet of C_n, and let b' and d' be two vectors in $\mathbb{R}^{n+1}$ given by

$$b' = (2(n+1)-13, 3, 2, -1, -1, -1, -2, \ldots, -2),$$
$$d' = ((n+1)-7, 1, 1, 0, 0, 0, -1, \ldots, -1).$$

Write

$$(v')^{\mathrm{T}}x = \sum_{1 \leqslant i < j \leqslant n} (b_i' b_j' - d_i' d_j') x_{ij}.$$

We want to show that the inequality

$$(v')^{\mathrm{T}}x \leqslant 0$$

defines a facet of C_{n+1}, i.e. we want to exhibit $\binom{n+1}{2}-1$ linearly independent roots of the vector v'. For this purpose, let $\delta(S)$ be a root of the vector v. Without loss of generality, we may assume that $1 \notin S$, and so every root of v is also a root of v'. By the inductive hypothesis, $\dim(v) = \binom{n}{2}-1$, and so there exist $\binom{n}{2}-1$ linearly independent roots of v'; let R_1 be the set containing such roots. Since $\binom{n+1}{2} = \binom{n}{2}+n$, we only need find n additional roots. For this purpose, let S' be a subset of $[1, n+1]$. Since

$$(v')^{\mathrm{T}}\delta(S) = b'(S)(1-b'(S)) - d'(S)(1-d'(S))$$
$$= (b'(S)-d'(S))(1-b'(S)-d'(S)),$$

it follows that $\delta(S')$ is a root of v' if and only if either $b'(S') = d'(S')$ or $b'(S') + d'(S') = 1$. Let S' be a subset of $[1, n+1]$ such that $1 \notin S'$ and such that $n+1 \in S'$; set $S = S' - \{n+1\}$. We have

$$b'(S') = b(S) - 2, \qquad d'(S') = d(S) - 1,$$

and so $\delta(S')$ is a root of v' if and only if

$$\text{either } b(S)=d(S)+1 \quad \text{or} \quad b(S)+d(S)=4. \tag{11}$$

Hence, to find n additional roots $\delta(S')$ of v', we only need find n subsets S of $[1,n]-\{1\}$ satisfying (11). Our choice for such sets is as follows.

$$S=\{2\}, \quad S=\{3\}, \quad S=\{2,4\}, \quad S=\{2,5\},$$

$$S=\{2,6\}, \quad S=\{2,k\}, \text{ for every } k=7,\dots,n, \quad S=\{2,3,n-1,n\}.$$

Clearly, every set S listed above satisfies (11). Let R_2 denote the set of these n new roots of v'. Now it is easy to verify that all the roots in $R_1 \cup R_2$ are linearly independent. □

We end this section by exhibiting a class of hypermetric inequalities defining faces of C_n, which are not contained in the face of C_n defined by any triangle inequality.

Proposition 5.6. *Let n be an odd integer greater than or equal to seven, and let b be the vector in $\mathbb{R}^n$ given by $b=(c,c,c,-c,-1,\dots,-1)$, where $c=(n-3)/2$. Then the face of C_7 defined by* Hyp(b) *is not contained in the face of C_n defined by any triangle inequality. Furthermore, let $d=(1,1,1,-1,-1,0,\dots,0)$ be a vector in $\mathbb{R}^n$. Then the inequality*

$$\sum_{ij}(b_i b_j - d_i d_j)x_{ij} \leqslant 0$$

is valid over C_n.

Proof. Let i,j, and k be arbitrary distinct elements in $[1,n]$. Clearly, to show that the face of C_n defined by Hyp(b) is not contained in the face of C_n defined by the triangle inequality $x_{jk}-x_{ij}-x_{ik}\leqslant 0$, we only need exhibit a root $\delta(S)$ of Hyp(b) such that $S\cap\{i,j,k\}=\{i\}$. If $i\in\{1,2,3\}$ then it is easy to verify that the desired root is $\delta(\{i\}\cup T)$, where T is any subset of $\{5,\dots,n\}-\{j,k\}$ of size $c-1$; if $i=4$ then it is easy to verify that the desired root is $\delta(\{1,i\})$; if $i=5,\dots,n$ then it is easy to verify that the desired root is $\delta(\{1,i\}\cup T)$, where T is any subset of $\{5,\dots,n\}-\{i,j,k\}$ of size $c-2$.

The proof of validity of $\sum_{ij}(b_i b_j - d_i d_j)x_{ij}\leqslant 0$ is similar to the proof of validity in Theorem 5.5. □

6. The cut cone on seven points

In 1960, Deza [12, 14] proved that all the facet-defining inequalities of C_4 and C_5 are hypermetric; C_4 has 12 triangle facets and C_5 has 40 facets (30 triangle facets and 10 facets of the type Hyp$(1,1,1,-1,-1)$ called *pentagonal* facets). In 1988, Avis and Mutt [7] proved using computer that all the facet-defining inequalities of C_6 are hypermetric; there are precisely 210 of them (60 triangle facets, 60 pentagonal facets, 60 facets of the type Hyp$(2,1,1,-1,-1,-1)$, and 30 facets of the type

Hyp(1, 1, 1, 1, −1, −2)). This is not true for C_7: Avis [4, 5] and Assouad [1] were the first to prove this. In 1989, Grishukhin [21] proved using computer that all the facet-defining inequalities of C_7 are (up to switching by a root and permutation) of four types: hypermetric inequalities, cycle inequalities, *parachute inequalities* and *Grishukhin* inequalities; the cut cone C_7 has precisely 38780 facets [19]. Let S be a subset of $[1,7]$; in this section, for every vector v in $\mathbb{R}^{\binom{7}{2}}$, v^S denotes the vector obtained from v by switching a root $\delta(S)$ of v.

Below we give a list of 36 facet defining inequalities of C_7; they are split into four groups.

(1) The first group consists of the following ten hypermetric facet-defining inequalities:

(H1): Hyp(1, 1, −1, 0, 0, 0, 0);

(H2): Hyp(1, 1, 1, −1, −1, 0, 0);

(H3): Hyp(1, 1, 1, 1, −1, −1, −1);

(H4): Hyp(2, 1, 1, −1, −1, −1, 0);

(H5): Hyp(−2, 1, 1, 1, 1, −1, 0);

(H6): Hyp(2, 2, 1, −1, −1, −1, −1);

(H7): Hyp(−2, 2, 1, 1, 1, −1, −1);

(H8): Hyp(−2, −2, 1, 1, 1, 1, 1);

(H9): Hyp(3, 1, 1, −1, −1, −1, −1);

(H10): Hyp(−3, 1, 1, 1, 1, 1, −1).

Note that inequality (H5) arises from inequality (H4) by switching the root $\delta(\{1,4,5\})$; (H7) arises from (H6) by switching the root $\delta(\{1,4,5\})$; (H8) arises from (H6) by switching the root $\delta(\{3\})$; (H10) arises from (H9) by switching the root $\delta(\{2,3,4\})$.

(2) The second group consists of 16 inequalities obtained from the following three cycle facet-defining inequalities by switching:

(C1): Cyc(1, 1, 1, 1, 1, −1, −1),
(C2): Cyc(2, 2, 1, 1, −1, −1, −1),
(C3): Cyc(3, 2, 2, −1, −1, −1, −1).

Let $u^{T}x \leqslant 0$, $v^{T}x \leqslant 0$, and $w^{T}x \leqslant 0$ denote inequalities (C1), (C2), and (C3), respectively. Switching roots of inequalities (C1), (C2), and (C3), yields the following (noncycle) inequalities:

(C4): $(u^{\{1\}})^{\mathrm{T}}x \leqslant 0$,

(C5): $(u^{\{1,2\}})^{\mathrm{T}}x \leqslant 0$,

(C6): $(u^{\{1,2,6\}})^{\mathrm{T}}x \leqslant 0$,

(C7): $(v^{\{1\}})^{\mathrm{T}}x \leqslant 0$,

(C8): $(v^{\{3\}})^{\mathrm{T}}x \leqslant 0$,

(C9): $(v^{\{1,5\}})^{\mathrm{T}}x \leqslant 0$,

(C10): $(v^{\{3,4\}})^{\mathrm{T}}x \leqslant 0$,

(C11): $(v^{\{1,4,5\}})^{\mathrm{T}}x \leqslant 0$,

(C12): $(v^{\{3,4,5\}})^{\mathrm{T}}x \leqslant 0$;

(C13): $(w^{\{2\}})^{\mathrm{T}}x \leqslant 0$,

(C14): $(w^{\{2,4\}})^{\mathrm{T}}x \leqslant 0$,

(C15): $(w^{\{1,4\}})^{\mathrm{T}}x \leqslant 0$,

(C16): $(w^{\{1,4,5\}})^{\mathrm{T}}x \leqslant 0$.

(3) The third group consists of a parachute inequalilty and its two switchings. This parachute inequality is the inequality

(P1) $p^{\mathrm{T}}x \leqslant 0$,

where the vector $p=(p_{12}, \ldots, p_{17}; \ldots; p_{67})^{\mathrm{T}}$ is given by

$$(0, -1, -1, -1, -1, 0;\ 1, 0, -1, -1, -1;\ 1, 0, 0, -1;\ 1, 0, -1;\ 1, 0;\ 1)^{\mathrm{T}}.$$

Switching roots of the inequality $(P1)$, yields the following two inequalities:

(P2): $(p^{\{3,7\}})^{\mathrm{T}}x \leqslant 0$,

(P3): $(p^{\{1,3,6\}})^{\mathrm{T}}x \leqslant 0$,

The graphs P_1, P_2 and P_3 in Fig. 1 are the supporting graphs of the vectors p, $p^{\{3,7\}}$, and $p^{\{1,3,6\}}$, respectively: a plain line ij corresponds to an edge ij with weight equal to 1, a dashed line ij corresponds to an edge ij with weight equal to -1.

(4) The fourth group consists of the Grishukhin inequality and its six switchings. The Grishukhin is the inequality

(G1): $g^{\mathrm{T}}x \leqslant 0$,

where the vector $g=(g_{12}, \ldots, g_{17}; \ldots; g_{67})^{\mathrm{T}}$ is given by

$$(1, 1, 1, -2, -1, 0;\ 1, 1, -2, 0, -1;\ 1, -2, -1, 0;\ -2, 0, -1;\ 1, 1;\ -1)^{\mathrm{T}}.$$

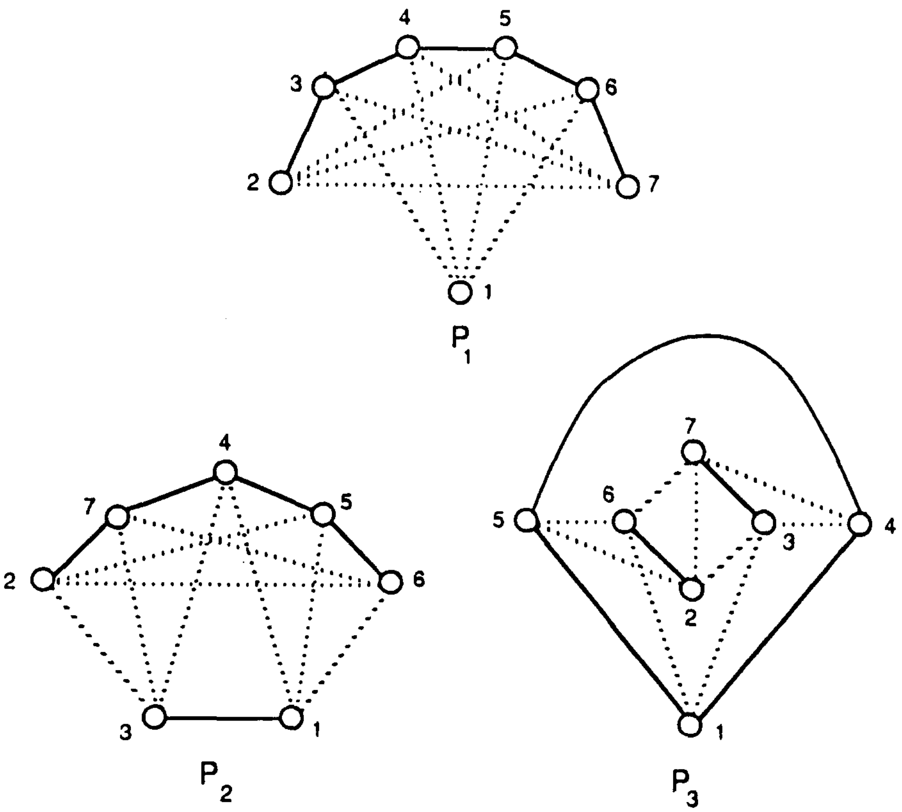

Fig. 1.

The graph in Fig. 2 is the supporting graph of the vector g: a plain line ij corresponds to an edge ij with weight equal to 1, a dashed line ij corresponds to an edge ij with weight equal to -1, a double dashed line ij corresponds to an edge ij with weight equal to -2.

Switching roots of the inequality $(G1)$, yields the following six inequalities:

(G2): $(g^{\{1\}})^{\mathrm{T}}x \leqslant 0$,

(G3): $(g^{\{1,6\}})^{\mathrm{T}}x \leqslant 0$,

(G4): $(g^{\{1,6,7\}})^{\mathrm{T}}x \leqslant 0$,

(G5): $(g^{\{1,3,5\}})^{\mathrm{T}}x \leqslant 0$,

(G6): $(g^{\{1,3,6\}})^{\mathrm{T}}x \leqslant 0$,

(G7): $(g^{\{1,2,5\}})^{\mathrm{T}}x \leqslant 0$,

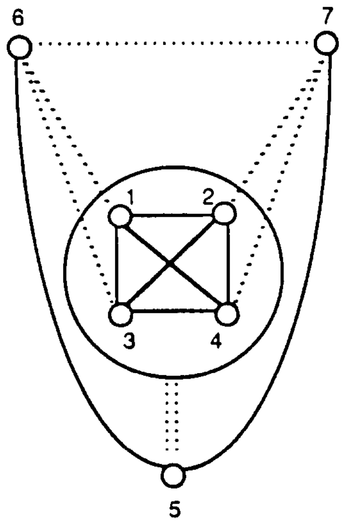

Fig. 2

Inequality (C3) was found by Avis [4, 5]; Assouad [1, 3] found the inequalities (Ci) ($i = 1, \ldots, 6, 13$) and the inequalities (Pi) ($i = 1, 2, 3$). Inequality (P1) is called parachute inequality since it belongs to the class of parachute inequalities introduced in [17]; inequality (G1) is called Grishukhin inequality since it was found by Grishukhin along with its six switchings [20, 21].

Let L denote the set of the 36 inequalities listed above. Grishukhin [21] proved that every facet-defining inequality of C_7 is switching or permutation equivalent to some inequality in L. We now show that the list L is (up to permutation) complete.

Theorem 6.1. *Every facet-defining inequality of C_7 is permutation equivalent to some inequality in L.*

Proof. We only need verify that every inequality obtained by switching a root of some inequality in L is also in L. For this purpose, let S and S' be two subsets of $[1, 7]$; clearly, $(v^S)^{S'} = v^{S \Delta S'}$ for every vector v in $\mathbb{R}^{\binom{7}{2}}$, where $S \Delta S' = (S - S') \cup (S' - S)$. It follows that every inequality obtained from some inequality $(v^S)^T x \leqslant 0$ by switching a root of v^S belongs to the family of all the inequalities $(v^S)^T x \leqslant 0$ obtained from the inequality $v^T x \leqslant 0$ by switching all the roots $\delta(S)$ of v.

First, consider an arbitrary hypermetric inequality $\text{Hyp}(b_1, \ldots, b_n)$, and let $\delta(S)$ be one of its roots; assume that $\sum_{i \in S} b_i = 0$. It is easy to verify that switching $\text{Hyp}(b_1, \ldots, b_n)$ by $\delta(S)$ yields the hypermetric inequality $\text{Hyp}(b'_1, \ldots, b'_n)$ with $b'_i = -b_i$ if $i \in S$ and $b'_i = b_i$ if $i \notin S$. Now it is easy to verify that every switching of an (Hi) ($i = 1, 2, 3, 4, 6, 9$) is permutation equivalent to one of (H_i), $i = 1, \ldots, 10$.

Secondly, consider an arbitrary cycle inequality $\mathrm{Cyc}(b_1,\dots,b_f,\dots,b_n)$ with cycle $C=(1,\dots,f)$, and let $\delta(S)$ be one of its roots such that $\sum_{i\in S} b_i=1$. Recall that $b_1,\dots,b_f>0>b_{f+1},\dots,b_n$, that $f\geqslant 3$, and that $E(C)$ stands for the edge set of the cycle C. It is easy to show that switching $\mathrm{Cyc}(b_1,\dots,b_n)$ by $\delta(S)$ yields the inequality

$$\sum_{i=1}^{n} b_i' b_j' x_{ij} - \Bigg(- \sum_{ij\in\delta(S)\cap E(C)} x_{ij} + \sum_{ij\in E(C)-\delta(S)} x_{ij} \Bigg) \leqslant 0, \tag{12}$$

with $b_i'=-b_i$ if $i\in S$ and $b_i'=b_i$ if $i\notin S$. Since $b_1',\dots,b_n'=1$, (12) is the sum of two inequalities one of which is the hypermetric inequality $\mathrm{Hyp}(b_1',\dots,b_n')$ and the other one is a 'switched' cycle. We simply write (12) as

$$\mathrm{Hyp}(b_1',\dots,b_n') - \Bigg(- \sum_{ij\in\delta(S)\cap E(C)} x_{ij} + \sum_{ij\in E(C)-\delta(S)} x_{ij} \Bigg) \leqslant 0.$$

Now consider the cycle inequality (C1); set

$$R_1=\{\{1\},\{2\},\{3\},\{4\},\{5\}\};$$

$$R_2=\{\{3,4,5,6,7\},\{1,4,5,6,7\},\{1,2,5,6,7\},\{1,2,3,6,7\},\{2,3,4,6,7\}\};$$

$$R_3=\{\{1,2,6\},\{1,2,7\},\{2,3,6\},\{2,3,7\},\{3,4,6\},\{3,4,7\},\{4,5,6\},\{4,5,7\},$$
$$\{1,5,6\},\{1,5,7\}\}.$$

It is easy to verify that the set R of all the roots of (C1) is the union of all sets $\delta(S)$ with S in $\bigcup_{i=1}^{3} R_i$, and that every switching of the inequality (C1) by a root in R yields an inequality that is permutation equivalent to one of the following three inequalities:

$$\mathrm{Hyp}(-1,1,1,1,1,-1,-1)-(-x_{12}+x_{23}+x_{34}+x_{45}-x_{15}),$$

$$\mathrm{Hyp}(1,1,-1,-1,-1,1,1)-(x_{12}-x_{23}+x_{34}+x_{45}-x_{15}),$$

$$\mathrm{Hyp}(-1,-1,1,1,1,1,-1)-(x_{12}-x_{23}+x_{34}+x_{45}-x_{15}),$$

which are not permutation equivalent (they are (C4), (C5), and (C6), respectively). A similar proof holds for (C2) and (C3).

Thirdly, consider the parachute inequality (P1); set

$$R_1=\{\{3\},\{6\},\{3,5\},\{4,6\},\{2,4,6\},\{3,5,7\}\};$$

$$R_2=\{\{4\},\{5\},\{2,6\},\{3,6\},\{3,7\},\{2,3,5,7\},\{2,4,6,7\}\};$$

$$R_3=\{\{2,5\},\{4,7\},\{2,5,7\},\{2,4,7\},\{2,5,6\},\{3,4,7\},\{2,4,5,7\}\}.$$

It is easy to verify that the set of all the roots of (P1) is the union of all sets $\delta(S)$ with S in $\bigcup_{i=1}^{3} R_i$ [17], and that every switching of the nequality (P1) by a root $\delta(S)$ with S in R_i $(i=1,2,3)$ yields an inequality that is permutation equivalent to inequality (Pi) $(i=1,2,3)$; in addition, inequalities (P1), (P2), and (P3) are not permutation equivalent.

Finally, consider the Grishukhin inequality (G1); let R denote the set of all its roots. Set

$$R_1 = \{\{1\},\{2\},\{3\},\{4\}\},$$

$$R_2 = \{\{1,6,7\},\{2,6,7\},\{3,6,7\},\{4,6,7\}\},$$

$$R_3 = \{\{1,6\},\{3,6\},\{2,7\},\{4,7\}\},$$

$$R_4 = \{\{2,4,5\},\{1,3,5\}\},$$

$$R_5 = \{\{1,3,6\},\{2,4,7\}\},$$

$$R_6 = \{\{1,4,5\},\{3,4,5\},\{1,2,5\},\{2,3,5\}\}.$$

It is easy to verify that R is the union of all sets $\delta(S)$ with S in $\bigcup_{i=1}^{6} R_i$, and that every switching of the inequality (G1) by a root in R yields an inequality that is permutation equivalent to one of (Gi) ($i=2,\ldots,7$). □

Theorem 6.2. *Every facet-defining inequality of* C_7 *collapses to some triangle inequality.*

Proof. By Theorem 6.1, we only need verify that every inequality in L collapses to some triangle inequality. For this purpose, recall that for every partition π of $[1,7]$ and for every vector v in $\mathbb{R}^{\binom{7}{2}}$, the vector v^π is the π-collapsing of v.

First, consider an arbitrary hypermetric inequality $\mathrm{Hyp}(b_1, b_2, \ldots, b_n)$, and observe that, for every nonnegative integer n greater than or equal to three, $\mathrm{Hyp}(b_1, b_2, \ldots, b_n)$ collapses to a triangle inequality if and only if the set $[1,n]$ can be partitioned into three subsets, say V_1, V_2, and V_3, in such a way that

$$\sum_{i\in V_1} b_i = \sum_{i\in V_2} b_i = -\sum_{i\in V_3} b_i = 1.$$

Now it is easy to verify that all hypermetric inequalities in L collapse to some triangle inequality.

To show that every cycle inequality in L and all its switchings collapse to some triangle inequality, set

$$u_1 = u^{\{1\}},\quad u_2 = u^{\{1,2\}},\quad u_3 = u^{\{1,2,6\}},$$

$$v_1 = v^{\{1\}},\quad v_2 = v^{\{3\}},\quad v_3 = v^{\{1,5\}},\quad v_4 = v^{\{3,4\}},\quad v_5 = v^{\{1,4,5\}},\quad v_6 = v^{\{3,4,5\}},$$

$$w_1 = w^{\{2\}},\quad w_2 = w^{\{2,4\}},\quad w_3 = w^{\{1,4\}},\quad w_4 = w^{\{1,4,5\}}.$$

Now it is easy to verify that the inequalities

$$(u^\pi)^{\mathrm{T}} x \leqslant 0,\quad \text{with } \pi = \{\{1\},\{3\},\{2,4,5,6,7\}\},$$

$$(v^\pi)^{\mathrm{T}} x \leqslant 0,\quad \text{with } \pi = \{\{1\},\{4\},\{2,3,5,6,7\}\},$$

$$(w^\pi)^{\mathrm{T}} x \leqslant 0,\quad \text{with } \pi = \{\{2\},\{3,4,5\},\{1,6,7\}\},$$

and the inequalities

$$(u_1^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{1\}, \{3\}, \{2,4,5,6,7\}\},$$

$$(u_2^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{6\}, \{7\}, \{1,2,3,4,5\}\},$$

$$(u_3^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{6\}, \{7\}, \{1,2,3,4,5\}\},$$

$$(v_1^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(v_2^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(v_3^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(v_4^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(v_5^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(v_6^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(w_1^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(w_2^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(w_3^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(w_4^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\}$$

are triangle inequalities.

To show that the inequalities (P1), (P2), and (P3) collapse to some triangle inequality, set

$$p_1 = p^{\{3,7\}}, \qquad p_2 = p^{\{1,3,6\}}.$$

Now it is easy to verify that the inequality

$$(p^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

and the inequalities

$$(p_1^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\},$$

$$(p_2^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{4\}, \{5\}, \{1,2,3,6,7\}\}$$

are triangle inequalities.

Finally, consider the inequalities (Gi) ($i = 1, \ldots, 7$). Set

$$g_1 = g^{\{1\}}, \qquad g_2 = g^{\{1,6\}}, \qquad g_3 = g^{\{1,6,7\}}, \qquad g_4 = g^{\{1,3,5\}}, \qquad g_5 = g^{\{1,3,6\}},$$
$$g_6 = g^{\{1,2,5\}}.$$

Now it is easy to verify that the inequality

$$(g^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{1\}, \{3\}, \{2,4,5,6,7\}\}$$

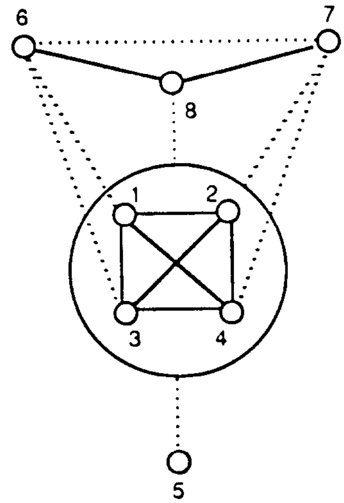

Fig.3

and the inequalities

$$(g_1^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{1\},\{6\},\{2,3,4,5,7\}\},$$

$$(g_2^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{6\},\{7\},\{1,2,3,4,5\}\},$$

$$(g_3^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{2\},\{3\},\{1,4,5,6,7\}\},$$

$$(g_4^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{2\},\{4\},\{1,3,5,6,7\}\},$$

$$(g_5^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{5\},\{7\},\{1,2,3,4,6\}\},$$

$$(g_6^\pi)^T x \leqslant 0, \quad \text{with } \pi = \{\{3\},\{4\},\{1,2,5,6,7\}\}$$

are triangle inequalities. □

We do not know any facet-defining inequality of C_n which does not collapse to some triangle inequality. Moreover, we do not know any facet-defining inequality of C_n which does not admit a purification; in other words, every facet-defining inequality that we know has an expansion that is pure and facet-defining. In particular, the facet-defining inequality (G1) admits a purification; the graph in Fig. 3 is the supporting graph corresponding to this pure inequality: a plain line ij corresponds to an edge ij with weight equal to 1, a dashed line ij corresponds to an edge ij with weight equal to -1.

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