Uniformity in Association schemes and Coherent Configurations

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UNIFORMITY IN ASSOCIATION SCHEMES AND COHERENT CONFIGURATIONS: COMETRIC Q-ANTIPODAL SCHEMES AND LINKED SYSTEMS

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UNIFORMITY IN ASSOCIATION SCHEMES AND COHERENT CONFIGURATIONS: COMETRIC Q-ANTIPODAL SCHEMES AND LINKED SYSTEMS

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Dedicated to the memory of Donald G. Higman

Abstract. Inspired by some intriguing examples, we study uniform association schemes and uniform coherent configurations. Perhaps the most important subclass of these objects is the class of cometric Q-antipodal association schemes. The concept of a cometric association scheme (the dual version of a distance-regular graph) is well-known; however, until recently it has not been studied well outside the area of distance-regular graphs. Uniformity is a concept introduced by Higman, but this likewise has not been well-studied. After a review of imprimitivity, we show that an imprimitive association scheme is uniform if and only if it is dismantlable, and we cast these schemes in the broader context of certain — uniform — coherent configurations. We also give a third characterization of uniform schemes in terms of the Krein parameters, and derive information on the primitive idempotents of such a scheme.

In the second half of the paper, we apply these results to cometric association schemes. We show that each such scheme is uniform if and only if it is Q-antipodal, and derive results on the parameters of the subschemes and dismantled schemes of cometric Q-antipodal schemes. We revisit the correspondence between uniform indecomposable three-class schemes and linked systems of symmetric designs, and show that these are cometric Q-antipodal. We obtain a characterization of cometric Q-antipodal four-class schemes in terms of only a few parameters, and show that any strongly regular graph with a ("non-exceptional") strongly regular decomposition gives rise to such a scheme. Hemisystems in generalized quadrangles provide interesting examples of such decompositions. We finish with a short discussion of five-class schemes as well as a list of all feasible parameter sets for cometric Q-antipodal four-class schemes with at most six fibres and fibre size at most 2000, and describe the known examples.

1. Introduction

Motivated by the search for cometric (Q-polynomial) association schemes, we study uniform association schemes. Cometric association schemes are the "dual version" of distance-regular graphs (metric schemes), and the latter are well-studied objects, cf. [7]. Classical metric schemes such as Hamming schemes and Johnson schemes are in fact also cometric. Bannai and Ito [5, p312] conjectured that for large enough d, a primitive d-class scheme is metric if and only if it is cometric. Partly because of this conjecture, the topic of cometric association schemes was

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studied mainly in connection to distance-regular graphs, at least until the end of last century. An exception to this is the work of Delsarte [21] (and others building on this) who showed the importance of cometric schemes in design theory.

This slowly changed when De Caen and Godsil raised the challenging problem of constructing cometric schemes that are not metric or duals of metric schemes (cf. [22, p234], [35, Acknowledgments]). Around the same time, Suzuki derived fundamental results on imprimitive cometric schemes [48] and on cometric schemes with multiple Q-polynomial orderings [49], but examples of the above type were still missing. In the last few years, however, there has been considerable activity in the area, with the first new constructions of cometric (but not metric) schemes given by Martin, Muzychuk, and Williford [35]. For a recent overview of results on cometric schemes we refer to the survey on association schemes by Martin and Tanaka [36]. Very recent is the work of Suda [45], [46], [47], and Penttila and Williford [41].

Meanwhile, in [26]–[30], Higman obtained numerous results on imprimitive association schemes and coherent configurations. In his paper on four-class schemes and triality [28] and also in an unpublished manuscript [30], he introduced the concept of uniformity of an imprimitive scheme, and he mentioned several examples of such uniform schemes. It turns out that many of these examples are cometric Q-antipodal. Inspired by this, we work out the concept of uniformity, and apply it to cometric Q-antipodal schemes.

This paper is organized as follows. We finish this introduction with an intriguing introductory example: the linked system of partial λ-geometries that is related to the Hoffman-Singleton graph. This example gives rise to a cometric Q-antipodal association scheme, and illustrates many of the interesting features we will consider in the paper. In Section 2, we remind the reader of basic background material on association schemes, focusing in particular on the natural subschemes and quotient schemes of an imprimitive association scheme. The main results for the first half of the paper are to be found in Sections 3 and 4. We first show in Section 3.1 that the dismantlability property introduced in [35] is implied by Higman's uniformity property [30]. In order to establish the reverse implication, we need to consider a fission of our uniform association scheme whose adjacency algebra is necessarily non-commutative. So we introduce coherent configurations at this point to draw out the deeper structure that occurs here. Only at the level of this more detailed structure do we see the full equivalence of the dismantlable and uniform properties in Theorem 4.3. We finish the first half of the paper with another characterization of the same phenomenon in Section 4.3, this time cast in terms of Krein parameters only. We introduce Q-Higman schemes and show that these, too, are equivalent to uniform schemes. To place the main concepts discussed here in perspective, we summarize them in the Venn diagram of Figure 1 below.

The second half of the paper returns to the cometric case and explores the implications of the results discussed above for cometric Q-antipodal schemes. In Section 5, as in Sections 2 and 3, we strive to make the paper fairly self-contained; we include all definitions that are not available in the standard literature. We show that each cometric scheme is uniform if and only if it is Q-antipodal, and derive results on the parameters of the subschemes and dismantled schemes of cometric Q-antipodal schemes. This general discussion of cometric Q-antipodal schemes is followed by three more detailed sections focusing on such association schemes with a small number of classes. In Section 6, we show that uniform indecomposable three-class schemes are always cometric Q-antipodal, and that these correspond naturally to linked systems of symmetric designs. In Section 7, we study the more complicated case of four-class schemes. We obtain a characterization of cometric Q-antipodal four-class schemes in terms of just a few of their parameters, and show that any strongly regular graph with a ("non-exceptional") strongly regular decomposition gives rise to such a scheme. An exciting special case of recent interest is that of hemisystems in generalized quadrangles. To facilitate future work on such problems, we generate a list of all feasible parameter sets for cometric Q-antipodal four-class schemes with at most six fibres and fibre size at most 2000, and describe the known examples from this table. In the short Section 8, we mention some examples of five-class schemes that are cometric Q-antipodal. The final section, Section 9, collects some miscellaneous remarks.

As background we refer to Cameron [10] and Higman [26] for coherent configurations, and to Bannai and Ito [5], Brouwer, Cohen, and Neumaier [7], Godsil [22], and Martin and Tanaka [36] for association schemes.

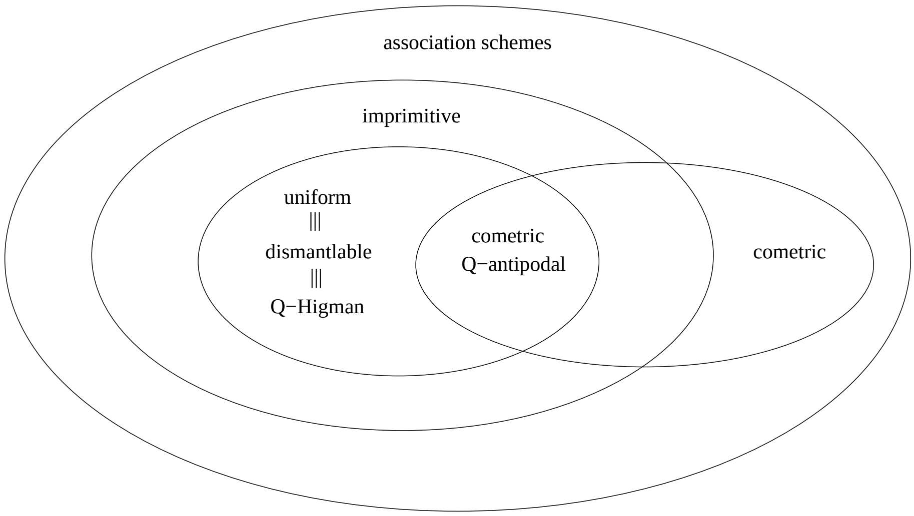

Figure 1. Venn diagram of relevant types of association schemes

1.1. **A linked system of partial λ-geometries related to the Hoffman-Singleton graph.** The maximum size of a coclique in the Hoffman-Singleton graph is 15. There are 100 cocliques of this size, and it is known that one can define a bipartite cometric distance-regular graph Γ with diameter four and valency 15 on these 100 cocliques by calling two cocliques adjacent whenever they intersect in eight vertices, cf. [7, p393]. Miraculously, the distance-four graph Γ_4 of this graph forms a Hoffman-Singleton graph on each part of the bipartition. Moreover, the union of Γ and Γ_4 is the so-called Higman-Sims graph. In fact, in this way it is clear that the Higman-Sims graph can be decomposed into two Hoffman-Singleton graphs; here we have a strongly regular decomposition of a strongly regular graph, in the sense of Haemers and Higman [25]. The incidence structure that Γ induces between the two parts of the bipartition is a so-called strongly regular design as defined by Higman [27], and more specifically a partial λ-geometry as defined by

Cameron and Drake [12]. Building on a description of the Hoffman-Singleton graph by Haemers [24], Neumaier [40] describes this partial λ-geometry — and hence the graph Γ — using the points, lines, and planes of $PG(3, 2)$. So far, so good.

Neumaier goes on to describe how Γ can be constructed in the Leech lattice. Using the group $2 \cdot U_3(5) \cdot S_3$, he finds three types of 50 vectors each, and between each two types of 50 vectors the above partial λ-geometry. Moreover, these geometries are linked: we have a linked system of partial λ-geometries.

What is going on combinatorially is that one can extend the distance-regular graph Γ by the 50 vertices of the Hoffman-Singleton graph, by calling a coclique adjacent to a vertex whenever the coclique contains the vertex. This gives a 30-regular graph on 150 vertices, and it generates a uniform imprimitive four-class association scheme. This association scheme turns out to be cometric too (but it is not metric); in fact it is Q-antipodal with three fibres of size 50. Here (again) one of the relations forms a Hoffman-Singleton graph on each fibre, and between each pair of fibres is the incidence structure of a partial λ-geometry (strongly regular design).

One natural question is whether you can throw in another 50 vertices, and get yet another cometric association scheme. We address this specific case in Section 7.6.2, and give a general bound on the number of fibres in Section 7.6.1.

Higman also gives the above example in his paper on four-class imprimitive schemes [28], and in his unpublished manuscript on uniform schemes [30]. This fairly small example illustrates most of the central features considered in this paper and, in our view, the attractive interplay of combinatorial subjects that one sees in the study of cometric Q-antipodal association schemes.

2. Association schemes

Our goal in this section is to review briefly the basic definitions from the theory of association schemes that we will need and to summarize some necessary material from the theory of imprimitive schemes. We defer our review of coherent configurations to Section 4 since their role will become clear at that point in the narrative.

2.1. **Definitions.** A (symmetric) d-class association scheme $(X, \mathcal{R})$ consists of a finite set X of size v and a set $\mathcal{R}$ of relations on X satisfying

- $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a partition of $X \times X$;
- $R_0 = \Delta_X := \{(x, x) | x \in X\}$ is the identity relation;
- $R_i^\top = R_i$ for each i, where $R_i^\top := \{(x, y) | (y, x) \in R_i\}$;
- there exist integers p_{ij}^h such that

$$|\{z \in X \,|\, (x, z) \in R_i \text{ and } (z, y) \in R_j\}| = p_{ij}^h$$

  whenever $(x, y) \in R_h$, for each $i, j, h \in \{0, \ldots, d\}$.

The integers p_{ij}^h are called the intersection numbers of the scheme.

The adjacency matrix A_R of a relation R on X is a $v \times v$ $(0, 1)$-matrix defined by $(A_R)_{xy} = 1$ if $(x, y) \in R$, and zero otherwise. In this case, we abbreviate by $A_i := A_{R_i}$ the adjacency matrix of relation R_i and consider $\mathcal{A} := \langle A_i | i = 0, ..., d\rangle$. Then this vector space is a $(d + 1)$-dimensional commutative algebra of symmetric matrices; this is called the Bose-Mesner algebra of the association scheme. Such an algebra admits a basis of pairwise orthogonal primitive idempotents (a nonzero idempotent E of $\mathcal{A}$ is called primitive if AE is proportional to E for each $A \in \mathcal{A}$). We denote these by $E_0, E_1, \ldots, E_d$ with the convention that $E_0 = \frac{1}{v}J$ where $J = \sum_i A_i$ is the all-ones matrix. The first and second eigenmatrices of the scheme are denoted by P and Q, respectively, and are defined by the change-of-basis equations

$$A_i = \sum_j P_{ji} E_j \qquad \text{and} \qquad E_j = \frac{1}{v} \sum_i Q_{ij} A_i.$$

We abbreviate $v_i := P_{0i} = p^0_{ii}$ and call this the i^{th} valency; likewise, $m_j := Q_{0j} = q^0_{jj}$ is called the j^{th} multiplicity of the scheme.

The algebra $\mathcal{A}$ is also closed under entrywise (Schur-Hadamard) multiplication $\circ$ of matrices because $A_i \circ A_j = \delta_{ij} A_i$. (We call these $(0,1)$-matrices the Schur idempotents of $\mathcal{A}$.) The (nonnegative) Krein parameters (or dual intersection numbers) q^h_{ij} are the structure constants for this multiplication with respect to the basis of primitive idempotents:

$$E_i \circ E_j = \frac{1}{v} \sum_h q^h_{ij} E_h.$$

2.2. **Metric schemes and cometric schemes.** The association scheme $(X, \mathcal{R})$ is called metric (or "P-polynomial") if there exists an ordering $R_0, R_1, \ldots, R_d$ of the relations for which

- $p^h_{ij} = 0$ whenever $0 \le h < |i-j|$ or $i+j < h$, and
- $p^{i+j}_{ij} > 0$ whenever p^{i+j}_{ij} is defined.

An ordering with respect to which these properties hold is called a P-polynomial ordering. In this case, R_i can be interpreted as the distance-i relation in the simple graph (X, R_1) which is necessarily distance-regular. Metric schemes with given P-polynomial orderings are in one-to-one correspondence with distance-regular graphs.

The association scheme $(X, \mathcal{R})$ is called cometric (or "Q-polynomial") if there exists an ordering of the primitive idempotents $E_0, E_1, \ldots, E_d$ for which

- $q^h_{ij} = 0$ whenever $0 \le h < |i-j|$ or $i+j < h$, and
- $q^{i+j}_{ij} > 0$ whenever q^{i+j}_{ij} is defined.

It is well known (cf. [7, Prop. 2.7.1]) that to check that a scheme is cometric it suffices to check these properties for $i = 1$. An ordering with respect to which these hold is called a Q-polynomial ordering, and E_1 is called a Q-polynomial generator. There is no known simple combinatorial or geometric interpretation of the cometric property. Suzuki [49] showed that, while it is possible to have two distinct Q-polynomial orderings, there can be no more than two such orderings for a given association scheme, with the exception of the cycles. Several important families of association schemes, such as the Hamming schemes and Johnson schemes, are both cometric and metric. But our study here does not assume the metric property at all.

Let $c^*_i := q^i_{1,i-1}, a^*_i := q^i_{1i}$, and $b^*_i := q^i_{1,i+1}$. Then $c^*_i + a^*_i + b^*_i = q^0_{11}$ and the Krein array of the cometric association scheme is defined as

$$\{b^*_0, b^*_1, ..., b^*_{d-1}; c^*_1, c^*_2, ..., c^*_d\}.$$

Using the Krein array, we define a sequence of orthogonal polynomials q_j, $j = 0, 1, ..., d+1$ by $q_0(x) = 1$, $q_1(x) = x$, and the three-term recurrence $xq_j(x) = c^*_{j+1} q_{j+1}(x) + a^*_j q_j(x) + b^*_{j-1} q_{j-1}(x)$, where we let $c^*_{d+1} := 1$. It follows that $vE_j = q_j(vE_1)$, $j = 0, 1, ..., d$ where matrix multiplication is entrywise (and hence the empty product is J). Moreover, because $vE_1 \circ E_d = b^*_{d-1}E_{d-1} + a^*_d E_d$, we have that the roots of $q_{d+1}(x)$ are precisely Q_{i1} for $i = 0, \ldots, d$. It is now easy to see that, whenever E_1 is a Q-polynomial generator for the Bose-Mesner algebra, column one of the matrix Q has $d+1$ distinct entries.

2.3. **Imprimitive schemes.** The association scheme $(X, \mathcal{R})$ with Bose-Mesner algebra $\mathcal{A}$, adjacency matrices $A_i, i = 0, 1, ..., d$, and primitive idempotents $E_j, j = 0, 1, ..., d$ is called imprimitive if at least one of its nontrivial relations is disconnected (as a graph). It was first shown by Cameron, Goethals, and Seidel [14] (and not hard to verify, cf. [5, Thm 9.3, Thm. 4.6]) that imprimitivity is equivalent to each of the following properties:

- there is a set $\mathcal{I}$ with $\{0\} \subsetneq \mathcal{I} \subsetneq \{0, 1, ..., d\}$ such that $\langle A_i | i \in \mathcal{I}\rangle$ is a matrix subalgebra of $\mathcal{A}$;
- there is a set $\mathcal{J}$ with $\{0\} \subsetneq \mathcal{J} \subsetneq \{0, 1, ..., d\}$ such that $\langle E_j | j \in \mathcal{J}\rangle$ is a $\circ$-subalgebra of $\mathcal{A}$;
- there is a matrix[1] $E \in \mathcal{A}$, not 0, I, or J, such that $E^2 = nE$ and $E \circ E = E$ for some n;
- the matrix E_j has repeated columns for some $j > 0$.

For an imprimitive scheme, the sets $\mathcal{I}$ and $\mathcal{J}$ may not be unique, however the various index sets $\mathcal{I}$ and $\mathcal{J}$ are paired by the following equation:

$$\sum_{i \in \mathcal{I}} A_i = n \sum_{j \in \mathcal{J}} E_j = I_w \otimes J_n \tag{2.1}$$

for some choice of ordering of the vertices. Thus the v vertices are partitioned into w fibres of size n. Like $\mathcal{I}$ and $\mathcal{J}$, this partitioning $\mathcal{F}$ into fibres — the so-called imprimitivity system — may not be unique, but each of $\mathcal{I}$, $\mathcal{J}$, $\mathcal{F}$ is well-defined given any other one of the three. In the remainder of the paper we will always assume however that $\mathcal{I}$, $\mathcal{J}$, and the imprimitivity system are fixed and given, unless mentioned otherwise. Of the equivalent statements of imprimitivity, the last one could be explained as "dual imprimitivity". In fact, in this case each of the matrices E_j, $j \in \mathcal{J}$ is constant on each fibre $U \in \mathcal{F}$ (i.e., columns x and y of E_j are identical when $x, y \in U$). This is analogous to the fact that each relation $R_i, i \in \mathcal{I}$ is disconnected.

It easily follows that on each fibre $U \in \mathcal{F}$, there is an association scheme — a so-called subscheme — induced by the relations indexed by $\mathcal{I}$. In fact, the intersection numbers $\tilde{p}^h_{ij}$ of the subscheme are the same as the corresponding ones in the original scheme, i.e.,

$$\tilde{p}^h_{ij} = p^h_{ij}, i, j, h \in \mathcal{I}.$$

To put things differently, $\mathcal{B} := \langle A_i | i \in \mathcal{I}\rangle$ is a Bose-Mesner subalgebra of the Bose-Mesner algebra $\mathcal{A}$ (i.e., $\mathcal{B}$ is a subalgebra under both ordinary and entrywise multiplication). For later purpose, we define a linear (projection) operator $\pi : \mathcal{A} \to \mathcal{A}$ by

$$\pi(A) = A \circ (I_w \otimes J_n) \tag{2.2}$$

for $A \in \mathcal{A}$. It is clear that $\pi(A \circ A') = \pi(A) \circ \pi(A')$ for all $A, A' \in \mathcal{A}$. Because the map π sends $A = \sum_{i=0}^d c_i A_i$ to $\sum_{i \in \mathcal{I}} c_i A_i$, it is also clear that $\pi(\mathcal{A}) = \mathcal{B}$. Note also

[1] This matrix is given by Equation (2.1).

that $\mathcal{B}$ is a $\circ$-ideal in $\mathcal{A}$, because if $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $A \circ B = A \circ \pi(B) = \pi(A) \circ B \in \mathcal{B}$.

Each imprimitivity system also gives us a quotient association scheme. Dual to $\mathcal{B}$, consider

$$\mathcal{C} := \langle E_j | j \in \mathcal{J} \rangle = \{A(I_w \otimes J_n) | A \in \mathcal{A}\};$$

this is also a Bose-Mesner subalgebra of $\mathcal{A}$. It is the image of $\mathcal{A}$ under the projection π^* which sends $A = \sum_{j=0}^{d} c_j E_j$ to $\pi^*(A) := \frac{1}{n} A(I_w \otimes J_n) = \sum_{j \in \mathcal{J}} c_j E_j$. Each Schur idempotent of $\mathcal{C}$ must be a sum of certain A_i and if $A = \sum_{i \in \mathcal{H}} A_i$ satisfies $A = \frac{1}{n} A(I_w \otimes J_n)$, then $A_{xy} = A_{x'y'}$ whenever x is in the same fibre as x', and y is in the same fibre as y'. So, for each $C \in \mathcal{C}$, there exists a well-defined $w \times w$ matrix $\iota(C)$ satisfying

$$C = \iota(C) \otimes J_n.$$

It is not hard to verify that the set $\{\iota(C) | C \in \mathcal{C}\}$ is a Bose-Mesner algebra also; this gives an association scheme — the so-called quotient scheme — on the set of fibres. In this case, the Krein parameters of this quotient scheme are the same as the corresponding ones in the original scheme (cf. [7, Sec. 2.4]). For completeness we mention that Rao, Ray-Chaudhuri, and Singhi [42] obtained results on the composition factors of imprimitive schemes.

For the topic of this paper — uniform schemes and, later, cometric Q-antipodal schemes — our main interest is in the relation between the scheme and its subschemes. The corresponding quotient scheme is in this case trivial, that is, a one-class scheme corresponding to a complete graph. The relationship between the scheme and its subschemes and quotient schemes is essentially worked out by Bannai and Ito [5, Thm. II.9.9] (see also [7, Section 2.4] for some information on the relation between the parameters). However, to get a better understanding of what is going on, we include some of their arguments and results (and those of others) applied to subschemes here. (Moreover, Bannai and Ito treated the dual case, which, even though it is analogous, may sometimes be confusing.) By doing this, we derive in Lemma 2.4 another (and new, as far as we know) relation between the parameters.

Following Bannai and Ito, we define the relation $\sim^*$ on the index set $\{0, 1, ..., d\}$ (indexing the primitive idempotents) by

$$i \sim^* j :\Leftrightarrow q_{ij}^h \neq 0 \text{ for some } h \in \mathcal{J}.$$

Lemma 2.1. *The relation $\sim^*$ is an equivalence relation.*

Proof. If $i \sim^* j \sim^* l$, say $q_{ij}^h \neq 0$ and $q_{jl}^{h'} \neq 0$ with $h, h' \in \mathcal{J}$, then by using a standard identity (cf. [5, Prop. II.3.7(vii)], [7, Lem. 2.3.1(vi)]) and the fact that $q_{hh'}^{h''} = 0$ if $h'' \notin \mathcal{J}$, we obtain that

$$\sum_{h'' \in \mathcal{J}} q_{ih''}^l q_{hh'}^{h''} = \sum_{h''=0}^{d} q_{ih''}^l q_{hh'}^{h''} = \sum_{j'=0}^{d} q_{ih}^{j'} q_{j'h'}^l \geq q_{ih}^j q_{jh'}^l > 0,$$

and it follows that for some $h'' \in \mathcal{J}$ we have $q_{il}^{h''} \neq 0$, i.e., $i \sim^* l$. $\square$

One of the equivalence classes of this relation must be $\mathcal{J} =: \mathcal{J}_0$, and we label the others by $\mathcal{J}_1, ..., \mathcal{J}_e$.

Example 2.2. In the linked system of geometries described in the introduction, we obtained a four-class imprimitive association scheme on 150 vertices. In that example, if we use the Q-polynomial ordering of the eigenspaces, the relation $\sim^*$ has equivalence classes $\mathcal{J}_0 = \{0, 4\}$, $\mathcal{J}_1 = \{1, 3\}$ and $\mathcal{J}_2 = \{2\}$, which, as we shall later see, is indicative of the Q-antipodal case.

Now we claim that the idempotents

$$F_j := \sum_{j' \in \mathcal{J}_j} E_{j'}, j = 0, 1, ..., e$$

are the primitive idempotents of $\mathcal{B}$, and hence by restricting these to a fibre we obtain the primitive idempotents of the subscheme on that fibre. To prove this claim, and to obtain a useful relation between the Krein parameters of $\mathcal{A}$ and $\mathcal{B}$, we define the nonnegative parameter

$$\rho_j^i := \sum_{h \in \mathcal{J}} q_{jh}^i, \tag{2.3}$$

and note that $\rho_j^i \neq 0$ if and only if $i \sim^* j$. We abbreviate $\rho_j^j =: \rho_j$.

Lemma 2.3. *The primitive idempotents of* $\mathcal{B}$ *are* F_j, $j = 0, 1, ..., e$, *so* $\mathcal{B}$ *has dimension* $|\mathcal{I}| = e + 1$. *Moreover, if* $j' \in \mathcal{J}_j$, *then* $\pi(E_{j'}) = \frac{\rho_{j'}}{w} F_j$.

Proof. We first note that each primitive idempotent of $\mathcal{B}$ is a sum of primitive idempotents of $\mathcal{A}$, and because $\sum_{j=0}^d E_j = I \in \mathcal{B}$, each E_j appears in exactly one such sum. Then for each $j = 0, 1, ..., d$, we use (2.2), (2.1), and (2.3) to find

$$\pi(E_j) = n \sum_{h \in \mathcal{J}} E_j \circ E_h = \frac{1}{w} \sum_{i=0}^d \rho_j^i E_i.$$

Thus, if $\mathcal{H} \subseteq \{0, \ldots, d\}$ and $F := \sum_{j \in \mathcal{H}} E_j$ is any idempotent of $\mathcal{B}$, then

$$F = \pi(F) = \sum_{j \in \mathcal{H}} \pi(E_j) = \frac{1}{w} \sum_{i=0}^d \sum_{j \in \mathcal{H}} \rho_j^i E_i.$$

This implies that if $i \notin \mathcal{H}$, then $\sum_{j \in \mathcal{H}} \rho_j^i = 0$, i.e., if $i \notin \mathcal{H}$ and $j \in \mathcal{H}$, then $i \nsim^* j$, which proves that $\mathcal{H}$ is a union of equivalence classes of $\sim^*$.

On the other hand, take any $0 \leq j \leq d$ and consider the primitive idempotent $F := \sum_{h \in \mathcal{H}} E_h$ for which $j \in \mathcal{H}$. Because $\frac{1}{w} \sum_{i=0}^d \rho_j^i E_i = \pi(E_j) \in \mathcal{B}$, it is a linear combination of primitive idempotents of $\mathcal{B}$ with a nonzero coefficient for F because $\rho_j^j > 0$. So, if $h \in \mathcal{H}$, then $\rho_j^h > 0$, which shows that h and j are in the same equivalence class. We may therefore conclude that $\mathcal{H}$ is an equivalence class of $\sim^*$.

Thus, the primitive idempotents of $\mathcal{B}$ are F_j, $j = 0, 1, ..., e$. For $j' \in \mathcal{J}_j$, it then also follows that $\pi(E_{j'}) = \frac{1}{w} \sum_{i \sim^* j'} \rho_{j'}^i E_i$ is a multiple of one of these idempotents. So $\rho_{j'}^i = \rho_{j'}$ for all $i \sim^* j'$, and $\pi(E_{j'}) = \frac{\rho_{j'}}{w} F_j$. $\square$

By working out the products $F_i \circ F_j$, the Krein parameters $\tilde{q}_{ij}^h$ of the subscheme can now be easily expressed in terms of those of the original scheme as

$$\tilde{q}_{ij}^h = \frac{1}{w} \sum_{i' \in \mathcal{J}_i, j' \in \mathcal{J}_j} q_{i'j'}^{h'},$$

for each $h' \in \mathcal{J}_h$. Moreover, it follows that the eigenmatrices $\tilde{P}$ and $\tilde{Q}$ of the subscheme are given by

$$\begin{array}{rclr} \tilde{P}_{ji} & = & P_{j'i}, & i \in \mathcal{I}, j' \in \mathcal{J}_j, j = 0,1,...,e; \\ \tilde{Q}_{ij} & = & \frac{1}{w}\sum_{j' \in \mathcal{J}_j} Q_{ij'}, & i \in \mathcal{I}, j = 0,1,...,e. \end{array} \tag{2.4}$$

However, the second part of Lemma 2.3 can be used to get another useful expression of the Krein parameters of the subschemes.

Lemma 2.4. *If $i' \in \mathcal{J}_i$, $j' \in \mathcal{J}_j$, then*

$$\tilde{q}_{ij}^h = \frac{1}{\rho_{i'}\rho_{j'}} \sum_{h' \in \mathcal{J}_h} \rho_{h'} q_{i'j'}^{h'}.$$

Proof. Let $i' \in \mathcal{J}_i$, $j' \in \mathcal{J}_j$, then

$$\rho_{i'}\rho_{j'} F_i \circ F_j = w^2 \pi(E_{i'} \circ E_{j'}) = \frac{w^2}{v} \sum_{h'=0}^{d} q_{i'j'}^{h'} \pi(E_{h'}) = \frac{1}{n} \sum_{h=0}^{e} \sum_{h' \in \mathcal{J}_h} \rho_{h'} q_{i'j'}^{h'} F_h,$$

which was to be proven. □

3. Uniform imprimitive schemes

So far we have given a selective review of imprimitive association schemes, focusing on the eigenspaces and the Krein parameters of subschemes. Exploring imprimitivity further, the main goal of this section is to reconcile the concept of dismantlable association scheme introduced in [35] with the concept of uniform association scheme introduced earlier in [30].

3.1. **Dismantlability and uniformity.** Besides the usual subschemes on each fibre, it was proven in [35, Thm. 4.7] that a cometric Q-antipodal scheme has so-called dismantled schemes on each union of fibres. To generalize this result, and to obtain more information on these dismantled schemes in the subsequent sections, we first define the following.

For a subset Y of the vertices, let I^Y be the $v \times v$ diagonal $(0,1)$-matrix with $(I^Y)_{xx} = 1$ if and only if $x \in Y$. For a matrix M, we let

$$M^{YZ} := I^Y M I^Z$$

for subsets Y and Z. Put differently, M^{YZ} is the $v \times v$ matrix containing the submatrix M_{YZ}, and that is zero everywhere else. Algebraically, in most of the following it turns out to be more convenient to work with the matrices M^{YZ} than with the usual submatrices M_{YZ}, although essentially they are the same. For a relation R we define related notation

$$R^{YZ} := R \cap (Y \times Z).$$

In case $Y = Z$, we often use shorthand notation $M^Y := M^{YY}$ and $R^Y := R^{YY}$. For a set $\mathcal{R}$ of either matrices or relations, we let $\mathcal{R}^Y := \{R^Y | R \in \mathcal{R}\}$.

Definition 3.1. An imprimitive association scheme $(X, \mathcal{R})$ is called dismantlable if $(Y, \mathcal{R}^Y)$ is an association scheme for each union Y of fibres. In this case, the association scheme $(Y, \mathcal{R}^Y)$ is called a dismantled scheme on Y, if Y is the union of at least two fibres.

This definition first appears in [35] where the structure of cometric Q-antipodal association schemes is considered. We shall see in Corollary 4.4 that two dismantled schemes $(Y, \mathcal{R}^Y)$ and $(Y', \mathcal{R}^{Y'})$ of $(X, \mathcal{R})$ with $|Y| = |Y'|$ always have the same parameters.

Bipartite schemes, i.e., imprimitive schemes with two fibres, are trivially dismantlable. Other examples of dismantlable schemes are the so-called uniform association schemes, as defined by Higman in his paper on four-class imprimitive schemes [28] and more generally in an unpublished manuscript [30]. Informally speaking, an imprimitive scheme is uniform if the intersection numbers are divided uniformly over the fibres whereas, in the general case, only the valencies enjoy this property.

To define uniform schemes precisely, we first introduce a bit of notation. Consider an imprimitive scheme with a trivial quotient scheme, i.e., where the quotient is a complete graph. As in Equation (2.1), let $\mathcal{I}$ denote the indices of relations that occur in the subschemes. For fibres U and V, we denote by $\mathcal{I}(U, V)$ the index set of relations that occur between U and V; so A_i^{UV} is nonzero precisely if $i \in \mathcal{I}(U, V)$. Because we are assuming that the quotient is a complete graph, $\mathcal{I}(U, V)$ equals $\mathcal{I}$ if $U = V$, and $\mathcal{I}(U, V) = \overline{\mathcal{I}}$ (the complement of $\mathcal{I}$) if $U \neq V$.

Definition 3.2. An imprimitive association scheme is called uniform if its quotient scheme is trivial, and if there are integers a_{ij}^h such that for all fibres U, V, and W, and $i \in \mathcal{I}(U, V)$, $j \in \mathcal{I}(V, W)$, we have

$$A_i^{UV} A_j^{VW} = \sum_h a_{ij}^h A_h^{UW}. \tag{3.1}$$

It is easily seen that in this case $p_{ij}^h = a_{ij}^h$ if $i \in \mathcal{I}$ or $j \in \mathcal{I}$, $p_{ij}^h = (w-1)a_{ij}^h$ if $i, j \notin \mathcal{I}$ and $h \in \mathcal{I}$, and $p_{ij}^h = (w-2)a_{ij}^h$ if $i, j, h \notin \mathcal{I}$, i.e., the intersection numbers are divided uniformly over the relevant fibres. Note that bipartite schemes are trivially uniform. Also, any imprimitive d-class association scheme with only one relation across fibres (a complete multipartite graph) is uniform. Such a scheme can easily be constructed as a wreath product scheme [50, p44], [2, p69] of a trivial scheme and an arbitrary scheme. Also the tensor product [50, p44] of a one-class scheme and an arbitrary scheme is uniform. (This is also called the "direct product" [2, p62].) In this paper, we call a scheme decomposable if it is has the same parameters as a wreath product or tensor product scheme.

Theorem 3.3. *A uniform scheme is dismantlable. Any dismantled scheme of a uniform scheme is also uniform.*

Proof. These claims follow in a straightforward way from the definition of a uniform scheme. □

In Section 4.2 we will show the converse of this proposition, namely that every dismantlable scheme is uniform.

3.2. **Linked systems and triality.** In Section 1.1 we described what we (and Neumaier [40]) called a linked system of partial λ-geometries. This linked system is in fact a uniform association scheme with three fibres of size 50. The term linked system was coined by Cameron [9] for linked systems of symmetric designs (see also Section 6).

Example 3.4. There are three non-isomorphic $(16, 6, 2)$ symmetric block designs. Each incidence structure gives us a three-class bipartite association scheme with two fibres of size sixteen. But only one of these can be extended to a linked system of symmetric designs with eight fibres of size sixteen. This is a uniform cometric scheme on 128 vertices and is the first example in an infinite family which arises from the Kerdock codes [16] (see also [38, 35]).

Following Neumaier, and also Cameron and Van Lint [15] (see Section 7.1), we will use the term linked system informally for the combinatorial structure underlying a uniform association scheme. Note also that Higman [28] mentions the term "system of uniformly linked strongly regular designs". We will now describe an infinite family of such systems, which we refer to as Higman's "triality schemes".

Example 3.5. The dual polar graph $D_4(q)$ is a cometric bipartite distance-regular graph defined on the maximal isotropic (four-dimensional) subspaces in $GF(q)^8$ with a quadratic form of Witt index 4. One can extend this graph by a third fibre containing the isotropic one-dimensional subspaces. In this way one obtains a uniform association scheme that is cometric Q-antipodal. Higman [28] explains how this scheme is obtained from classical triality related to the group $O_8^+(q)$, and also how some other sporadic examples, such as the one in Section 1.1, have a triality related to some group. Higman also mentions that related to these examples are certain coherent configurations.

4. Coherent configurations and uniformity

To understand uniformity better, we will need to recall certain combinatorial structures that are more general than association schemes. As we will see, a (symmetric) d-class association scheme can be viewed as a homogeneous coherent configuration of rank $d+1$ in which all relations are symmetric.

4.1. **Definitions and algebraic automorphisms.** A coherent configuration is a pair $(X, \mathcal{S})$ consisting of a finite set X of size v and a set $\mathcal{S}$ of binary relations on X such that

- $\mathcal{S}$ is a partition of $X \times X$;
- the diagonal relation Δ_X is the union of some relations in $\mathcal{S}$;
- for each $R \in \mathcal{S}$ it holds that $R^\top \in \mathcal{S}$;
- there exist integers p_{ST}^R such that
$$|\{z \in X|(x,z) \in S \text{ and } (z,y) \in T\}| = p_{ST}^R$$
whenever $(x,y) \in R$, for each $R, S, T \in \mathcal{S}$.

The relations of $\mathcal{S}$ are called basic relations of the configuration. A basic relation R is called a diagonal relation if $R \subseteq \Delta_X$. Each diagonal relation is of the form Δ_U for some $U \subseteq X$. Because the relations of $\mathcal{S}$ form a partition of $X \times X$, the diagonal relations of $\mathcal{S}$ form a partition of Δ_X. Thus there exists a uniquely determined partition of X into a set $\mathcal{F}_\mathcal{S}$ of w fibres such that $\Delta_U \in \mathcal{S}$ for each $U \in \mathcal{F}_\mathcal{S}$. The numbers $v = |X|$ and $|\mathcal{S}|$ are called the order and the rank of the configuration, respectively.

Given $R \in \mathcal{S}$ and $x \in X$ we define $R(x) := \{y \in X \,|(x,y) \in R\}$. For any basic relation R we define its projections onto the first and second coordinates as $\mathsf{pr}_1(R) := \{x \in X \,|\, R(x) \neq \emptyset\}$ and $\mathsf{pr}_2(R) := \mathsf{pr}_1(R^\top)$. One can show that these projections are fibres. So, each basic relation R is contained in $\mathsf{pr}_1(R) \times \mathsf{pr}_2(R)$. We write $\mathcal{S}^{UV}$ for the set of all basic relations $R \in \mathcal{S}$ with $\mathsf{pr}_1(R) = U, \mathsf{pr}_2(R) = V$, and $r_{UV} := |\mathcal{S}^{UV}|$. Note that $r_{UV} = r_{VU}$ and $|\mathcal{S}| = \sum_{U,V} r_{UV}$. The $w \times w$ integer symmetric matrix (r_{UV}) is called the type of the configuration.

The last axiom of the definition of coherent configuration implies that

$$A_S A_T = \sum_R p_{ST}^R A_R.$$

It thus follows that the vector subspace of $M_X(\mathbb{C})$ spanned by the adjacency matrices A_R, $R \in \mathcal{S}$ is a subalgebra of the full matrix algebra $M_X(\mathbb{C})$. It also explains why the intersection numbers p_{ST}^R are sometimes called structure constants. The subalgebra is called the adjacency algebra of $\mathcal{S}$ and will be denoted by $\mathbb{C}[\mathcal{S}]$. This algebra has the following properties:

- it is closed with respect to (ordinary) matrix multiplication;
- it is closed with respect to entrywise (Schur-Hadamard) multiplication $\circ$;
- it is closed with respect to transposition $^\top$;
- it contains the identity matrix I and the all-ones matrix J.

Any subspace of $M_X(\mathbb{C})$ which satisfies these conditions is called a coherent algebra. There is a one-to-one correspondence between coherent configurations on X and coherent algebras in $M_X(\mathbb{C})$, i.e., each coherent algebra is the adjacency algebra of a uniquely determined coherent configuration.

An algebraic automorphism of $\mathcal{S}$ is a permutation $\sigma \in \mathsf{Sym}(\mathcal{S})$ which preserves the structure constants, that is, $p_{ST}^R = p_{\sigma(S)\sigma(T)}^{\sigma(R)}$ for all $R, S, T \in \mathcal{S}$ (an algebraic automorphism of an association scheme is also called a pseudo-automorphism, cf. [32]). One can extend such a σ to a linear map from $\mathbb{C}[\mathcal{S}]$ into itself by setting $\sigma(\sum_{R\in\mathcal{S}} \alpha_R A_R) := \sum_{R\in\mathcal{S}} \alpha_R A_{\sigma(R)}$. This yields an automorphism of the adjacency algebra; the linear map defined in this way preserves the ordinary matrix product, Schur-Hadamard product, and matrix transposition, i.e., $\sigma(AB) = \sigma(A)\sigma(B), \sigma(A\circ B) = \sigma(A) \circ \sigma(B)$, and $\sigma(A^\top) = \sigma(A)^\top$ for all $A, B \in \mathbb{C}[\mathcal{S}]$. Vice versa, each permutation σ which preserves these three operations is an algebraic automorphism of $\mathcal{S}$.

The algebraic automorphisms of $\mathcal{S}$ form a group (which is a subgroup of $\mathsf{Sym}(\mathcal{S})$), which will be denoted by $\mathsf{AAut}(\mathcal{S})$. Any subgroup $G \leq \mathsf{AAut}(\mathcal{S})$ gives rise to a fusion configuration $\mathcal{S}/G$ whose basic relations are $\cup_{R\in O} R$, $O \in \Omega$, where Ω is the set of orbits of $\mathcal{S}$ under the action of G. The adjacency algebra of $\mathcal{S}/G$ can be characterized as the subspace of $\mathbb{C}[\mathcal{S}]$ consisting of all G-invariant elements of $\mathbb{C}[\mathcal{S}]$.

The matrices $I^U, U \in \mathcal{F}_\mathcal{S}$ are the only idempotent matrices of the standard basis $\{A_R | R \in \mathcal{S}\}$ of $(X, \mathcal{S})$. Therefore any algebraic automorphism σ of $\mathcal{S}$ permutes these diagonal matrices, hereby also inducing a permutation $U \mapsto \sigma(U)$ on the set of fibres. So, instead of $\sigma(I^U)$, we could also write $I^{\sigma(U)}$.

If $G \leq \mathsf{AAut}(\mathcal{S})$ acts transitively on the set of fibres, then $\mathcal{S}/G$ is homogeneous, that is, it is a coherent configuration with one fibre, or in other words, a — possibly nonsymmetric — association scheme.

4.2. **Uniformity in coherent configurations.** We now make a fundamental observation about uniform association schemes. Consider such a scheme $(X, \mathcal{R})$, with related (generic) notation as above. It follows immediately from (3.1) that the set of relations $\mathcal{S} := \{R_i^{UV} | i \in \mathcal{I}(U, V); U, V \in \mathcal{F}\}$ forms a coherent configuration, with the same fibres as those of the association scheme, i.e., $\mathcal{F}_\mathcal{S} = \mathcal{F}$. Moreover, any $\sigma \in \mathsf{Sym}(\mathcal{F}_\mathcal{S})$ acts as a permutation on $\mathcal{S}$ by $\sigma(R_i^{UV}) := R_i^{\sigma(U)\sigma(V)}$, for

$i \in \mathcal{I}(U,V) = \mathcal{I}(\sigma(U),\sigma(V))$. In this way, σ is an algebraic automorphism of $\mathcal{S}$, because if $i \in \mathcal{I}(U,V)$, $j \in \mathcal{I}(V,W)$, $h \in \mathcal{I}(U,W)$, then

$$p_{R_i^{UV}R_j^{VW}}^{R_h^{UW}} = a_{ij}^h = p_{R_i^{\sigma(U)\sigma(V)}R_j^{\sigma(V)\sigma(W)}}^{R_h^{\sigma(U)\sigma(W)}} = p_{\sigma(R_i^{UV})\sigma(R_j^{VW})}^{\sigma(R_h^{UW})}.$$

Moreover, the fusion scheme $\mathcal{S}/\mathsf{Sym}(\mathcal{F}_\mathcal{S})$ is the association scheme that we started from. These observations are the motivation for the definition of a uniform coherent configuration. But first we need a little more terminology. We say that two triples (U,V,W) and (U',V',W') of fibres have the "same type" if and only if there is a permutation σ of the fibres such that $\sigma((U,V,W)) = (U',V',W')$.

Definition 4.1. A coherent configuration $(X,\mathcal{S})$ with at least two fibres is called uniform if there are complementary sets of indices $\mathcal{I}_\mathcal{S} \ni 0$ and $\overline{\mathcal{I}_\mathcal{S}}$ of sizes $e_\mathcal{S}+1$ and $\ell_\mathcal{S}$ (say), respectively, such that the basic relations $R \in \mathcal{S}$ can be relabeled as $R = S_i^{UV}$ ($U = \mathsf{pr}_1(R)$, $V = \mathsf{pr}_2(R)$, $i \in \mathcal{I}_\mathcal{S} \cup \overline{\mathcal{I}_\mathcal{S}}$) such that

- $S_0^{UU} = \Delta_U$ for each fibre U;
- $\mathcal{S}^{UU} = \{S_i^{UU} | i \in \mathcal{I}_\mathcal{S}\}$ for each fibre U and $\mathcal{S}^{UV} = \{S_i^{UV} | i \in \overline{\mathcal{I}_\mathcal{S}}\}$ for all fibres $U \neq V$;
- $(S_i^{UV})^\top = S_i^{VU}$ for all fibres $U \neq V$;
- for any two triples (U,V,W) and (U',V',W') of the same type and any $i \in \mathcal{I}_\mathcal{S}(U,V)$, $j \in \mathcal{I}_\mathcal{S}(V,W)$, $h \in \mathcal{I}_\mathcal{S}(U,W)$, it holds that

$$p_{S_i^{UV}S_j^{VW}}^{S_h^{UW}} = p_{S_i^{U'V'}S_j^{V'W'}}^{S_h^{U'W'}}. \tag{4.1}$$

In this definition $\mathcal{I}_\mathcal{S}(U,V)$ is defined in the same way as before: it equals $\mathcal{I}_\mathcal{S}$ if $U=V$, and $\overline{\mathcal{I}_\mathcal{S}}$ otherwise. Without loss of generality we will assume that $\mathcal{I}_\mathcal{S} \cup \overline{\mathcal{I}_\mathcal{S}} = \{0,...,e_\mathcal{S}+\ell_\mathcal{S}\}$.

It is clear from the above observations that from a uniform association scheme one obtains a uniform coherent configuration with $\mathcal{F}_\mathcal{S} = \mathcal{F}$, $\mathcal{I}_\mathcal{S} = \mathcal{I}$, $\overline{\mathcal{I}_\mathcal{S}} = \overline{\mathcal{I}}$, $e_\mathcal{S} = e$, $\ell_\mathcal{S} = d-e$, and $S_i^{UV} = R_i^{UV}$.

Conversely, given a uniform coherent configuration, any permutation σ of the fibres acts — just as in Section 4.1 — as an algebraic automorphism of $\mathcal{S}$, by (4.1). Thus, the relations $R_i := \cup_{U,V} S_i^{UV}$ are the relations of the $(e_\mathcal{S}+\ell_\mathcal{S})$-class association scheme $\mathcal{S}/\mathsf{Sym}(\mathcal{F}_\mathcal{S})$. It is clear that this scheme is imprimitive with $\mathcal{F} = \mathcal{F}_\mathcal{S}$ and $\mathcal{I} = \mathcal{I}_\mathcal{S}$, and that its quotient scheme is trivial. Because (3.1) follows from (4.1), this scheme is uniform. We have thus shown a one-to-one correspondence between uniform association schemes and uniform coherent configurations.

Proposition 4.2. *If $(X,\mathcal{R})$ is a uniform association scheme, then $(X, \{R_i^{UV} | i \in \mathcal{I}(U,V); U,V \in \mathcal{F}\})$ is a uniform coherent configuration. Conversely, if $(X,\mathcal{S})$ is a uniform coherent configuration, then after relabeling $(X, \{\cup_{U,V} S_i^{UV} | i = 0,...,e_\mathcal{S}+\ell_\mathcal{S}\})$ is a uniform association scheme on X.*

We will now use this one-to-one correspondence to show that every dismantlable association scheme is uniform.

Theorem 4.3. *An association scheme is dismantlable if and only if it is uniform.*

Proof. One direction has already been shown in Theorem 3.3.

Let $(X,\mathcal{R})$ be a dismantlable association scheme. Because bipartite schemes are uniform, we may assume that $w \geq 3$. We must first check that the quotient scheme is trivial. To see this, it suffices to show that, for any three distinct fibres U, V and

W, $\mathcal{I}(U,V)=\mathcal{I}(V,W)$. But this is clear since the dismantled scheme on vertex set $Y=U\cup V\cup W$ is still imprimitive and there is only one choice for its quotient: the trivial scheme on three vertices. So $\mathcal{I}(U,V)=\mathcal{I}(V,W)=\overline{\mathcal{I}}$.

Next, we claim that $\mathcal{S}:=\{R_i^{UV}|i\in\mathcal{I}(U,V);U,V\in\mathcal{F}\}$ forms a coherent configuration on X. In order to do this, we will have to consider the intersection numbers of the dismantled schemes $(Y,\mathcal{R}^Y)$ where Y is a union of fibres, which we denote by $p_{ij}^h(Y)$. To establish the claim, we first observe that the non-empty relations R_i^{UV} form a partition of $X\times X$ and that $(R_i^{UV})^\top=R_i^{VU}$.

Now pick an arbitrary triple of relations $R_i^{UV},R_j^{VW},R_h^{UW}$, with $i\in\mathcal{I}(U,V),j\in\mathcal{I}(V,W),h\in\mathcal{I}(U,W)$. We have to show that the number

$$\lambda_{ijh}^{UVW}(u,x):=|R_i^{UV}(u)\cap(R_j^{VW})^\top(x)|=|R_i(u)\cap R_j(x)\cap V|$$

does not depend on the pair $(u,x)\in R_h^{UW}$.

If $U=V$ then $i\in\mathcal{I}$, implying $R_i(u)\cap V=R_i(u)$. Therefore $\lambda_{ijh}^{UVW}(u,x)=|R_i(u)\cap R_j(x)|=p_{ij}^h$. Analogously, $\lambda_{ijh}^{UVW}(u,x)=p_{ij}^h$ if $V=W$.

Next, we consider the case that $U\neq V$ and $U=W$. In this case $h\in\mathcal{I}$, while $i,j\in\overline{\mathcal{I}}$. Consider the scheme $(Y,\mathcal{R}^Y)$, where $Y=U\cup V$. For $u,x\in U=W$, we have

$$p_{ij}^h(Y)=|R_i^Y(u)\cap(R_j^Y)^\top(x)|=|R_i(u)\cap R_j(x)\cap Y|. \tag{4.2}$$

Because $i,j\in\overline{\mathcal{I}}$, the intersections $R_i(u)\cap U$ and $R_j(x)\cap U$ are empty. Therefore $R_i(u)\cap R_j(x)\cap Y=R_i(u)\cap R_j(x)\cap V$, and hence $\lambda_{ijh}^{UVW}(u,x)=p_{ij}^h(U\cup V)$.

The last case is the one in which the fibres U,V,W are pairwise distinct. Then $i,j,h\in\overline{\mathcal{I}}$. Consider the scheme $(Y,\mathcal{R}^Y)$, where $Y=U\cup V\cup W$. As before, we have (4.2) for $u\in U,x\in W$. Because $i,j\in\overline{\mathcal{I}}$, we obtain $R_i(u)\cap Y\subseteq V\cup W$ and $R_j(x)\cap Y\subseteq U\cup V$. This implies that $R_i(u)\cap R_j(x)\cap Y=R_i(u)\cap R_j(x)\cap V$, and therefore $\lambda_{ijh}^{UVW}(u,x)=p_{ij}^h(U\cup V\cup W)$.

Thus we proved that the relations in $\mathcal{S}$ form a coherent configuration, with intersection numbers

$$p_{R_i^{UV}R_j^{VW}}^{R_h^{UW}}=\lambda_{ijh}^{UVW}=\begin{cases} p_{ij}^h & \text{if } U=V \text{ or } V=W;\\ p_{ij}^h(U\cup V) & \text{if } U\neq V,U=W;\\ p_{ij}^h(U\cup V\cup W) & \text{if } U\neq V,V\neq W,W\neq U.\end{cases} \tag{4.3}$$

Finally, we shall show that the coherent configuration is uniform. By the above one-to-one correspondence between uniform association schemes and uniform coherent configurations this proves the theorem. To show that the configuration is uniform, we have to prove that $\lambda_{ijh}^{UVW}=\lambda_{ijh}^{U'V'W'}$ whenever the triples (U,V,W) and (U',V',W') have the same type.

For $U=V$ (and, therefore, $U'=V'$), or if $V=W$, this is clear. If $U\neq V,U=W$ and $U'\neq V',U'=W'$, then $i,j\in\overline{\mathcal{I}},h\in\mathcal{I}$. In this case we have to show that $p_{ij}^h(U\cup V)=p_{ij}^h(U'\cup V')$. To prove this, it is sufficient to show that $p_{ij}^h(U\cup V)=p_{ij}^h(V\cup W)$ holds for any triple (U,V,W) of pairwise distinct fibres. So, consider a scheme $(Y,\mathcal{R}^Y)$, where $Y=U\cup V\cup W$. Because $h\in\mathcal{I}$, $R_h^Y=R_h^U\cup R_h^V\cup R_h^W$. Pick an arbitrary pair $(u,u')\in R_h^U$, that is, $(u,u')\in R_h$ and $u,u'\in U$. Because $i,j\in\overline{\mathcal{I}}$, we have that

$$p_{ij}^h(Y)=|R_i(u)\cap R_j(u')\cap Y|=|R_i(u)\cap R_j(u')\cap V|+|R_i(u)\cap R_j(u')\cap W|=$$

$$|R_i(u)\cap R_j(u')\cap(U\cup V)|+|R_i(u)\cap R_j(u')\cap(U\cup W)|=p_{ij}^h(U\cup V)+p_{ij}^h(U\cup W).$$

The same argument with $(x,x') \in R_h^W$ shows that $p_{ij}^h(W \cup V) + p_{ij}^k(W \cup U) = p_{ij}^h(Y)$, and hence $p_{ij}^h(U \cup V) = p_{ij}^h(V \cup W)$.

Consider now the remaining case where the triples (U,V,W) and (U',V',W') consist of pairwise distinct fibres. In this case $i,j,h \in \overline{\mathcal{I}}$ and we have to show that $p_{ij}^h(U \cup V \cup W) = p_{ij}^h(U' \cup V' \cup W')$. If $w = 3$, then there is nothing to prove, so we may assume that $w \geq 4$. In this case it is sufficient to show that $p_{ij}^h(U \cup V \cup W) = p_{ij}^h(V \cup W \cup Z)$ holds for each quadruple U,V,W,Z of pairwise distinct fibres. The arguments for this are similar as in the previous case. Consider the scheme $(Y, \mathcal{R}^Y)$, where $Y = U \cup V \cup W \cup Z$. Then it follows from considering pairs $(u,y) \in R_h^{UV}$ and $(y,z) \in R_h^{VZ}$ that

$$p_{ij}^h(U \cup V \cup W) + p_{ij}^h(U \cup V \cup Z) = p_{ij}^h(Y) = p_{ij}^h(V \cup Z \cup W) + p_{ij}^h(V \cup Z \cup U),$$

which finishes the proof. □

As an immediate consequence, we obtain important structural information about dismantled schemes.

Corollary 4.4. *Let $(X,\mathcal{R})$ be a dismantlable association scheme with w fibres. If $2 \leq w' \leq w$ and each of $Y, Y' \subseteq X$ are expressible as a union of w' fibres, then the dismantled schemes $(Y, \mathcal{R}^Y)$ and $(Y', \mathcal{R}^{Y'})$ have the same parameters (i.e., same eigenmatrices P and Q and same intersection numbers and Krein parameters, with appropriate orderings of their relations and idempotents).*

Proof. It follows from Definition 3.2 that the parameters of the dismantled scheme $(Y, \mathcal{R}^Y)$ depend only on w' and the parameters a_{ij}^h and not on the choice of Y itself. □

4.3. **Q-Higman schemes.** In the previous section, we have seen that uniformity of a scheme is equivalent to dismantlability. In this section, we give a characterization of uniform schemes in terms of the Krein parameters (through so-called Q-Higman schemes) and study the idempotents of uniform schemes.

4.3.1. *Krein parameters of Q-Higman schemes.* With cometric Q-antipodal association schemes in mind, we consider an imprimitive association scheme with $\mathcal{J} = \{0,d\}$, $\mathcal{J}_j = \{j, d-j\}$ for $j = 0,1,...,\ell-1 < \frac{d}{2}$, and $\mathcal{J}_j = \{j\}$ for $j = \ell,...,d-\ell$ (for some ℓ).

For such a scheme we consider the dual intersection matrix L_d^* with entries $(L_d^*)_{ij} := q_{dj}^i$. First note that $\rho_j = 1 + q_{dj}^j$. If $j < \ell$ or $j > d-\ell$, then from $E_j \circ (E_0 + E_d) = \frac{1}{n}\pi(E_j) = \frac{1}{v}(1 + q_{dj}^j)(E_j + E_{d-j})$, we find that $E_j \circ E_d = \frac{1}{v}(q_{dj}^j E_j + (1 + q_{dj}^j)E_{d-j})$, and hence that $q_{dj}^{d-j} = 1 + q_{dj}^j$, and $q_{dj}^i = 0$ for $i \neq j, d-j$.

For $\ell \leq j \leq d-\ell$, we find from $E_j \circ (E_0 + E_d) = \frac{1}{v}(1 + q_{dj}^j)E_j$ that $q_{dj}^i = 0$ for $i \neq j$, and hence $q_{dj}^j = w - 1$. In other words, the only nonzero entries of L_d^* are on the diagonal and the antidiagonal.

For $j < \ell$ or $j > d-\ell$, we may combine the facts $q_{dj}^{d-j} = 1 + q_{dj}^j$ and $q_{dj}^j + q_{d,d-j}^j = m_d = w-1$ to find $(1 + q_{dj}^j)m_{d-j} = (w - 1 - q_{dj}^j)m_j$. This implies that $m_{d-j} \leq (w-1)m_j$ with equality if and only if $q_{dj}^j = 0$. We thus obtain the following

Lemma 4.5. *Consider an imprimitive association scheme with $\mathcal{J} = \{0,d\}$, $\mathcal{J}_j = \{j, d-j\}$ for $j = 0,1,...,\ell-1 < \frac{d}{2}$, and $\mathcal{J}_j = \{j\}$ for $j = \ell,...,d-\ell$. If $\ell \leq j \leq d-\ell$, then $q_{dj}^i = 0$ for $i \neq j$ and $\rho_j = q_{dj}^j + 1 = w$. If $j < \ell$ or $j > d-\ell$, then*

$\rho_j = q_{dj}^{d-j} = 1 + q_{dj}^j$ and $q_{dj}^i = 0$ for $i \neq j, d-j$, and moreover, $m_{d-j} \leq (w-1)m_j$ with equality if and only if $q_{dj}^j = 0$.

The case of equality is one of the motivations for the following definition.

Definition 4.6. An imprimitive association scheme is called Q-Higman if for some ℓ such that $1 \leq \ell < \frac{d}{2}+1$ and for some ordering of the primitive idempotents, we have that $\mathcal{J} = \{0, d\}$, $\mathcal{J}_j = \{j, d-j\}$ for $j = 0, 1, ..., \ell-1$, $\mathcal{J}_j = \{j\}$ for $j = \ell, ..., d-\ell$, and $q_{jj}^d = 0$ (or equivalently $m_{d-j} = (w-1)m_j$) for $j = 0, 1, ..., \ell-1$.

It is important to note that this Q-Higman property is formulated entirely in terms of the Krein parameters, in particular in terms of the dual intersection matrix L_d^*.

Proposition 4.7. *An association scheme is Q-Higman if and only if for some ℓ such that $1 \leq \ell < \frac{d}{2}+1$, for some w, and some ordering of the idempotents it holds that $q_{d,d-j}^j = w-1$ for $j < \ell$, $q_{d,d-j}^j = 1$ and $q_{dj}^j = w-2$ for $j > d-\ell$, $q_{dj}^j = w-1$ for $\ell \leq j \leq d-\ell$, and $q_{dj}^i = 0$ for all other values of i and j. Moreover, if this is the case, then $\rho_j = 1$ if $j < \ell$, $\rho_j = w$ if $\ell \leq j \leq d-\ell$, and $\rho_j = w-1$ if $j > d-\ell$.*

Proof. If the scheme is Q-Higman, then the stated properties follow from the above considerations. On the other hand, suppose that these properties hold. Then it follows that $v(E_0+E_d)\circ(E_0+E_d) = w(E_0+E_d)$ and that $\langle E_0, E_d\rangle$ is a $\circ$-subalgebra. This means that the scheme is imprimitive with $\mathcal{J} = \{0, d\}$ and fibres of size $\frac{v}{w}$. The equivalence classes of $\sim^*$ then easily follow, and so does the conclusion that the scheme is Q-Higman. $\square$

We note that the standard relations between the Krein parameters of a scheme (e.g., see [7, Lemma 2.3.1]) give some more specific information on those of Q-Higman schemes. It can for example be derived (from [7, Lemma 2.3.1] or directly by working out the product $E_i \circ E_j \circ E_d$ in different ways) that if $j < \ell$ and i is arbitrary, then $q_{i,d-j}^h = (w-1)q_{ij}^{d-h}$ for $h < \ell$, $q_{i,d-j}^h = (w-1)q_{ij}^h$ for $\ell \leq h \leq d-\ell$, and $q_{i,d-j}^h = q_{ij}^{d-h} + (w-2)q_{ij}^h$ for $h > d-\ell$. It also follows that $q_{ij}^h = q_{ij}^{d-h}$ for all i, $\ell \leq j \leq d-\ell$ and $h < \ell$. In the cometric Q-antipodal case, we include these observations in Lemma 5.5 below.

4.3.2. *The idempotents of uniform schemes.* In this section we shall show one of our main results, i.e., that Q-Higman schemes and uniform schemes are the same. For this we will again use the correspondence to uniform coherent configurations.

We remind the reader that $\mathcal{A} = \langle A_i \,|\, i = 0, ..., d\rangle$ is the Bose-Mesner algebra of the association scheme under consideration, and that $\mathcal{B} = \langle A_i \,|\, i \in \mathcal{I}\rangle$ is the Bose-Mesner subalgebra on the fibres. Moreover, we let

$$\mathcal{D} := \langle A_i \,|\, i \notin \mathcal{I}\rangle.$$

In order to show that a uniform scheme is Q-Higman, and to find relations with its dismantled schemes, we study its idempotents. We start off with the case of bipartite schemes, i.e., imprimitive schemes with two fibres.

Lemma 4.8. *A bipartite scheme is Q-Higman. Each primitive idempotent of $\mathcal{B}$ that is not a primitive idempotent of $\mathcal{A}$ is of the form $E + E'$, where E and E' are primitive idempotents of $\mathcal{A}$, and $E - E' \in \mathcal{D}$.*

Proof. Consider a bipartite scheme with fibres U and V. Because all relations $R_i, i \notin \mathcal{I}$ are bipartite, it follows that $E = \begin{bmatrix} E_{UU} & E_{UV} \\ E_{VU} & E_{VV} \end{bmatrix}$ is a primitive idempotent if and only if $E' = \begin{bmatrix} E_{UU} & -E_{UV} \\ -E_{VU} & E_{VV} \end{bmatrix}$ is a primitive idempotent. Moreover, then $E + E'$ is a primitive idempotent of the Bose-Mesner subalgebra $\mathcal{B}$, and $E - E' \in \mathcal{D}$. This implies that the primitive idempotents of $\mathcal{B}$ that are not primitive idempotents of $\mathcal{A}$ are of the form $E + E'$, where E and E' are primitive idempotents of $\mathcal{A}$, and $E - E' \in \mathcal{D}$. Thus, all sets $\mathcal{J}_j$ have size at most two. Moreover, the multiplicities of the idempotents E and E' are equal, because $\mathrm{trace}(E) = \mathrm{trace}(E')$. Thus, the scheme is Q-Higman. □

Lemma 4.9. *Consider a uniform association scheme. Let $F \in \mathcal{B}$ be a primitive idempotent of $\mathcal{B}$. Then F is a primitive idempotent of $\mathcal{A}$ if and only if $F\mathcal{D} = \{0\}$. Let Y be a union of at least two fibres. Then F^Y is a primitive idempotent of $\mathcal{B}^Y$. Moreover, F^Y is a primitive idempotent of $\mathcal{A}^Y$ if and only if F is a primitive idempotent of $\mathcal{A}$.*

Proof. An idempotent F of $\mathcal{A}$ is primitive if and only if FA is proportional to F for each $A \in \mathcal{A}$. Because F is a primitive idempotent of $\mathcal{B}$, FA is proportional to F for each $A \in \mathcal{B}$. Therefore F is a primitive idempotent of $\mathcal{A}$ if and only if FA is proportional to F for each $A \in \mathcal{D}$. So consider $A \in \mathcal{D}$. Because F is block-diagonal and $A^U = 0$ for $U \in \mathcal{F}$, we obtain $(FA)^U = 0$. Therefore FA is proportional to F if and only if $FA = 0$.

Because of the block-diagonal structure of $\mathcal{B}$, F^Y is clearly a primitive idempotent of $\mathcal{B}^Y$, and $(F\mathcal{D})^Y = F^Y\mathcal{D}^Y$. Because the linear map $A \mapsto A^Y$ is a bijection between $\mathcal{A}$ and $\mathcal{A}^Y$, it follows that $F\mathcal{D} = \{0\}$ if and only if $F^Y\mathcal{D}^Y = \{0\}$, hereby proving the final statement of the lemma. □

Theorem 4.10. *Consider a uniform association scheme. Let $F_0, ..., F_e \in \mathcal{B}$ be a complete set of primitive idempotents of $\mathcal{B}$, ordered such that $F_0, ..., F_{\ell-1}$ are not primitive in $\mathcal{A}$, and $F_\ell, ..., F_e$ are primitive in $\mathcal{A}$. Then for each $j = 0, ..., \ell-1$ there exists a matrix $D_j \in \mathcal{D}$ such that for each union Y of $w' \geq 2$ fibres, the matrices $\frac{1}{w'}(F_j^Y + D_j^Y)$ and $F_j^Y - \frac{1}{w'}(F_j^Y + D_j^Y), j = 0, ..., \ell - 1$, and $F_\ell^Y, ..., F_e^Y$ are the primitive idempotents of $\mathcal{A}^Y$.*

Proof. First of all it follows from Lemma 4.9 that the matrices $F_\ell^Y, ..., F_e^Y$ are primitive idempotents of $\mathcal{A}^Y$. Secondly, we fix $j \in \{0, ..., \ell - 1\}$ for the moment, and let $F := F_j$. We then claim that there is a matrix $D \in \mathcal{D}$, which is unique up to sign, such that for any two distinct fibres U, V, we have

$$\begin{aligned} F^{UU}D^{UV} &= D^{UV}F^{VV} = D^{UV}, \\ D^{UV}D^{VU} &= F^{UU}, \\ D^{VU}D^{UV} &= F^{VV}. \end{aligned} \tag{4.4}$$

To prove this claim, we first fix two fibres U and V, let $Z := U \cup V$, and consider the bipartite dismantled scheme on Z. By Lemma 4.9 we have that F^Z is a primitive idempotent of $\mathcal{B}^Z$ which is not a primitive idempotent of $\mathcal{A}^Z$. From Lemma 4.8 we obtain that $F^Z = E + E'$, where E and E' are primitive idempotents of $\mathcal{A}^Z$ such that $E - E' \in \mathcal{D}^Z$. Because the map $D \mapsto D^Z$ is a bijection between $\mathcal{D}$ and $\mathcal{D}^Z$, there is a matrix $D \in \mathcal{D}$ such that $D^Z = E - E'$. Because E and E' are orthogonal, this matrix D satisfies $F^Z D^Z = D^Z F^Z = D^Z$ and $(D^Z)^2 = F^Z$. It then follows that D satisfies (4.4) for the fixed fibres U and V. Now we use the fact that $\mathsf{Sym}(\mathcal{F})$ acts doubly transitively on $\mathcal{F}$: by applying algebraic automorphisms $\sigma \in \mathsf{Sym}(\mathcal{F})$ to these equations, we find that they hold for all fibres U, V.

It remains to prove uniqueness of D. Let $M \in \mathcal{D}$ be a matrix satisfying (4.4), i.e., $F^Z M^Z = M^Z F^Z = M^Z$ and $(M^Z)^2 = F^Z$. Because $F^Z = E + E'$ and E, E' are primitive idempotents of $\mathcal{A}^Z$, there exist four solutions of the equation $(M^Z)^2 = F^Z$ with $M^Z \in \mathcal{A}^Z$, namely $\pm E \pm E'$ (this easily follows by writing M^Z as a linear combination of primitive idempotents of $\mathcal{A}^Z$). On the other hand, the matrices $\pm F^Z, \pm D^Z$ satisfy this equation. Therefore $M^Z = \pm D^Z$. Again, because the map $D \mapsto D^Z$ is a bijection between $\mathcal{D}$ and $\mathcal{D}^Z$, we obtain that $M = \pm D$, and the claim is proven.

The above considerations show the existence of $D \in \mathcal{D}$ such that $FD = D$ and $D^2 = (w-1)F + (w-2)D$ for the case $w = 2$. Now let us assume that $w \geq 3$. Fix three arbitrary but distinct fibres, say U, V, W, and consider the product $D^{UV} D^{VW}$. Because of uniformity this product belongs to $\mathcal{A}^{UW}$. Therefore there exists a $G \in \mathcal{D}$ such that $G^{UW} = D^{UV} D^{VW}$. It follows from (4.4) that $F^{UU} G^{UW} = G^{UW} F^{WW} = G^{UW}$, $G^{UW} G^{WU} = F^{UU}$, and $G^{WU} G^{UW} = F^{WW}$. From the above claim it then follows that $G = \varepsilon D$, where $\varepsilon = \pm 1$. Thus $D^{UV} D^{VW} = \varepsilon D^{UW}$, and after replacing D by εD this becomes $D^{UV} D^{VW} = D^{UW}$. Applying — as before — algebraic automorphisms $\sigma \in \mathsf{Sym}(\mathcal{F})$ to this equality we obtain that $D^{U'V'} D^{V'W'} = D^{U'W'}$ for any triple of pairwise distinct fibres U', V', W'.

If Y is a union of $w' \geq 2$ fibres, then a routine calculation shows that $(D^Y)^2 = (w'-1)F^Y + (w'-2)D^Y$. After releasing the fixation of j by indexing F and D, we thus obtain that

$$F_j^Y D_j^Y = D_j^Y \text{ and } (D_j^Y)^2 = (w'-1)F_j^Y + (w'-2)D_j^Y. \tag{4.5}$$

For fixed Y, it remains to show that the matrices $E_j := \frac{1}{w'}(F_j^Y + D_j^Y)$ and $E'_j := F_j^Y - \frac{1}{w'}(F_j^Y + D_j^Y), j = 0, ..., \ell-1$, and $F_\ell^Y, ..., F_e^Y$ are the primitive idempotents of $\mathcal{A}^Y$. It follows from (4.5) that E_j, E'_j are pairwise orthogonal idempotents. To show that E_j, E'_j are orthogonal to E_h, E'_h for $h \neq j$, and to F_h^Y for $h \geq \ell$, it is sufficient to check that $F_j^Y D_h^Y = F_h^Y D_j^Y = D_j^Y D_h^Y = 0$. These equations hold because $F_h^Y D_j^Y = F_h^Y F_j^Y D_j^Y = 0$ and $D_j^Y D_h^Y = F_j^Y D_j^Y F_h^Y D_h^Y = F_j^Y F_h^Y D_j^Y D_h^Y = 0$.

Thus we have $2\ell + e + 1 - \ell = e + 1 + \ell$ pairwise orthogonal idempotents of $\mathcal{A}^Y$. It remains to show that $d + 1 = e + 1 + \ell$. Because d, e, ℓ do not depend on w' (for $w' \geq 2$; for ℓ this follows from Lemma 4.9), it is enough to check this equality for $w' = 2$. But in the case of $w' = 2$ each primitive idempotent of $\mathcal{B}$ is either primitive in $\mathcal{A}$ or splits into a sum of two primitive idempotents of $\mathcal{A}$, as we saw in Lemma 4.8. This implies that $d + 1 = e + 1 + \ell$. □

Corollary 4.11. *A uniform association scheme is Q-Higman.*

Proof. Consider a uniform association scheme. Apply Theorem 4.10 with $Y = X$ and $w' = w$ to see that the sets $\mathcal{J}_j$ have size at most two, i.e., its primitive idempotents that are not primitive idempotents of $\mathcal{B}$ come in pairs E_j, E'_j. The corresponding multiplicities satisfy $m'_j = \mathrm{trace}(E'_j) = \frac{w-1}{w}\mathrm{trace}(F_j) = (w-1)\mathrm{trace}(E_j) = (w-1)m_j$, which concludes the proof. □

The next result also follows easily from Theorem 4.10.

Corollary 4.12. *Consider a uniform association scheme, with primitive idempotents $E_j, j = 0, ..., d$ (ordered as in Definition 4.6), and let Y be a union of $w' \geq 2$ fibres. Then the primitive idempotents of the dismantled scheme on Y are $\overline{E}_j := \frac{w}{w'}E_j^Y$ and $\overline{E}_{d-j} := E_{d-j}^Y + E_j^Y - \frac{w}{w'}E_j^Y$, $j = 0, ..., \ell - 1$, and $\overline{E}_j := E_j^Y$, $j = \ell, ..., d - \ell$.*

To show the converse of Corollary 4.11, i.e., that a Q-Higman scheme is uniform, we use the following lemma, whose proof is similar to the dismantlability proof of a cometric Q-antipodal scheme in [35, Thm 4.7].

Lemma 4.13. *Consider a Q-Higman scheme. Then for each fibre U,*

$$E_jI^UE_h = \begin{cases} w^{-1}E_j & \text{if} \quad h = j \text{ and } j = 0, ..., \ell - 1; \\ E_jI^U - w^{-1}E_j & \text{if} \quad h = d - j \text{ and } j = 0, ..., \ell - 1; \\ I^UE_{d-j} - w^{-1}E_{d-j} & \text{if} \quad h = d - j \text{ and } j = d - \ell + 1, ..., d; \\ E_jI^U - I^UE_{d-j} + w^{-1}E_{d-j} & \text{if} \quad h = j \text{ and } j = d - \ell + 1, ..., d; \\ E_jI^U & \text{if} \quad h = j \text{ and } j = \ell, ..., d - \ell; \\ 0 & \quad \text{otherwise.} \end{cases}$$

Proof. Similar as in the proof of [35, Thm 4.7], it follows from [7, p61, Eq. 9] that

$$\| vE_jI^UE_h - n\delta_{jh}E_j \|^2 = q_{jh}^d n^2(w - 1). \tag{4.6}$$

To start with the bottom line of the expression for $E_jI^UE_h$: if $h \nsim^* j$ then $h \neq j$ and $q_{jh}^d = 0$, and we obtain from (4.6) that $E_jI^UE_h = 0$.

If $h = j$ with $j = 0, ..., \ell - 1$, then $q_{jh}^d = 0$ and so $E_jI^UE_j = w^{-1}E_j$.

If $h = d - j$ with $j = 0, ..., \ell - 1$, then

$$E_jI^UE_{d-j} = E_jI^U(I - \sum_{i \neq d-j} E_i) = E_jI^U - E_jI^UE_j = E_jI^U - w^{-1}E_j.$$

For $j = d - \ell + 1, ..., d$, we have that $0 \leq d - j \leq \ell - 1$, hence from the above it follows that $E_{d-j}I^UE_j = E_{d-j}I^U - w^{-1}E_{d-j}$. By transposing this expression we obtain that $E_jI^UE_{d-j} = I^UE_{d-j} - w^{-1}E_{d-j}$.

Also for $j = d - \ell + 1, ..., d$ we have that

$$E_jI^UE_j = E_jI^U(I - \sum_{i \neq j} E_i) = E_jI^U - E_jI^UE_{d-j} = E_jI^U - I^UE_{d-j} + w^{-1}E_{d-j}.$$

For $j = \ell, ..., d - \ell$, the idempotent E_j is block-diagonal, implying that $E_jI^UE_j = E_jI^U$. □

Theorem 4.14. *Consider a Q-Higman association scheme. Then*

$$\mathfrak{M} := \left\langle E_j^{UV} \mid j = 0, ..., d - \ell \text{ and } U, V \in \mathcal{F} \right\rangle$$

is a coherent algebra corresponding to a uniform coherent configuration.

Proof. We shall show that $\mathfrak{M}$ is closed with respect to transposition, ordinary matrix multiplication, and entrywise multiplication, and contains I and J, thus proving it is a coherent algebra.

First however, we claim that $E_j^{UV} \in \mathfrak{M}$ also for $j = d-\ell+1, ..., d$. Indeed, in this case $0 \le d-j \le \ell-1$ and $v^{-1}E_j = E_d \circ E_{d-j}$. Therefore

$$\begin{aligned} E_j^{UV} = vE_d^{UV} \circ E_{d-j}^{UV} &= \begin{cases} -J^{UV} \circ E_{d-j}^{UV} & \text{if } U \neq V \\ (w-1)J^{UV} \circ E_{d-j}^{UV} & \text{if } U = V \end{cases} \\ &= \begin{cases} -E_{d-j}^{UV} & \text{if } U \neq V \\ (w-1)E_{d-j}^{UV} & \text{if } U = V \end{cases} \in \mathfrak{M}. \end{aligned}$$

Hence $E_j^{UV} \in \mathfrak{M}$ for each j, U, V. This implies that $E_j \in \mathfrak{M}$ for each j, and hence $I, J \in \mathfrak{M}$.

Concerning the closure properties, note that closure with respect to transposition is evident. Closure with respect to matrix multiplication follows from Lemma 4.13, because it implies that

$$E_i^{UV} E_j^{WZ} = \delta_{VW}\, \delta_{ij}\, \lambda\, E_i^{UZ} \in \mathfrak{M}, \tag{4.7}$$

where $\lambda = w^{-1}$ for $i = 0, ..., \ell-1$ and $\lambda = \delta_{WZ}$ for $i = \ell, ..., d-\ell$ (here δ is the Kronecker delta). Closure with respect to entrywise multiplication follows from

$$E_j^{UV} \circ E_h^{UV} = (E_j \circ E_h)^{UV} = v^{-1} \sum_{i=0}^{d} q_{jh}^i E_i^{UV} \in \mathfrak{M}.$$

It remains to show uniformity. Note that it is clear from the above that $\mathfrak{M}$ contains all the matrices A_i^{UV}; the nonzero matrices among these form a basis of Schur idempotents for the corresponding coherent configuration. Because A_i^{UV} can be expressed as a linear combination of the $E_j^{UV}, j = 0, ..., d-\ell$, it follows from (4.7) that the coherent configuration is uniform. $\square$

Corollary 4.15. *A Q-Higman scheme is uniform. Any dismantled scheme of such a scheme is also Q-Higman.*

Proof. The first statement follows from Theorem 4.14 and the correspondence between uniform coherent configurations and uniform schemes (Proposition 4.2). The second statement follows from dismantlability (Proposition 3.3) and the converse of the first part (Corollary 4.11). $\square$

We thus have proven the following.

Theorem 4.16. *An association scheme is uniform if and only if it is Q-Higman.*

5. Cometric Q-antipodal schemes

A cometric association scheme (with a Q-polynomial ordering $E_0, E_1, ..., E_d$) is called Q-antipodal if it is imprimitive with $\mathcal{J} = \{0, d\}$. It is called Q-bipartite if it is imprimitive with $\mathcal{J} = \{0, 2, 4, ...\}$, or equivalently if $a_i^* = 0$ for all i, cf. [48].

It was shown by Suzuki [48] that an imprimitive cometric d-class association scheme is Q-antipodal, Q-bipartite, or both, unless possibly when $d = 4$ or $d = 6$. The exceptional case for $d = 4$ was later ruled out by Cerzo and Suzuki [17]. Here we will consider the Q-antipodal case.

5.1. **Uniformity.** Consider a cometric Q-antipodal association scheme. In this case, it follows that the equivalence classes of the relation $\sim^*$ are $\mathcal{J}_j = \{j, d-j\}, j = 0, 1, ..., \lfloor \frac{d}{2} \rfloor$. So the primitive idempotents of the Bose-Mesner subalgebra $\mathcal{B}$ are $F_j = E_j + E_{d-j}, j < \frac{d}{2}$, and $F_{\frac{d}{2}} = E_{\frac{d}{2}}$ for d even. Note also that $q^j_{dj} = 0$ for $j < \frac{d}{2}$, hence a cometric Q-antipodal scheme is Q-Higman (with $\ell = \lceil \frac{d}{2} \rceil$), and therefore it is also uniform, and dismantlable. On the other hand, we will show now that a uniform cometric scheme is Q-antipodal.

Theorem 5.1. *A cometric association scheme is uniform if and only if it is Q-antipodal.*

Proof. One direction is clear from the above. Consider now a cometric scheme that is uniform with imprimitivity system $\mathcal{F}$. So the scheme is Q-Higman, and let us assume that the idempotents are ordered as in Definition 4.6; in particular we have $\mathcal{J} = \{0, d\}$. In order to show that the scheme is cometric Q-antipodal, it suffices to show that E_d is last in a Q-polynomial ordering too. In the case $d = 3$, however, a somewhat degenerate case also arises where E_d is second in the Q-polynomial ordering, but in this ordering E_1 is last and there is a second imprimitivity system $\mathcal{F}'$ with subscheme corresponding to $\mathcal{J}' = \{0, 1\}$.

We first note that it is clear that E_d cannot be a Q-polynomial generator, and that this proves the case $d = 2$.

Next, consider the case $d > 3$. Then E_d must take the last position in any Q-polynomial ordering as $E_i \circ E_d \in \langle E_i, E_{d-i} \rangle$ eliminates positions from three up to $d - 1$ (taking E_i to be the Q-polynomial generator) and position two (taking $i = d$ and some E_j, $j \in \{1, 2, ..., d - 1\}$, in position four).

For the case $d = 3$, we apply several properties of the Krein parameters from Proposition 4.7. Consider a Q-polynomial ordering, and assume that E_3 is not in its last position. Because $q^3_{11} = 0$, this ordering cannot be E_0, E_1, E_3, E_2, hence it must be E_0, E_2, E_3, E_1. In this latter case, the scheme is cometric Q-bipartite, hence $q^i_{2i} = 0$ for all i. Because $q^2_{32} = w - 2$, it follows that $m_2 = \sum_j q^2_{2j} = w - 1$, which in turn shows that $m_1 = 1$. Thus $\{E_0, E_1\}$ induces another imprimitivity system $\mathcal{F}'$ with $\mathcal{J}' = \{0, 1\}$. Because E_1 is last in the Q-polynomial ordering under consideration, this implies that also in this case the scheme is cometric Q-antipodal. □

An interesting consequence of Theorem 5.1 is that among the cometric association schemes, the Q-antipodal ones can be recognized combinatorially.

The exceptional case in the above proof is realized only by the rectangular scheme $R(w, 2), w > 2$ (the direct product of two trivial schemes; on w and 2 vertices). Note that this cometric Q-antipodal Q-bipartite scheme has one Q-polynomial ordering, but two "uniform" imprimitivity systems; for one such system there is a uniform ordering of the idempotents (as in Definition 4.6) that matches the Q-polynomial ordering, for the other not. The proof of Theorem 5.1 thus implies the following.

Corollary 5.2. *Consider a uniform d-class association scheme with $\mathcal{J} = \{0, d\}$. If the scheme is cometric then E_d is in the last position in any cometric ordering, unless possibly when $d = 3$ and the scheme is isomorphic to the rectangular scheme $R(w, 2), w > 2$.*

We next obtain some (known) results for the parameters of cometric Q-antipodal schemes. These are used, for example, to show that the dismantled schemes are also cometric.

Lemma 5.3. *A cometric Q-antipodal scheme has $b^*_j = c^*_{d-j}$ for all $j \neq \lfloor \frac{d}{2} \rfloor$, $a^*_j = a^*_{d-j}$ for all $j \neq \frac{d-1}{2}, \frac{d+1}{2}$, and $m_{d-j} = (w-1)m_j$ for $j < \frac{d}{2}$. Moreover, for $j = \lfloor \frac{d}{2} \rfloor$, it holds that $b^*_j = (w-1)c^*_{d-j}$.*

Proof. From the fact that $E_1 \circ F_j \in \mathcal{B}$, it follows that this matrix is a linear combination of the F_i. From the expressions of $E_1 \circ E_j$ and $E_1 \circ E_{d-j}$ in terms of Krein parameters and idempotents, we then find that $b^*_j = c^*_{d-j}$ and $a^*_j = a^*_{d-j}$ for all $j \neq \lfloor \frac{d}{2} \rfloor$. It follows from Lemma 4.5 that $m_{d-j} = (w-1)m_j$ for $j < \frac{d}{2}$.

For odd d, and $j = \frac{d-1}{2}$, we have that $b^*_j = \frac{m_{j+1}}{m_j} c^*_{j+1} = (w-1)c^*_{d-j}$. For even d, and $j = \frac{d}{2}$, we have $b^*_j = \frac{m_{j+1}}{m_j} c^*_{j+1} = (w-1)\frac{m_{j-1}}{m_j} b^*_{j-1} = (w-1)c^*_j = (w-1)c^*_{d-j}$. □

Before we compute the Krein parameters of the subscheme, we determine the dual intersection matrix L^*_d and the values of ρ_j. These follow immediately from Proposition 4.7.

Lemma 5.4. *The Krein parameters of a cometric Q-antipodal scheme satisfy the following properties:*

(i) $q^j_{d,d-j} = w - 1$ *for* $j \leq \frac{d}{2}$;
(ii) $q^j_{d,d-j} = 1$ *and* $q^j_{dj} = w - 2$ *for* $j > \frac{d}{2}$;
(iii) $q^i_{dj} = 0$ *for all other values of i and j.*

Moreover, $\rho_j = 1$ if $j < \frac{d}{2}$, $\rho_{\frac{d}{2}} = w$, and $\rho_j = w - 1$ if $j > \frac{d}{2}$.

For convenient reference, we also collect here a few equations involving the remaining Krein parameters that were obtained in Section 4.3.1 above.

Lemma 5.5. *The Krein parameters of a cometric Q-antipodal scheme satisfy the following properties: if $0 \leq j < \frac{d}{2}$ and $0 \leq i \leq d$, then*

(i) $q^h_{i,d-j} = (w-1)q^{d-h}_{ij}$ *for* $h \leq \frac{d}{2}$;
(ii) $q^h_{i,d-j} = q^{d-h}_{ij} + (w-2)q^h_{ij}$ *for* $h > \frac{d}{2}$; *and*
(iii) $q^h_{i,\frac{d}{2}} = q^{d-h}_{i,\frac{d}{2}}$ *for all h when d is even.*

5.2. **Subschemes.** Lemma 2.4 can now be used to show that the subschemes are cometric.

Proposition 5.6. *Let $(X, \mathcal{R})$ be a cometric Q-antipodal association scheme with w fibres, and Krein array $\{b^*_0, b^*_1, ..., b^*_{d-1}; c^*_1, c^*_2, ..., c^*_d\}$, where $d \geq 3$. Then the subschemes induced on the fibres are cometric with Krein array*

$$\{b^*_0, b^*_1, ..., b^*_{\frac{d-1}{2}-1}; c^*_1, c^*_2, ..., c^*_{\frac{d-1}{2}}\}$$

for d odd, and Krein array

$$\{b^*_0, b^*_1, ..., b^*_{\frac{d}{2}-1}; c^*_1, c^*_2, ..., wc^*_{\frac{d}{2}}\}$$

for d even.

Proof. We use Lemma 2.4 with $i' = i = 1$ and $j' = j \leq \frac{d}{2}$ and $h \leq \frac{d}{2}$.

First, let $h < \frac{d}{2} - 1$. Then we have that $q_{1j}^{d-h} = 0$ because the scheme is cometric, and hence $\tilde{q}_{1j}^h = 0$ if $j < h-1$ or $j > h+1$. Moreover, $\tilde{q}_{1,h-1}^h = q_{1,h-1}^h = c_h^*$, and $\tilde{q}_{1,h+1}^h = q_{1,h+1}^h = b_h^*$. Similarly, it follows for $h \geq \frac{d}{2} - 1$ that $\tilde{q}_{1j}^h = 0$ if $j < h-1$.

For $h = \frac{d}{2} - 1$, we have $\tilde{q}_{1,h-1}^h = q_{1,h-1}^h = c_h^*$, and $\tilde{q}_{1,h+1}^h = \frac{1}{w}(q_{1,h+1}^h + (w-1)q_{1,h+1}^{d-h}) = \frac{1}{w}(b_h^* + (w-1)c_{d-h}^*) = b_h^*$.

For $h = \frac{d}{2}$, we obtain that $\tilde{q}_{1,h-1}^h = wq_{1,h-1}^h = wc_h^*$, and finally, for $h = \frac{d-1}{2}$, we obtain that $\tilde{q}_{1,h-1}^h = q_{1,h-1}^h = c_h^*$. Thus, it follows that the scheme is cometric, and the Krein array follows. □

Note that it follows from the proof that the Q-polynomial ordering of idempotents is $F_0, F_1, ..., F_{\lfloor \frac{d}{2} \rfloor}$. The multiplicities $\tilde{m}_j =$ rank F_j of a subscheme follow for example as follows: $\tilde{m}_j = \tilde{q}_{jj}^0 = \frac{1}{w}(q_{jj}^0 + q_{d-j,d-j}^0) = m_j$ for $j \neq \frac{d}{2}$, and $\tilde{m}_{\frac{d}{2}} = \frac{1}{w} m_{\frac{d}{2}}$.

5.3. **Dismantled schemes.** Proposition 5.6 is a well-known result. In [32, Thm. 4.7] it was shown that a cometric Q-antipodal scheme is dismantlable, with its dismantled schemes being cometric Q-antipodal too. The proof of the latter is not complete however, because incorrect idempotents are suggested there. The fact that such a dismantled scheme is Q-Higman is clear from Corollary 4.15. That it is cometric Q-antipodal can be shown as follows using Corollary 4.12.

Theorem 5.7. *Let $(X, \mathcal{R})$ be a cometric Q-antipodal association scheme with w fibres, and Krein array $\{b_0^*, b_1^*, ..., b_{d-1}^*; c_1^*, c_2^*, ..., c_d^*\}$, where $d \geq 3$, and let $\ell = \lceil \frac{d}{2} \rceil$. Then the dismantled scheme induced on a union Y of $w' \geq 2$ fibres is cometric Q-antipodal with Krein array $\{\overline{b}_0^*, \overline{b}_1^*, ..., \overline{b}_{d-1}^*; \overline{c}_1^*, \overline{c}_2^*, ..., \overline{c}_d^*\}$, where*

$$\overline{c}_j^* = c_j^* \text{ for } j \neq \ell, \text{ and } \overline{c}_\ell^* = \frac{w}{w'} c_\ell^* \ ,$$

$$\overline{b}_j^* = b_j^* \text{ for } j \neq d-\ell, \text{ and } \overline{b}_{d-\ell}^* = \frac{w}{w'} \frac{w'-1}{w-1} b_{d-\ell}^* \ .$$

Proof. The stated result follows from working out the products $\overline{E}_1 \circ \overline{E}_j$ for all j, where we use the expressions for $\overline{E}_j$ in Corollary 4.12, and the expressions for the dual intersection numbers b_j^*, a_j^*, and c_j^* in Lemma 5.3. For most cases this is rather straightforward; for readability we will therefore only give the details of one of the more complicated cases, i.e., that of d even and $j = \ell + 1$. In this case, with $v' = w'n = \frac{w'}{w} v$ being the number of vertices in Y, we have that

$$\begin{aligned} v' \overline{E}_1 \circ \overline{E}_{\ell+1} &= v E_1^Y \circ (E_{\ell+1}^Y + \tfrac{w'-w}{w'} E_{\ell-1}^Y) \\ &= b_\ell^* E_\ell^Y + a_{\ell+1}^* E_{\ell+1}^Y + c_{\ell+2}^* E_{\ell+2}^Y \\ &\quad + \tfrac{w'-w}{w'} (b_{\ell-2}^* E_{\ell-2}^Y + a_{\ell-1}^* E_{\ell-1}^Y + c_\ell^* E_\ell^Y) \\ &= (b_\ell^* + \tfrac{w'-w}{w'} c_\ell^*) E_\ell^Y + a_{\ell+1}^* (E_{\ell+1}^Y + \tfrac{w'-w}{w'} E_{\ell-1}^Y) \\ &\quad + c_{\ell+2}^* (E_{\ell+2}^Y + \tfrac{w'-w}{w'} E_{\ell-2}^Y) \\ &= \tfrac{w}{w'} \tfrac{w'-1}{w-1} b_\ell^* \overline{E}_\ell + a_{\ell+1}^* \overline{E}_{\ell+1} + c_{\ell+2}^* \overline{E}_{\ell+2}. \end{aligned}$$

Because $\ell = d - \ell$, it thus follows that $\overline{b}_{d-\ell}^* = \frac{w}{w'} \frac{w'-1}{w-1} b_{d-\ell}^*$, $\overline{a}_{\ell+1}^* = a_{\ell+1}^*$, and $\overline{c}_{\ell+2}^* = c_{\ell+2}^*$. The other parameters follow similarly, and prove the statement. □

Corollary 5.8. *Let $(X,\mathcal{R})$ be a cometric Q-antipodal d-class association scheme with $w \geq 3$ fibres, with d odd and $\ell = \frac{d+1}{2}$. Then $a^*_\ell \neq 0$. Moreover, if Y is a union of w' fibres, where $w > w' \geq 2$, then $\overline{a}^*_{\ell-1} \neq 0$.*

Proof. If $w > w' \geq 2$, then $\overline{a}^*_\ell = \overline{b}^*_0 - \overline{c}^*_\ell - \overline{b}^*_\ell = b^*_0 - \frac{w}{w'}c^*_\ell - b^*_\ell < a^*_\ell$, and similarly $\overline{a}^*_{\ell-1} > a^*_{\ell-1}$. The result follows from these inequalities. □

So, if d is odd, and the scheme is cometric Q-antipodal Q-bipartite, then $w = 2$. Moreover, it cannot be a dismantled scheme of a cometric Q-antipodal scheme with more fibres.

5.4. **The natural ordering of relations.** For a cometric scheme, we define the natural ordering of relations as the one satisfying $Q_{01} > Q_{11} > \cdots > Q_{d1}$. Recall that $Q_{ij}A_i = vE_j \circ A_i$. Because $\sum_{i\in\mathcal{I}} A_i = n(E_0 + E_d) = I_w \otimes J_n$ for Q-antipodal schemes, it follows that in this case Q_{id} equals $w-1$ if $i \in \mathcal{I}$, and -1 otherwise.

The orthogonal polynomials $q_j, j = 0,1,...,d+1$ associated to the cometric scheme have the property that $Q_{ij} = q_j(Q_{i1}), j = 0,1,...,d$ and $q_{d+1}(Q_{i1}) = 0$. Because the roots of q_j and q_{j+1} interlace (a standard and easily proven property of orthogonal polynomials, cf. [18, Thm. 5.3]), it follows that the values of Q_{id} alternate in sign. Thus for cometric Q-antipodal schemes it follows that

$$\mathcal{I} = \{0,2,4,....\}.$$

6. Three-class uniform schemes; linked systems of symmetric designs

Every two-class imprimitive association scheme is uniform and cometric. It has one (nontrivial) relation within the fibres and one across the fibres (it is a wreath product of two trivial schemes), and may thus be seen as a linked system of complete designs. Likewise, an imprimitive three-class scheme with one relation across the fibres is uniform (and decomposable), but such a scheme clearly cannot be cometric.

It is well-known that (homogeneous) linked systems of symmetric designs give three-class association schemes, and in fact, these are uniform, almost by definition, and cometric Q-antipodal (for information on such linked systems we refer to [20], [35], and the references therein). In [20, Thm. 5.8] it was conversely shown (in a different context though) that imprimitive indecomposable three-class schemes with one extra condition on the multiplicities must come from such linked systems. We can derive this easily now from the results in the previous sections.

Indeed, let us consider a three-class imprimitive association scheme that is indecomposable. Such a scheme must have two relations across the fibres and have a trivial quotient scheme. Thus we may assume that $\mathcal{J} = \{0,3\}$, $\mathcal{J}_1 = \{1,2\}$, and $\mathcal{I} = \{0,2\}$. Moreover we may assume that $m_2 \geq m_1$. It then follows that the scheme is uniform (Q-Higman) if and only if $m_2 = (w-1)m_1$ (which is the case if and only if $m_1 = n-1$). It is clear (straight from the definition) that such a uniform scheme corresponds to a linked system of symmetric designs. We thus obtain the same result as in [20, Thm. 5.8]. The eigenmatrices of a three-class uniform scheme can be written as

$$P = \begin{bmatrix} 1 & (w-1)k_1 & n-1 & (w-1)(n-k_1) \\ 1 & P_{11} & -1 & -P_{11} \\ 1 & -\frac{1}{w-1}P_{11} & -1 & \frac{1}{w-1}P_{11} \\ 1 & -k_1 & n-1 & -(n-k_1) \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1 & n-1 & (w-1)(n-1) & w-1 \\ 1 & Q_{11} & -Q_{11} & -1 \\ 1 & -1 & -(w-1) & w-1 \\ 1 & -\frac{k_1}{n-k_1}Q_{11} & \frac{k_1}{n-k_1}Q_{11} & -1 \end{bmatrix},$$

where k_1 is the block size of the symmetric designs in the corresponding linked system. If we order the relations such that $P_{11} > 0$, then $P_{11} = (w-1)\sqrt{\frac{k_1(n-k_1)}{n-1}}$ and $Q_{11} = \sqrt{\frac{(n-1)(n-k_1)}{k_1}}$. We remark that Noda [38, Prop. 0] showed that $\frac{k_1(n-k_1)}{n-1}$ is a square (integer) if $w \geq 3$.

Because the equality $m_2 = (w-1)m_1$ is equivalent to $q^3_{11} = 0$, it follows that such a uniform scheme is cometric except possibly when $k_1 = 1$ (note that $q^3_{12} > 0$ because $1 \sim^* 2$; and $q^2_{11} > 0$ follows except when $k_1 = 1$; we omit the derivation). In case $k_1 = 1$ however, the scheme is decomposable: it is a rectangular scheme $R(w, n)$ (the direct product of two trivial schemes), which is cometric (and metric) if and only if exactly one of w and n equals 2. We thus conclude the following.

Proposition 6.1. *Consider an imprimitive three-class association scheme that is indecomposable, and assume without loss of generality that $\mathcal{J} = \{0, 3\}$, $\mathcal{I} = \{0, 2\}$, and $m_2 \geq m_1$. Then it is uniform if and only if $m_2 = (w-1)m_1$. If so, then it is cometric Q-antipodal and corresponds to a linked system of symmetric designs.*

7. Four-class cometric Q-antipodal association schemes

We next consider the four-class schemes, comparing the "class I" imprimitive schemes of Higman with the cometric Q-antipodal schemes.

7.1. **A linked system of Van Lint-Schrijver partial geometries.** Uniform association schemes with three classes and more than one relation across fibres thus turn out to be cometric. For four classes this is not the case. There are several examples with just two fibres that are not cometric, such as those (non-cometric) schemes generated by bipartite distance-regular graphs with diameter four. The following example of a system of linked partial geometries by Cameron and Van Lint [15] is perhaps more interesting because it has three fibres.

Example 7.1. Consider the ternary repetition code C of length 6. The vertices of the association scheme are the 243 cosets of C in $GF(3)^6$, and these can be partitioned into three fibres according to the sum of the coordinates of any vector in the coset. Consider the graph where two cosets in different fibres are adjacent if one can be obtained from the other by adding a vector of weight one. This defines one of the two relations across fibres, and it generates the entire four-class scheme. The incidence structure between two fibres is a partial geometry that is isomorphic to the one constructed by Van Lint and Schrijver [34] (with parameters $pg(5+1, 5+1, 2)$), which has as a point graph (and line graph) a strongly regular graph with parameters $(81, 30, 9, 12)$; this gives the two (nontrivial) relations on the fibres. The scheme is not cometric because $q^1_{13} \neq 0$.

7.2. **Higman's imprimitive four-class schemes.** Higman [28] studied imprimitive four-class association schemes, and classified these according to the dimensions of the subalgebras $\mathcal{B}$ and $\mathcal{C}$ associated to a fixed imprimitivity system (or "parabolic") as outlined in Section 2.3 above. Since we showed that $\mathcal{B}$ has dimension $|\mathcal{I}|$

and $\mathcal{C}$ has dimension $|\mathcal{J}|$, we may say that a four-class scheme falls into Higman's "class I" (relative to a given imprimitivity system) if it has $|\mathcal{I}| = 3$ and $|\mathcal{J}| = 2$. It is known that the cometric Q-antipodal four-class association schemes fall into this "class I". In the next section we shall characterize the cometric schemes in this class.

Let us consider a "class I" scheme. Although Higman ordered relations and idempotents differently, we will assume (without loss of generality) that $\mathcal{J} = \{0, 4\}$ and $\mathcal{I} = \{0, 2, 4\}$. Then, using Lemma 2.3, we may assume that $\mathcal{J}_1 = \{1, 3\}$ and $\mathcal{J}_2 = \{2\}$. So the subscheme on each fibre is a strongly regular graph, on n vertices with valency k, say. Let r and s denote the nontrivial eigenvalues of this graph and let f and g denote the multiplicities of r and s, respectively. The eigenmatrices $\tilde{P}$ and $\tilde{Q}$ for this strongly regular graph are related to the eigenmatrices of this four-class scheme by Equation (2.4). Using this, we claim (and Higman [28] obtained the same) that the eigenmatrices for a "class I" scheme can be written as

$$P = \begin{bmatrix} 1 & (w-1)k_1 & k & (w-1)(n-k_1) & n-1-k \\ 1 & P_{11} & r & -P_{11} & -1-r \\ 1 & 0 & s & 0 & -1-s \\ 1 & -\frac{m_1}{m_3}P_{11} & r & \frac{m_1}{m_3}P_{11} & -1-r \\ 1 & -k_1 & k & -(n-k_1) & n-1-k \end{bmatrix} \tag{7.1}$$

and

$$Q = \begin{bmatrix} 1 & m_1 & wg & m_3 & w-1 \\ 1 & Q_{11} & 0 & -Q_{11} & -1 \\ 1 & \frac{m_1}{k}r & \frac{wg}{k}s & \frac{m_3}{k}r & w-1 \\ 1 & -\frac{k_1}{n-k_1}Q_{11} & 0 & \frac{k_1}{n-k_1}Q_{11} & -1 \\ 1 & -\frac{m_1}{n-1-k}(1+r) & -\frac{wg}{n-1-k}(1+s) & -\frac{m_3}{n-1-k}(1+r) & w-1 \end{bmatrix}, \tag{7.2}$$

where $k_1 := 1 + p^1_{12} + p^1_{14}$ and the remaining unknowns are related by

$$m_1 + m_3 = wf, \qquad P_{11}m_1 = Q_{11}v_1, \qquad v_1 = (w-1)k_1.$$

Indeed, for a given vertex x and a fibre U not containing x, k_1 equals the number of 1-neighbors of x in U. So the incidence structure between any two fibres induced by relation R_1 is a square 1-design with block size k_1. Thus the total number of 1-neighbors of x equals $v_1 = (w-1)k_1$. We also have $Q_{12} = Q_{32} = 0$ because $\mathcal{J}_2 = \{2\}$ forces $E_2 \in \mathcal{B}$. The remaining simplifications in (7.1) and (7.2) can easily be checked using the orthogonality relations $Q_{ij} = P_{ji}\frac{m_j}{v_i}$ and (column zero of) $PQ = QP = vI$.

It will benefit us to make the expressions (7.1) and (7.2) as unambiguous as possible. Let us agree to order the idempotents E_1 and E_3 by $m_1 \leq m_3$. Unless otherwise noted, we will order the relations R_2 and R_4 by assuming that $r \geq 0$, and the relations R_1 and R_3 by assuming $P_{11} \geq 0$. We now verify that, if such a scheme is cometric, then E_0, E_1, E_2, E_3, E_4 must be the Q-polynomial ordering, except possibly when $w = 2$.

Since columns two and four of Q have repeated entries, neither E_2 nor E_4 can be a Q-polynomial generator. In fact, E_4 must take the last position in any Q-polynomial ordering by the same argument as that in the proof of Theorem 5.1. Finally, E_2 cannot take position three because $q^4_{13} > 0$ follows from $1 \sim^* 3$. The last two possibilities for our Q-polynomial ordering are E_0, E_1, E_2, E_3, E_4 and E_0, E_3, E_2, E_1, E_4.

But $q_{43}^3 = 0$ then gives $m_1 = (w-1)m_3$ in the second case (by Lemma 4.5) and, with our conventions above, this can only happen if $w = 2$. In fact, when $w = 2$, we find that either one of these orderings – or both of them – can be Q-polynomial orderings. But in the case where E_3 is the Q-polynomial generator, the natural ordering of relations described in Section 5.4 is instead R_0, R_3, R_2, R_1, R_4.

From the 13-entry of the equation $PQ = vI$ and the 11-entry from the similar equation for the subscheme, we find that

$$P_{11} = \sqrt{\frac{m_3(w-1)k_1(n-k_1)}{m_1 f}}.$$

By using the expression [5, Thm. II.3.6(i)]

$$q_{ij}^h = \frac{m_i m_j}{v} \sum_l \frac{P_{il} P_{jl} P_{hl}}{v_l^2}, \tag{7.3}$$

and the similar expression

$$\tilde{q}_{11}^1 = \frac{f^2}{n}\left(1 + \frac{r^3}{k^2} - \frac{(1+r)^3}{(n-1-k)^2}\right)$$

for the subscheme we then derive that

$$q_{13}^1 = \frac{m_1 m_3}{w f^2}\left(\tilde{q}_{11}^1 - \sqrt{\frac{m_3}{m_1(w-1)}} \frac{(n-2k_1)\sqrt{f}}{\sqrt{k_1(n-k_1)}}\right),$$

which, of course, must vanish when the scheme is cometric with respect to the ordering E_0, E_1, E_2, E_3, E_4.

7.3. **Linked systems of strongly regular designs.** Let us proceed with the expressions of the previous section. From Lemma 4.5, we know that $q_{14}^1 = 0$ if and only if $m_3 = m_1(w-1)$. By Definition 4.6 and Theorem 4.16, this happens if and only if the scheme is uniform. In this case, the incidence structure between two fibres is a so-called strongly regular design as defined by Higman [27], and the scheme corresponds to a linked system of strongly regular designs. Cameron and Van Lint [15] constructed such an example, as we saw, and also the example in Section 1.1 is a linked system of strongly regular designs.

Proposition 7.2. *An imprimitive four-class association scheme of Higman's "class I" is cometric (and therefore Q-antipodal) if and only if $r \neq k$, $m_3 = (w-1)m_1$, and*

$$\tilde{q}_{11}^1 = \frac{(n-2k_1)\sqrt{f}}{\sqrt{k_1(n-k_1)}}, \tag{7.4}$$

possibly after reordering the idempotents E_1 and E_3 and the relations R_1 and R_3 in the case $w = 2$.

Proof. We address the case $w \geq 3$. The same ideas work in the case where $w = 2$, but an extra case argument is involved.

First recall that a cometric Q-antipodal scheme is uniform and we have just shown that uniformity, the vanishing of q_{14}^1, and the equation $m_3 = (w-1)m_1$ are all equivalent.

We know from above that the scheme is cometric if and only if E_0, E_1, E_2, E_3, E_4 is a Q-polynomial ordering. So we need

$$q^1_{14} = 0, \quad q^1_{13} = 0, \quad q^2_{11} > 0, \quad q^3_{12} > 0, \quad q^4_{13} > 0.$$

Observing that $q^2_{11} = \frac{m_1^2}{wf^2}\tilde{q}^2_{11}$ and $q^2_{13} = \frac{m_1 m_3}{wf^2}\tilde{q}^2_{11}$, and that $\tilde{q}^2_{11} = 0$ if and only if the strongly regular graphs on the fibres are imprimitive with $r = k$, one easily works out the remaining implications in both directions. □

Thus, for a cometric Q-antipodal four-class association scheme, all parameters can be expressed in terms of the number of fibres, w, and the parameters of the strongly regular graph.

Corollary 7.3. *If $(X, \mathcal{R})$ is a cometric Q-antipodal four-class association scheme with w fibres, then there exists a strongly regular graph with $n = v/w$ vertices, eigenvalues k, r, and s having multiplicities* 1*, f, and g respectively, such that the eigenmatrices for $(X, \mathcal{R})$ are given by Equations (7.1) and (7.2) where $m_1 = f$, $m_3 = (w-1)f$,*

$$P_{11} = (w-1)\sqrt{k_1(n-k_1)/f}, \qquad Q_{11} = \sqrt{f(n-k_1)/k_1},$$

and

$$k_1 = \frac{n}{2}\left(1 - \frac{\tilde{q}^1_{11}}{\sqrt{4f + (\tilde{q}^1_{11})^2}}\right). \tag{7.5}$$

Proof. The expression for k_1 follows from (7.4). □

Moreover, because $\tilde{q}^1_{11}$ is always a rational number (even in the case when some entry of $\tilde{P}$ is irrational), we obtain that $\sqrt{k_1(n-k_1)/f} \in \mathbb{Q}$ for a cometric Q-antipodal scheme with four classes, unless perhaps when $n = 2k_1$ (equivalently, $\tilde{q}^1_{11} = 0$). On the other hand $(w-1)\sqrt{k_1(n-k_1)/f} = P_{11}$ is an algebraic integer. Therefore P_{11} is a rational integer if $n \neq 2k_1$. Because a Q-antipodal cometric scheme is dismantlable, we can take $w = 2$ and consider P_{11} for the dismantled scheme; now we see that $k_1(n-k_1)/f$ is a perfect square provided $n \neq 2k_1$.

It also follows from (7.5) that if $\tilde{q}^1_{11} \neq 0$, then $4f + (\tilde{q}^1_{11})^2$ is a square of a rational number. This immediately implies the following result, which we will use in Section 7.5.

Proposition 7.4. *For a cometric Q-antipodal four-class association scheme, the strongly regular graph on a fibre cannot be a conference graph.*

Proof. Assume the contrary. Then $n = 2k+1$, $f = k > 0$, $\tilde{q}^1_{11} = (k-2)/2$, k is even, and

$$4f + (\tilde{q}^1_{11})^2 = 4k + \left(\frac{k-2}{2}\right)^2.$$

Because n is odd, $\tilde{q}^1_{11} \neq 0$. Therefore $k^2 + 12k + 4$ is the square of an integer. But $k = -12, 0$ are the only even integers for which the expression $k^2 + 12k + 4$ is a perfect square. □

The rationality condition that follows from (7.5) turns out to be quite a strong one. It is possible to show, for example, that also the lattice graphs cannot occur as our strongly regular graph on the fibres, and probably many more graphs can be excluded in this way. We will employ this condition as well in the next section.

7.4. **Four-class cometric Q-antipodal Q-bipartite association schemes; linked systems of Hadamard symmetric nets.** Recently, four-class cometric Q-antipodal Q-bipartite association schemes were shown to be equivalent to so-called real mutually unbiased bases, and a connection to Hadamard matrices was found in [33]. We also refer to [1] for connections between real mutually unbiased bases and association schemes. Here we shall derive the connection to Hadamard matrices, and see cometric Q-antipodal Q-bipartite four-class schemes as linked systems of Hadamard symmetric nets.

So, let us consider a cometric Q-antipodal Q-bipartite four-class association scheme, and its eigenmatrix Q in (7.2) with $m_3 = (w-1)m_1 = (w-1)f$ and $Q_{11} = \sqrt{f(n-k_1)/k_1}$ from Corollary 7.3. Since the scheme is cometric Q-bipartite, the column of Q corresponding to a Q-polynomial generator has its $d+1$ distinct values symmetric about zero when ordered naturally [35, Cor. 4.2]. In our case, this is either column one or column three, and in both cases it follows that $r = 0$, $n = k+2$, and $n = 2k_1$. This implies that $s = -2$, $f = \frac{n}{2}$, and the strongly regular graphs on the fibres are cocktail party graphs (complements of matchings). Now restrict to any dismantled scheme on $w' = 2$ fibres; straightforward calculations show that this must correspond to a so-called Hadamard graph, an antipodal bipartite distance-regular graph of diameter four, cf. [7, p19, 425]. Such graphs correspond to Hadamard matrices; more precisely, the incidence structure between a pair of fibres is a Hadamard symmetric net (that is, a symmetric (m,μ)-net with $m = 2$). We thus obtain that cometric Q-antipodal Q-bipartite four-class association schemes are linked systems of Hadamard symmetric nets. Interesting examples of these are given by the extended Q-bipartite doubles of the Cameron-Seidel schemes (linked systems of symmetric designs) [16]. These have $n = 2^{2k+1}$ and $w = 2^{2k-1}+1$ (which is extremal) for $k \geq 2$. For more details and constructions, the correspondence to real mutually unbiased bases, and bounds on w, we refer to [33].

On the other hand, we can characterize the cometric Q-antipodal Q-bipartite four-class association schemes as follows.

Proposition 7.5. *Consider a cometric Q-antipodal four-class association scheme, such that the strongly regular graph on a fibre is imprimitive. Then it is Q-bipartite.*

Proof. By Proposition 7.2, we have $r \neq k$. So $r = 0$, and the strongly regular graph on a fibre must be a complete multipartite graph, say a t-partite graph with parts of size $\frac{n}{t}$ each. For such a graph $s = -\frac{n}{t}$, $f = n-t$, and $\tilde{q}_{11}^1 = n-2t$. So, if $\tilde{q}_{11}^1 \neq 0$ (which is equivalent to $s \neq -2$), then $t \leq \frac{n}{3}$, and $4f + (\tilde{q}_{11}^1)^2$ is square (as before by (7.5)). However, for $t \leq \frac{n}{3}$, we have $(n-2t+2)^2+4t-4 = 4f+(\tilde{q}_{11}^1)^2 < (n-2t+4)^2$, so $4f + (\tilde{q}_{11}^1)^2$ cannot be square. Thus, $\tilde{q}_{11}^1 = 0$ and $t = \frac{n}{2}$, so the strongly regular graph on a fibre is a cocktail party graph, and therefore $n = k+2$ and $n = 2k_1$. From the expression for the eigenmatrix P in (7.1) and Equation (7.3), one can now derive that the Krein parameters $a_i^* = q_{1i}^i$ are zero for all i. Thus the scheme is Q-bipartite. Note that, in this case, not only is column one, but also is column three of Q symmetric about zero. □

The same result may be derived by using the fact that there are two different imprimitivity systems and Suzuki's results on imprimitive cometric schemes [48] and cometric schemes with multiple Q-polynomial orderings [49]. It would be interesting to work this out more generally, that is, for any cometric scheme with multiple imprimitivity systems, but we leave this to the interested reader.

7.5. **Strongly regular graphs with a strongly regular decomposition.** One of the interesting features of the example in Section 1.1 is that there is a decomposition of the Higman-Sims graph into two Hoffman-Singleton graphs; thus a strongly regular graph decomposes into two strongly regular graphs. Such strongly regular graphs with a strongly regular decomposition were studied by Haemers and Higman [25] and Noda [39], and they occur in more examples of four-class cometric Q-antipodal association schemes, as we shall see.

Let $\Gamma_0 = (X, E)$ be a primitive strongly regular graph with adjacency matrix M, parameter set $(v, k_0, \lambda_0, \mu_0)$, and distinct eigenvalues $k_0 > r_0 > s_0$. A strongly regular decomposition of Γ_0 is a partition of X into two sets U_1 and U_2 such that the induced subgraphs $\Gamma_i := \Gamma_{U_i}, i = 1, 2$ are strongly regular.

For our purpose, the sets U_1 and U_2 will play the role of the $w = 2$ fibres of an imprimitive (bipartite) association scheme, and the disjoint union of the graphs Γ_1 and Γ_2 is one of the two relations in $\mathcal{I}$. Thus we will only consider the case that the sets U_1 and U_2 are of equal size, and the parameter sets of Γ_1 and Γ_2 are the same, say (n, k, λ, μ). The eigenvalues of both graphs will be denoted by $k \geq r > s$.

To make the connection between a strongly regular graph with a strongly regular decomposition and our four-class association schemes more precise, write $M = \begin{bmatrix} M_1 & C \\ C^\top & M_2 \end{bmatrix}$, where the blocks correspond to our partition of X. We then define relations by the following adjacency matrices:

$$\begin{aligned} A_0 &:= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad A_1 := \begin{bmatrix} 0 & C \\ C^\top & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \\ A_3 &:= \begin{bmatrix} 0 & J - C \\ J - C^\top & 0 \end{bmatrix}, \quad A_4 := \begin{bmatrix} J - M_1 - I & 0 \\ 0 & J - M_2 - I \end{bmatrix}. \end{aligned} \tag{7.6}$$

We shall determine when these relations form an association scheme, and if they do, we shall see that the scheme is cometric Q-antipodal. But first we make some more observations.

By taking the complements of Γ_i, $i = 0, 1, 2$, we obtain another strongly regular graph with a strongly regular decomposition; we call this the complementary decomposition. Note that this complementary decomposition determines the same relations, i.e., the same A_i, $i = 0, ..., 4$, but ordered differently. In case that these relations form an association scheme, it is not clear a priori which ordering corresponds to the one in the eigenmatrix P in (7.1). The straightforward choice that we make is that we consider that decomposition for which the eigenvalues k, r, s of $\Gamma_i, i = 1, 2$ correspond to the k, r, s in the eigenmatrix P (however, in the case of hemisystems in the next section we make an exception).

A strongly regular decomposition is called exceptional if $r_0 \neq r$ and $s_0 \neq s$. It was shown by Haemers and Higman [25, Thm. 2.7] (and it also follows from [39, Thm. 1]) that in this exceptional case the graphs Γ_1 and Γ_2 are conference graphs. Thus, Proposition 7.4 implies that such an exceptional decomposition does not correspond to a cometric scheme. An example of an exceptional decomposition is that of the Petersen graph into two pentagons.

Note that when the relations defined by (7.6) do form an association scheme, then it has a fusion scheme $\{A_0, A_1 + A_2, A_3 + A_4\}$. In that case it follows from the expression (7.1) for the eigenmatrix P that the strongly regular graph Γ_0 with adjacency matrix $M = A_1 + A_2$ has an eigenvalue $0 + s$, hence $s_0 = s$, and the decomposition is not exceptional. Note that in (7.1) the roles of A_1 and A_3 may be swapped, but this has no influence on the observation. Thus, in the case of an exceptional decomposition, (7.6) does not yield an association scheme.

We shall now show that if the relations defined by (7.6) form a scheme, then this scheme is cometric Q-antipodal. This follows from the following proposition, where we consider Higman's "class I" schemes with two fibres, i.e., $w = 2$; since the scheme is bipartite, it is Q-Higman and Lemma 4.5 applies, giving us $m_4 = 1$, $m_1 = m_3 = f$ and $q^j_{4,4-j} = 1$. Thus, $q^1_{41} = q^3_{43} = 0$, and the conditions from Proposition 7.2 for the scheme to be cometric reduce to $r \neq k$ and $q^3_{11} = 0$ or $q^1_{33} = 0$.

Proposition 7.6. *Consider an imprimitive four-class association scheme in Higman's "class I" with two fibres. Suppose that the scheme has a primitive two-class fusion scheme, and that $r \neq k$. Then the scheme is cometric Q-antipodal.*

Proof. From the form of the eigenmatrix P in (7.1) it follows that the only way to obtain a primitive two-class fusion (i.e., one where both nontrivial relations correspond to connected strongly regular graphs) is to fuse relation R_1 with either R_2 or R_4, and to fuse the remaining two nontrivial relations. But then there exists a corresponding partition $\{T_1, T_2\}$ of $\{1, 2, 3, 4\}$ such that E_0, $E_{T_1} := \sum_{j \in T_1} E_j$ and $E_{T_2} := \sum_{j \in T_2} E_j$ are the primitive idempotents of the fusion scheme. Depending on the fusion of relations, one of the fused relations has eigenvalue $\pm P_{11} + r$ corresponding to idempotent E_1, and eigenvalue $\mp P_{11} + r$ corresponding to idempotent E_3, these two eigenvalues differing by $2P_{11}$ in either case. In any case it follows that 1 and 3 are not in the same set T_i.

Now assume first that one of T_1, T_2 is a one-element set, say $T_1 = \{i\}$. From the above it follows that $i \neq 2, 4$. If $i = 1$, then $E_1 \circ E_1$ is a linear combination of $E_0, E_1, E_2 + E_3 + E_4$. But $q^4_{11} = 0$. Therefore $E_1 \circ E_1 \in \langle E_0, E_1 \rangle$ implying that the fusion is imprimitive, which is a contradiction. The case $i = 3$ can be settled analogously.

Thus $|T_1| = |T_2| = 2$. Without loss of generality $T_1 = \{i, 4\}$ for $i = 1$, or $i = 3$ (the case $i = 2$ is eliminated by the above considerations). Assume without loss of generality that $i = 1$; then

$$v(E_1 + E_4) \circ (E_1 + E_4) = (m_1 + 1)E_0 + x(E_1 + E_4) + y(E_2 + E_3)$$

for some non-negative reals x, y. Because $q^4_{11} = 0$, $q^4_{14} = 0$, and $q^4_{44} = 0$ (by Proposition 4.7 and using $w = 2$), E_4 does not appear in the left-hand side. Therefore $x = 0$, implying $E_1 \circ E_1 \in \langle E_0, E_2, E_3 \rangle$. Together with $E_3 \circ E_3 = E_1 \circ E_1$ (which follows from the equations $vE_4 \circ E_4 = E_0$ and $vE_3 \circ E_4 = E_1$) we obtain $E_3 \circ E_3 \in \langle E_0, E_2, E_3 \rangle$. So $q^1_{33} = 0$, which yields the claim. □

What remains is to show that a decomposition that is not exceptional gives an association scheme. This gives the following result.

Proposition 7.7. *Consider a strongly regular graph with a strongly regular decomposition into parts with the same parameters. Then the above-mentioned relations form an association scheme if and only if the decomposition is not exceptional. If so, then for $r \neq k$, the scheme is cometric Q-antipodal.*

Proof. We showed before that an exceptional decomposition does not correspond to an association scheme. So suppose that the decomposition is not exceptional.

From the parameters of the strongly regular graphs it follows that

$$M^2 = (r_0 + s_0)M - r_0 s_0 I + (k_0 + r_0 s_0)J, \qquad MJ = k_0 J,$$

$$M_i^2 = (r + s)M_i - rsI + (k + rs)J, \qquad \text{and} \qquad M_i J = kJ, \quad i = 1, 2.$$

By working out the first equation, it follows that

$$\begin{aligned} CJ = (k_0 - k)J,\ C^\top J &= (k_0 - k)J, \\ M_1 C + CM_2 &= (r_0 + s_0)C + (k_0 + r_0 s_0)J, \\ CC^\top &= (r_0 + s_0 - r - s)M_1 - (r_0 s_0 - rs)I + (k_0 + r_0 s_0 - k - rs)J, \\ C^\top C &= (r_0 + s_0 - r - s)M_2 - (r_0 s_0 - rs)I + (k_0 + r_0 s_0 - k - rs)J, \end{aligned} \tag{7.7}$$

and this implies that $(r_0 + s_0 - r - s)(M_1 C - CM_2) = 0$. If $r_0 + s_0 = r + s$, then it follows from a result of Noda [39, Thm. 1] that the decomposition is exceptional, hence we must have that $M_1 C = CM_2$. From (7.7) it then follows that $M_1 C = CM_2 = \frac{r_0+s_0}{2}C + \frac{k_0+r_0 s_0}{2}J$. Now a routine check shows that the matrices $A_i, i = 0, ..., 4$ form an association scheme, and by Proposition 7.6 this scheme is cometric Q-antipodal. □

For the non-exceptional case, Noda [39] found that all parameters of the decomposition can be expressed in terms of r_0 and s_0. In our case, we have that $s = s_0$, which is the complementary case to the one considered in [39, Thm. 1]. From this result, it follows for example that $r = \frac{r_0+s_0}{2}$. Note that this also follows by considering the eigenvalues of the fusion scheme using (7.1): indeed, we have $s_0 = 0 + s = -P_{11} + r$, and $r_0 = P_{11} + r$.

Haemers and Higman [25] give a list of parameter sets of non-exceptional decompositions on at most 300 vertices. The smallest example is the Clebsch graph that decomposes into two perfect matchings on 8 vertices. The association scheme corresponding to this decomposition (consider the complementary one for the parameters) is the four-class binary Hamming scheme $H(4, 2)$ (which is (co-)metric, (Q-)bipartite, (Q-)antipodal). Note that this is a dismantled scheme of the cometric Q-bipartite Q-antipodal scheme (with $w = 3$) related to the so-called 24-cell. The next example is the Higman-Sims graph decomposing into two Hoffman-Singleton graphs, and there are two more examples: on 112 vertices and 162 vertices. The one on 112 vertices is a decomposition of a generalized quadrangle into two Gewirtz graphs, and it is part of an infinite family of decompositions coming from hemisystems.

7.5.1. *Hemisystems of generalized quadrangles.* Segre [43] introduced the concept of hemisystems on the Hermitian surface H in $PG(3, q^2)$ as a set of lines of H such that every point in H lies on exactly $(q+1)/2$ such lines. This point-line geometry, denoted $H(3, q^2)$, gives an important classical family of generalized quadrangles, called the Hermitian generalized quadrangles. It is now well-known [11] that the incidence relation on lines in this hemisystem yields a strongly regular subgraph of the line graph of the geometry. Thus we obtain a strongly regular decomposition of the (strongly regular) line graph of this generalized quadrangle. In fact, this holds for any hemisystem in a generalized quadrangle $GQ(t^2, t)$.

Let $(\mathcal{P}, \mathcal{L})$ be the point-line incidence structure of a generalized quadrangle $GQ(t^2, t)$ with t odd. Let Γ_0 be the line graph: its vertex set is $X = \mathcal{L}$ with two vertices adjacent if the lines have a point in common. This is a strongly regular graph with parameters $((t^3 + 1)(t + 1), t(t^2 + 1),\ t - 1, t^2 + 1)$ and with eigenvalues $k_0 = t(t^2 + 1)$, $r_0 = t - 1$, and $s_0 = -1 - t^2$. A hemisystem in $(\mathcal{P}, \mathcal{L})$ is a subset

$U_1 \subseteq \mathcal{L}$ with the property that every point in $\mathcal{P}$ lies on exactly $(t+1)/2$ lines in U_1 and $(t+1)/2$ lines in $U_2 = X - U_1$. Cameron, Delsarte, and Goethals [11] showed that any hemisystem in a generalized quadrangle of order (t^2, t) corresponds to a strongly regular decomposition of the line graph of the corresponding generalized quadrangle. Because the complementary set U_2 of lines of a hemisystem is also a hemisystem, this decomposition $X = U_1 \cup U_2$ has equally sized parts. Moreover, the parameters of the parts are the same: each U_i induces a subgraph Γ_i which is strongly regular with parameters

$$(n, k, \lambda, \mu) = \left(\frac{1}{2}(t^3+1)(t+1), \frac{1}{2}(t^2+1)(t-1),\ \frac{1}{2}(t-3), \frac{1}{2}(t-1)^2\right)$$

and eigenvalues $k = \frac{1}{2}(t^2+1)(t-1)$, $r = t-1$, and $s = -\frac{1}{2}(t^2-t+2)$. The decomposition is clearly not exceptional (note though that here we have the complementary setting as in the previous section because $r = r_0$), so by Proposition 7.7, we have

Corollary 7.8. *Let $(\mathcal{P}, \mathcal{L})$ be a generalized quadrangle $GQ(t^2, t)$ with t odd and let $\mathcal{C}$ denote the set of all ordered pairs of distinct intersecting lines from $\mathcal{L}$. Suppose $\mathcal{L} = U_1 \cup U_2$ is a partition of the lines into hemisystems. Then the relations $R_0 = \{(\ell, \ell) | \ell \in \mathcal{L}\}$, $R_1 = \mathcal{C} \cap (U_1 \times U_2 \cup U_2 \times U_1)$, $R_2 = \mathcal{C} \cap (U_1 \times U_1 \cup U_2 \times U_2)$, $R_3 = (U_1 \times U_2 \cup U_2 \times U_1) - R_0 - R_1$, $R_4 = (U_1 \times U_1 \cup U_2 \times U_2) - R_0 - R_2$ give a cometric Q-antipodal association scheme on $X = \mathcal{L}$. This scheme has Krein array $\{(t^2+1)(t-1), (t^2-t+1)^2/t, (t^2-t+1)(t-1)/t, 1\,; 1, (t^2-t+1)(t-1)/t, (t^2-t+1)^2/t,\ (t^2+1)(t-1)\}$.*

Segre [43] constructed a hemisystem in $H(3, q^2)$ (a $GQ(q^2, q)$) for $q = 3$; it corresponds to the above-mentioned example on 112 vertices with a decomposition into Gewirtz graphs. A breakthrough was made by Cossidente and Penttila [19], who constructed hemisystems in $H(3, q^2)$ for all odd prime powers q. Bamberg, De Clerck, and Durante [3] constructed a hemisystem for a nonclassical generalized quadrangle of order $(25, 5)$ (which has the same parameters as $H(3, 25)$), and recently Bamberg, Giudici, and Royle [4] showed that every flock generalized quadrangle has a hemisystem. Currently, all known generalized quadrangles of order (t^2, t) are flock generalized quadrangles.

7.6. **Classification, parameter sets, and examples.** We saw in Section 7.3 that the parameters of a four-class cometric Q-antipodal scheme are completely determined by those of the strongly regular graph on the fibres, together with the number of fibres w. We used this to generate "feasible" parameter sets for four-class cometric Q-antipodal schemes that are not Q-bipartite, and that have $n \leq 2000$ and $w \leq 6$. These parameter sets are listed in the appendix. Standard conditions such as integrality of parameters p^h_{ij} and nonnegativity of the Krein parameters q^h_{ij} were checked. Once a parameter set failed, we did not search for the corresponding parameter set with larger w (because dismantlability would exclude such a parameter set). We also checked one of the so-called absolute bounds on multiplicities, i.e., the one in Proposition 7.9 in the next section.

7.6.1. *Absolute bound on the number of fibres.* By the absolute bound we obtain the following bound for w.

Proposition 7.9. *For a four-class cometric Q-antipodal scheme, we have* $w \le (f+1)(f-2)/2g$.

Proof. By the absolute bound (cf. [7, Thm. 2.3.3]) we have

$$f(f+1)/2 \ge \mathrm{rank}(E_1 \circ E_1) = \mathrm{rank}(E_0) + \mathrm{rank}(E_1) + \mathrm{rank}(E_2) = 1 + f + wg,$$

and the result follows. □

For example, for the parameter sets with $n = 81$ in the appendix, we obtain that $w \le 3$ from $f = 20$ and $g = 60$. In general, the bound does not appear to be very good though.

7.6.2. *The small examples.* The first family of parameter sets in the appendix ($n = 50$) corresponds to the examples (with $w = 2$ and $w = 3$) related to the Hoffman-Singleton graph in Section 1.1. The case $w = 2$ corresponds to a distance-regular graph that is uniquely determined by the parameters, cf. [7, p393]. Now consider more generally an association scheme with w fibres $V_i, i = 1, ..., w$ (in this family of parameter sets). Because also the Hoffman-Singleton graph is determined by its parameters, relation R_4 is such a graph on each fibre. Let us call two vertices in distinct fibres incident if they are related by relation R_1. Because $p^1_{14} = 0$ for all w, it follows that if we take a vertex $x \in V_i, i > 1$, then the 15 vertices in V_1 incident to x will form a coclique in the Hoffman-Singleton graph on V_1. Because distinct x are incident to distinct cocliques, and there are exactly 100 distinct cocliques of size 15 in the Hoffman-Singleton graph, it follows that $w \le 3$. Moreover, because the scheme with $w = 2$ is uniquely determined by its parameters, and is a dismantled scheme of a scheme with $w = 3$, this implies that the latter scheme is also uniquely determined by its parameters.

For the second family of parameter sets in the appendix ($n = 56$) a construction is known for $w = 3$. Higman [28, Ex. 3] for example mentions it can be constructed on the set of ovals in the projective plane of order 4. The fibres are the three orbits of ovals under the action of the group $L_3(4)$. The case $w = 2$ corresponds to a hemisystem of the generalized quadrangle of order $(9, 3)$, or equivalently, to a strongly regular decomposition of the point graph of $GQ(3, 9)$ into two Gewirtz graphs. It is known that such a decomposition, and hence the corresponding scheme, is unique (the uniqueness of the hemisystem in the generalized quadrangle is proven by Hirschfeld [31, Thm. 19.3.18], and the uniqueness of the point graph as a strongly regular graph was proven by Cameron, Goethals, and Seidel [13]). As in the first family of parameter sets, we can show here that $w \le 3$, and that the scheme with $w = 3$ is unique. In this case, the intersection number p^1_{14} equals one (for all w), which implies that the set of 20 neighbors in V_1 of any vertex $x \notin V_1$ must be an induced matching $10K_2$ in the Gewirtz graph induced on V_1. Brouwer and Haemers [8, p405] mention that there are exactly 112 such induced subgraphs in the Gewirtz graph, which implies that $w \le 3$ as well as the uniqueness of the scheme with $w = 3$.

The case $n = 64$ has $w \le 2$. Dismantlability implies that the schemes with $n = 64$ and $w > 2$ do not exist (a scheme with $w = 3$ does not occur because for example the intersection number $p^1_{11} = 4.5$ is not integer). The case $w = 2$ corresponds to the distance-regular folded 8-cube, which is uniquely determined by its parameters.

For the family of parameter sets with $n = 81$, the absolute bound implies that $w \le 3$. Goethals and Seidel [23, p156] give a decomposition of the strongly regular graph on 243 vertices from the ternary Golay code (also known as Delsarte graph) into three strongly regular graphs on 81 vertices. This gives a scheme with $w = 3$ and $n = 81$. According to Brouwer [6], the decomposition of the unique strongly 56-regular graph on 162 vertices into two strongly regular graphs on 81 vertices is unique, hence the association scheme with $w = 2$ is unique as well. We also expect the scheme with $w = 3$ to be unique.

Besides the above examples, and the examples related to triality or hemisystems, there occurs one more family of examples in the appendix. These are related to the Leech lattice, cf. [28, Ex. 4], and have $n = 1408$ and $w \leq 3$.

Curiously, the Krein array $\{176, 135, 24, 1; 1, 24, 135, 176\}$ is formally dual to the intersection array of a known graph, a cometric antipodal distance-regular double cover on 1344 vertices found by Meixner [37]. Likewise, the Krein array $\{56, 45, 16, 1; 1, 8, 45, 56\}$ is formally dual to the intersection array of an antipodal distance-regular triple cover found by Soicher [44] which is not cometric.

8. Five-class cometric Q-antipodal association schemes

In [29], Higman introduced so-called strongly regular designs of the second kind and showed that these are equivalent to coherent configurations of type [3 3; 3]. In case the two fibres have the same size, these designs thus give five-class uniform schemes.

A trivial way to obtain such schemes is by taking the bipartite double of a strongly regular graph (Higman calls the corresponding strongly regular design of the second kind trivial). Though trivial, there are some cometric (and also metric) schemes obtained in this way, such as the ones obtained from the Clebsch graph, Schläfli graph, Higman-Sims graph, the McLaughlin graph and both its subconstituents. These strongly regular graphs have in common that $q^1_{11} = 0$ and $q^2_{12} \neq 0$. It was in fact claimed by Bannai and Ito [5, p314] that the bipartite double of a scheme is cometric if and only if the (original) scheme is cometric with $q^i_{1i} = 0$ for $i \neq d$ and $q^d_{id} \neq 0$.

To obtain less trivial examples of cometric schemes, we checked the examples and table of parameter sets for nontrivial strongly regular designs of the second kind in [29]. Four parameter sets in the table there turn out to give cometric schemes. One with $n = 162$ (Higman's Example 4.4) is related to $U_4(3)$, and has Krein array $\{21, 20, 9, 3, 1; 1, 3, 9, 20, 21\}$. The second one (Higman's Example 4.5) has $n = 176$, and can be described using the Steiner 3-design on 22 points. It has Krein array $\{21, 19.36, 11, 2.64, 1; 1, 2.64, 11, 19.36, 21\}$. The parameter set with $n = 243$ can be realized as a dismantled scheme on two of the three fibres of a cometric scheme that is the dual of a metric scheme corresponding to the coset graph of the shortened extended ternary Golay code (cf. [7, p365]). Its Krein array is $\{22, 20, 13.5, 2, 1; 1, 2, 13.5, 20, 22\}$. The last cometric example from the table has $n = 256$ (second such parameter set in Higman's table) and corresponds to the distance-regular folded 10-cube.

Higman also mentions (in his Example 4.3) the strongly regular designs of the second kind related to the family of bipartite cometric distance-regular dual polar graphs $D_5(q)$. We did not bother to completely check all other examples mentioned by Higman [29], but we expect no other cometric examples among these.

9. Miscellaneous

In his book on permutation groups, Cameron [10, p79] describes how to use the computer package GAP to construct the strongly regular decomposition of the Higman-Sims graph into two Hoffman-Singleton graphs. This description can easily be extended to get the linked system of partial λ-geometries of Section 1.1.

We checked whether any of the remaining examples mentioned in Higman's unpublished paper on uniform schemes [30] gives rise to a cometric scheme. Although we should mention that one of the examples (Example 6) is unclear to us, we found no cometric schemes among these examples.

Many of the examples mentioned in this paper, and also examples of other cometric association schemes, are listed on the website `http://users.wpi.edu/~martin/RESEARCH/QPOL/`. Included there are all parameters of the examples.

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Appendix

Below are putative parameter sets of four-class cometric Q-antipodal association schemes with fibre size $n \leq 2000$ and $w \leq 6$, and that are not Q-bipartite. The parameter sets are grouped according to the parameters of the strongly regular graph which would appear as the subscheme on the fibres. These "srg" parameters are given at the beginning of each group. For each group of parameter sets, we give the absolute bound of Proposition 7.9 (if relevant).

Each remaining line contains information on one parameter set. An exclamation mark (!) means that the scheme is unique, a plus sign (+) indicates existence, and a minus sign (-) non-existence. Next to this, the Krein array is given, then w, the partition $v = 1 + v_1 + v_2 + v_3 + v_4$, and the spectrum of R_1. At the end, some miscellaneous information is given. The notation `(P)` indicates that the scheme is (or, would be) also metric. The examples listed here appear in the body of the paper as follows.

- `Hoff-Singleton` – the linked system of partial λ-geometries related to the Hoffman-Singleton graph (Section 1.1)
- `hemisystem` – schemes arising from hemisystems (Corollary 7.8)
- `ovals of PG(2,4)` – Higman's scheme defined on the ovals of $PG(2,4)$ (Section 7.6.2)
- `folded 8-cube` – (Section 7.6.2)
- `ternary Golay code` – the decomposition of Goethals and Seidel in [23] (Section 7.6.2)
- `D_4(q)` and `O+(8,2), triality` – Higman's triality schemes and their dismantled schemes (Example 3.5)
- `Leech lattice` – Higman's Leech lattice example [28, Ex. 4] (Section 7.6.2)

```
-------------------
   srg(50,42,35,36)                          w <= 7
!{21, 16, 6,  1; 1, 6,  16, 21}              2  100=1+  15+ 42+  35+  7    15   5 0  -5  -15 Hoff-Singleton (P)
!{21, 16, 8,  1; 1, 4,  16, 21}              3  150=1+  30+ 42+  70+  7    30  10 0  -5  -15 Hoff-Singleton
-{21, 16, 9,  1; 1, 3,  16, 21}              4  200=1+  45+ 42+ 105+  7    45  15 0  -5  -15
-{21, 16, 9.6,1; 1, 2.4,16, 21}              5  250=1+  60+ 42+ 140+  7    60  20 0  -5  -15
-{21, 16, 10, 1; 1, 2,  16, 21}              6  300=1+  75+ 42+ 175+  7    75  25 0  -5  -15
-------------------
   srg(56,45,36,36)                          w <= 5
!{20, 16.333, 4.667, 1; 1, 4.667, 16.333, 20} 2  112=1+  20+ 45+  36+ 10    20   6 0  -6  -20 hemisystem
!{20, 16.333, 6.222, 1; 1, 3.111, 16.333, 20} 3  168=1+  40+ 45+  72+ 10    40  12 0  -6  -20 ovals of PG(2,4)
-{20, 16.333, 7,     1; 1, 2.333, 16.333, 20} 4  224=1+  60+ 45+ 108+ 10    60  18 0  -6  -20
-{20, 16.333, 7.467, 1; 1, 1.867, 16.333, 20} 5  280=1+  80+ 45+ 144+ 10    80  24 0  -6  -20
-------------------
   srg(64,28,12,12)
!{28, 15, 6, 1; 1, 6, 15, 28}                2  128=1+   8+ 28+  56+ 35     8   4 0  -4   -8 folded 8-cube (P)
-------------------
   srg(81,60,45,42)                          w <= 3
!{20, 18, 3,  1; 1, 3,  18, 20}              2  162=1+  36+ 60+  45+ 20    36   9 0  -9  -36 ternary Golay code
+{20, 18, 4,  1; 1, 2,  18, 20}              3  243=1+  72+ 60+  90+ 20    72  18 0  -9  -36 ternary Golay code
--------------------
   srg(135,70,37,35)                         w <= 14
+{50, 31.5,  9.375, 1; 1, 9.375, 31.5, 50}   2  270=1+  15+ 70+ 120+ 64    15   6 0  -6  -15 D_4(2) (P)
+{50, 31.5, 12.5,   1; 1, 6.25,  31.5, 50}   3  405=1+  30+ 70+ 240+ 64    30  12 0  -6  -15 O+(8,2), triality
 {50, 31.5, 14.0625,1; 1, 4.6875,31.5, 50}   4  540=1+  45+ 70+ 360+ 64    45  18 0  -6  -15
 {50, 31.5, 15,     1; 1, 3.75,  31.5, 50}   5  675=1+  60+ 70+ 480+ 64    60  24 0  -6  -15
 {50, 31.5, 15.625, 1; 1, 3.125, 31.5, 50}   6  810=1+  75+ 70+ 600+ 64    75  30 0  -6  -15
-----------------------
   srg(162,140,121,120)                      w <= 14
 {56, 45, 12,  1; 1, 12,  45, 56}            2  324=1+  36+140+ 126+ 21    36   9 0  -9  -36 (P)
 {56, 45, 16,  1; 1,  8,  45, 56}            3  486=1+  72+140+ 252+ 21    72  18 0  -9  -36
 {56, 45, 18,  1; 1,  6,  45, 56}            4  648=1+ 108+140+ 378+ 21   108  27 0  -9  -36
 {56, 45, 19.2,1; 1,  4.8,45, 56}            5  810=1+ 144+140+ 504+ 21   144  36 0  -9  -36
 {56, 45, 20,  1; 1,  4,  45, 56}            6  972=1+ 180+140+ 630+ 21   180  45 0  -9  -36
-----------------------
```

```
  srg(196,150,116,110)                   w <= 6
 {45, 40,  6,  1; 1, 6,  40, 45}         2  392=1+  70+150+ 126+ 45    70  14 0 -14  -70
 {45, 40,  8,  1; 1, 4,  40, 45}         3  588=1+ 140+150+ 252+ 45   140  28 0 -14  -70
 {45, 40,  9,  1; 1, 3,  40, 45}         4  784=1+ 210+150+ 378+ 45   210  42 0 -14  -70
 {45, 40,  9.6,1; 1, 2.4,40, 45}         5  980=1+ 280+150+ 504+ 45   280  56 0 -14  -70
 {45, 40, 10,  1; 1, 2,  40, 45}         6 1176=1+ 350+150+ 630+ 45   350  70 0 -14  -70
----------------------
  srg(243,176,130,120)                   w <= 4
 {44, 40.5, 4.5, 1; 1, 4.5, 40.5, 44}    2  486=1+  99+176+ 144+ 66    99  18 0 -18  -99
 {44, 40.5, 6,   1; 1, 3,   40.5, 44}    3  729=1+ 198+176+ 288+ 66   198  36 0 -18  -99
 {44, 40.5, 6.75,1; 1, 2.25,40.5, 44}    4  972=1+ 297+176+ 432+ 66   297  54 0 -18  -99
----------------------
  srg(320,220,156,140)                   w <= 3
 {44, 41.667, 3.333, 1; 1, 3.333, 41.667, 44} 2  640=1+ 144+220+ 176+ 99   144  24 0 -24 -144
 {44, 41.667, 4.444, 1; 1, 2.222, 41.667, 44} 3  960=1+ 288+220+ 352+ 99   288  48 0 -24 -144
----------------------
  srg(378,325,280,275)                   w <= 19
+{104, 88.2, 16.8, 1; 1, 16.8,88.2, 104}    2  756=1+  78+325+ 300+ 52    78  15 0 -15  -78 hemisystem
 {104, 88.2, 22.4, 1; 1, 11.2,88.2, 104}    3 1134=1+ 156+325+ 600+ 52   156  30 0 -15  -78
 {104, 88.2, 25.2, 1; 1, 8.4, 88.2, 104}    4 1512=1+ 234+325+ 900+ 52   234  45 0 -15  -78
 {104, 88.2, 26.88,1; 1, 6.72,88.2, 104}    5 1890=1+ 312+325+1200+ 52   312  60 0 -15  -78
 {104, 88.2, 28,   1; 1, 5.6, 88.2, 104}    6 2268=1+ 390+325+1500+ 52   390  75 0 -15  -78
----------------------
  srg(392,345,304,300)                   w <= 23
 {115, 96, 20,    1; 1, 20,    96, 115}     2  784=1+  70+345+ 322+ 46    70  14 0 -14  -70 (P)
 {115, 96, 26.667,1; 1, 13.333,96, 115}     3 1176=1+ 140+345+ 644+ 46   140  28 0 -14  -70
 {115, 96, 30,    1; 1, 10,    96, 115}     4 1568=1+ 210+345+ 966+ 46   210  42 0 -14  -70
 {115, 96, 32,    1; 1,  8,    96, 115}     5 1960=1+ 280+345+1288+ 46   280  56 0 -14  -70
 {115, 96, 33.333,1; 1,  6.667,96, 115}     6 2352=1+ 350+345+1610+ 46   350  70 0 -14  -70
----------------------
  srg(400,315,250,240)                   w <= 11
 {84, 75, 10,    1; 1, 10,   75, 84}     2  800=1+ 120+315+ 280+ 84   120  20 0 -20 -120
 {84, 75, 13.333,1; 1, 6.667,75, 84}     3 1200=1+ 240+315+ 560+ 84   240  40 0 -20 -120
 {84, 75, 15,    1; 1, 5,    75, 84}     4 1600=1+ 360+315+ 840+ 84   360  60 0 -20 -120
 {84, 75, 16,    1; 1, 4,    75, 84}     5 2000=1+ 480+315+1120+ 84   480  80 0 -20 -120
 {84, 75, 16.667,1; 1, 3.333,75, 84}     6 2400=1+ 600+315+1400+ 84   600 100 0 -20 -120
----------------------
  srg(540,385,280,260)                   w <= 6
 {77, 72,  6,  1; 1, 6,  72, 77}         2 1080=1+ 210+385+ 330+154   210  30 0 -30 -210
 {77, 72,  8,  1; 1, 4,  72, 77}         3 1620=1+ 420+385+ 660+154   420  60 0 -30 -210
 {77, 72,  9,  1; 1, 3,  72, 77}         4 2160=1+ 630+385+ 990+154   630  90 0 -30 -210
 {77, 72,  9.6,1; 1, 2.4,72, 77}         5 2700=1+ 840+385+1320+154   840 120 0 -30 -210
 {77, 72, 10,  1; 1, 2,  72, 77}         6 3240=1+1050+385+1650+154  1050 150 0 -30 -210
----------------------
  srg(672,440,292,280)
 {176, 135, 24, 1; 1, 24, 135, 176}      2 1344=1+  56+440+ 616+231    56  14 0 -14  -56 (P)
----------------------
  srg(704,475,330,300)                   w <= 4
 {76, 72.6, 4.4,  1; 1, 4.4,  72.6, 76}  2 1408=1+ 304+475+ 400+228   304  40 0 -40 -304
 {76, 72.6, 5.867,1, 1, 2.933,72.6, 76}  3 2112=1+ 608+475+ 800+228   608  80 0 -40 -304
 {76, 72.6, 6.6,  1; 1, 2.2,  72.6, 76}  4 2816=1+ 912+475+1200+228   912 120 0 -40 -304
----------------------
  srg(729,588,477,462)                   w <= 16
 {140, 126, 15,  1; 1,15,  126, 140}     2 1458=1+ 189+ 588+ 540+140   189  27 0 -27 -189
 {140, 126, 20,  1; 1,10,  126, 140}     3 2187=1+ 378+ 588+1080+140   378  54 0 -27 -189
 {140, 126, 22.5,1; 1, 7.5,126, 140}     4 2916=1+ 567+ 588+1620+140   567  81 0 -27 -189
 {140, 126, 24,  1; 1, 6,  126, 140}     5 3645=1+ 756+ 588+2160+140   756 108 0 -27 -189
 {140, 126, 25,  1; 1, 5,  126, 140}     6 4374=1+ 945+ 588+2700+140   945 135 0 -27 -189
----------------------
  srg(760,594,468,450)                   w <= 13
 {132, 120.333,12.667,1;1,12.667,120.333,132} 2 1520=1+ 220+ 594+ 540+165   220  30 0 -30 -220
 {132, 120.333,16.889,1;1, 8.444,120.333,132} 3 2280=1+ 440+ 594+1080+165   440  60 0 -30 -220
 {132, 120.333,19,    1;1, 6.333,120.333,132} 4 3040=1+ 660+ 594+1620+165   660  90 0 -30 -220
 {132, 120.333,20.267,1;1, 5.067,120.333,132} 5 3800=1+ 880+ 594+2160+165   880 120 0 -30 -220
 {132, 120.333,21.111,1;1, 4.222,120.333,132} 6 4560=1+1100+ 594+2700+165  1100 150 0 -30 -220
----------------------
  srg(800,714,638,630)                   w <= 34
 {204, 175, 30, 1; 1, 30, 175, 204}      2 1600=1+ 120+ 714+ 680+ 85   120  20 0 -20 -120 (P)
 {204, 175, 40, 1; 1, 20, 175, 204}      3 2400=1+ 240+ 714+1360+ 85   240  40 0 -20 -120
 {204, 175, 45, 1; 1, 15, 175, 204}      4 3200=1+ 360+ 714+2040+ 85   360  60 0 -20 -120
 {204, 175, 48, 1; 1, 12, 175, 204}      5 4000=1+ 480+ 714+2720+ 85   480  80 0 -20 -120
 {204, 175, 50, 1; 1, 10, 175, 204}      6 4800=1+ 600+ 714+3400+ 85   600 100 0 -20 -120
----------------------
  srg(875,570,385,345)                   w <= 3
 {76, 73.5, 3.5,   1; 1, 3.5,  73.5, 76}    2 1750=1+ 400+ 570+ 475+304   400  50 0 -50 -400
 {76, 73.5, 4.667, 1; 1, 2.333,73.5, 76}    3 2625=1+ 800+ 570+ 950+304   800 100 0 -50 -400
-----------------------
  srg(1120,390,146,130)                  w <= 54
+{300, 212.333,38.889,1;1,38.889,212.333,300} 2 2240=1+  40+ 390+1080+729    40  12 0 -12  -40 D_4(3) (P)
+{300, 212.333,51.852,1;1,25.926,212.333,300} 3 3360=1+  80+ 390+2160+729    80  24 0 -12  -40 O+(8,3), triality
 {300, 212.333,58.333,1;1,19.444,212.333,300} 4 4480=1+ 120+ 390+3240+729   120  36 0 -12  -40
 {300, 212.333,62.222,1;1,15.556,212.333,300} 5 5600=1+ 160+ 390+4320+729   160  48 0 -12  -40
 {300, 212.333,64.815,1;1,12.963,212.333,300} 6 6720=1+ 200+ 390+5400+729   200  60 0 -12  -40
```

```
------------------------
  srg(1210,819,568,525)                    w <= 6
 {117, 112, 6,  1; 1, 6,  112, 117}        2 2420=1+ 495+ 819+ 715+390   495  55 0 -55 -495
 {117, 112, 8,  1; 1, 4,  112, 117}        3 3630=1+ 990+ 819+1430+390   990 110 0 -55 -495
 {117, 112, 9,  1; 1, 3,  112, 117}        4 4840=1+1485+ 819+2145+390  1485 165 0 -55 -495
 {117, 112, 9.6,1; 1, 2.4,112, 117}        5 6050=1+1980+ 819+2860+390  1980 220 0 -55 -495
 {117, 112,10,  1; 1, 2,  112, 117}        6 7260=1+2475+ 819+3575+390  2475 275 0 -55 -495
-------------------------
  srg(1225,1008,833,812)                   w <= 23
 {216, 196, 21,  1; 1,21,  196, 216}       2 2450=1+ 280+1008+ 945+216   280  35 0 -35 -280
 {216, 196, 28,  1; 1,14,  196, 216}       3 3675=1+ 560+1008+1890+216   560  70 0 -35 -280
 {216, 196, 31.5,1; 1,10.5,196, 216}       4 4900=1+ 840+1008+2835+216   840 105 0 -35 -280
 {216, 196, 33.6,1; 1, 8.4,196, 216}       5 6125=1+1120+1008+3780+216  1120 140 0 -35 -280
 {216, 196, 35,  1; 1, 7,  196, 216}       6 7350=1+1400+1008+4725+216  1400 175 0 -35 -280
---------------------------
  srg(1376,1225,1092,1078)                 w <= 41
+{300, 264.143,36.857,1;1,36.857,264.143,300} 2 2752=1+ 200+1225+1176+150   200  28 0 -28 -200 hemisystem
 {300, 264.143,49.143,1;1,24.571,264.143,300} 3 4128=1+ 400+1225+2352+150   400  56 0 -28 -200
 {300, 264.143,55.286,1;1,18.429,264.143,300} 4 5504=1+ 600+1225+3528+150   600  84 0 -28 -200
 {300, 264.143,58.971,1;1,14.743,264.143,300} 5 6880=1+ 800+1225+4704+150   800 112 0 -28 -200
 {300, 264.143,61.429,1;1,12.286,264.143,300} 6 8256=1+1000+1225+5880+150  1000 140 0 -28 -200
------------------------
  srg(1408,567,246,216)                    w <= 27
+{252, 201.667,22,    1;1,22,    201.667,252} 2 2816=1+ 112+ 567+1296+840   112  24 0 -24 -112 Leech lattice
+{252, 201.667,29.333,1;1,14.667,201.667,252} 3 4224=1+ 224+ 567+2592+840   224  48 0 -24 -112 Leech lattice
 {252, 201.667,33,    1;1,11,    201.667,252} 4 5632=1+ 336+ 567+3888+840   336  72 0 -24 -112
 {252, 201.667,35.2,  1;1, 8.8,  201.667,252} 5 7040=1+ 448+ 567+5184+840   448  96 0 -24 -112
 {252, 201.667,36.667,1;1, 7.333,201.667,252} 6 8448=1+ 560+ 567+6480+840   560 120 0 -24 -112
---------------------------
  srg(1458,1316,1189,1176)                 w <= 47
 {329, 288, 42,  1; 1, 42,  288, 329}      2 2916=1+ 189+1316+1269+141   189  27 0 -27 -189 (P)
 {329, 288, 56,  1; 1, 28,  288, 329}      3 4374=1+ 378+1316+2538+141   378  54 0 -27 -189
 {329, 288, 63,  1; 1, 21,  288, 329}      4 5832=1+ 567+1316+3807+141   567  81 0 -27 -189
 {329, 288, 67.2,1; 1, 16.8,288, 329}      5 7290=1+ 756+1316+5076+141   756 108 0 -27 -189
 {329, 288, 70,  1; 1, 14,  288, 329}      6 8748=1+ 945+1316+6345+141   945 135 0 -27 -189
-------------------------
  srg(1625,1044,693,630)                   w <= 4
 {116, 112.667, 4.333,1;1,4.333,112.667, 116} 2 3250=1+ 725+1044+ 900+580   725  75 0 -75 -725
 {116, 112.667, 5.778,1;1,2.889,112.667, 116} 3 4875=1+1450+1044+1800+580  1450 150 0 -75 -725
 {116, 112.667, 6.5,  1;1,2.167,112.667, 116} 4 6500=1+2175+1044+2700+580  2175 225 0 -75 -725
-------------------------
  srg(1701,1190,847,798)                   w <= 9
 {170, 162,  9,  1; 1, 9,  162, 170}       2 3402=1+ 630+1190+1071+510   630  63 0 -63 -630
 {170, 162, 12,  1; 1, 6,  162, 170}       3 5103=1+1260+1190+2142+510  1260 126 0 -63 -630
 {170, 162, 13.5,1; 1, 4.5,162, 170}       4 6804=1+1890+1190+3213+510  1890 189 0 -63 -630
 {170, 162, 14.4,1; 1, 3.6,162, 170}       5 8505=1+2520+1190+4284+510  2520 252 0 -63 -630
 {170, 162, 15,  1; 1, 3,  162, 170}      6 10206=1+3150+1190+5355+510  3150 315 0 -63 -630
---------------------------
  srg(1936,1620,1360,1332)                 w <= 30
 {315, 288, 28,     1; 1, 28,    288, 315}   2 3872=1+ 396+1620+1540+315   396  44 0 -44 -396
 {315, 288, 37.333,1; 1, 18.667,288, 315}    3 5808=1+ 792+1620+3080+315   792  88 0 -44 -396
 {315, 288, 42,     1; 1, 14,    288, 315}   4 7744=1+1188+1620+4620+315  1188 132 0 -44 -396
 {315, 288, 44.8,  1; 1, 11.2,  288, 315}    5 9680=1+1584+1620+6160+315  1584 176 0 -44 -396
 {315, 288, 46.667,1; 1,  9.333,288, 315}   6 11616=1+1980+1620+7700+315  1980 220 0 -44 -396
-------------------------
  srg(1944,1218,792,714)                   w <= 3
 {116, 113.4, 3.6, 1; 1, 3.6, 113.4, 116}    2 3888=1+ 900+1218+1044+725   900  90 0 -90 -900
 {116, 113.4, 4.8, 1; 1, 2.4, 113.4, 116}    3 5832=1+1800+1218+2088+725  1800 180 0 -90 -900
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