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EQUILIBRIUM SELECTION IN 2 x 2 GAMES (*)

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Abstract
This paper presents several approaches to equilibrium selection in 2 x 2 bimatrix games with 2 strict Nash equilibria. Special emphasis is placed on the Harsanyi/Selten concept of risk dominance and two reasoning processes are presented that justify this concept. Both processes are based on approximations of a given game by games with lack of common knowledge.

1. INTRODUCTION

This author has not the slightest doubt that, when asked to play the coordination game $G_1$ from Figure 1 in a purely noncooperative fashion, and without receiving any outside guidance about how to play, each reader would choose $R$.

Common sense dictates that one should play $R$ in this game ($R$ is the focal point (Schelling [1960])), but until recently the literature offered no formal, purely noncooperative theories (i.e. theories based solely on considerations of individual utility maximization) that single out $R$ as the unique solution of $G_1$. Our purpose in this paper is to acquaint the reader with some recent theories that do indeed single out $R$ as the solution of $G_1$.

Note that also $L = (L_1, L_2)$ is a Nash equilibrium of $G_1$, in fact it is a strict equilibrium: each player strictly loses by unilaterally deviating from $L$. Consequently, $L$ satisfies the conditions imposed by the most refined equilibrium notions proposed today such as stra-
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tegic stability in the sense of Kohlberg and Mertens (1986). Kohlberg and Mertens do not argue that $L$ will necessarily be observed if $G_1$ is played, however, they claim that an agreement (or assignment) to play $L$ is self-enforcing, i.e. that no player will have an incentive to deviate from $L$ if it is recommended that $L$ should be played (and if this recommendation is common knowledge). Even this weaker justification of $L$ is challenged by experimental results reported in Van Huyck et al. (1988). In a coordination game similar to $G_1$, these authors found that only 1 pair of players (out of 30) followed the recommendation to play the Pareto dominated equilibrium: Without any communication, 47 from the 60 individuals (and 18 of the 30 pairs of players) deviated to the Pareto dominant equilibrium. Apparently, the players are so convinced that only $R$ makes sense in $G_1$ that they consider any suggestion to play something else to be nonsensical, and they ignore such a suggestion since they know that any other player will also ignore it. The obvious question of course is how players can know that only $R$ makes sense. In this paper we present the main ideas of some theories that address this question.

In the existing literature one approach to problems posed by games as $G_1$ has simply been to postulate collective rationality as an axiom, i.e. to assume that players will always coordinate on a Pareto optimal equilibrium. For example, Harsanyi and Selten (1988) impose such collective rationality as long as it does not conflict with symmetry or other more desirable properties (the reader is referred to the postscript of that book for insightful arguments in favor of collective rationality). Also the literature on renegotiation in repeated games (see Pearce (1990) for a survey) is based on the assumption of collective rationality. Although interesting insights can be obtained from that assumption, it is this author's opinion that collective rationality should not be taken as a basic postulate, hence, in this paper attention will be confined to a purely noncooperative (i.e. individualistic) framework. (The reader is referred to Van Damme (1988) for an example where collective rationality conflicts with more basic requirements).

The paper deals exclusively with 2 x 2 games, i.e. with 2-person normal form games in which each player has two pure strategies at his disposal. No doubt, the class of 2 x 2 games is very special. However, it is a very important class of games: Each situation in which players are uncertain about which of two equilibria should be played formally corresponds to a 2 x 2 game. The restriction to 2 x 2 games allows us to present some fundamental ideas in their purest form. These ideas can also be applied to more general games. For an investigation of the extent to which the results generalize to larger classes of games, the reader should consult the papers cited in the text. For general attacks on the equilibrium selection problem the reader is referred to Harsanyi and Selten (1988) or to Güth and Kalkofen (1989).

The remainder of the paper is organized as follows. In Section 2 we investigate whether adding a communication stage will enable (or force) players to play the Pareto dominant equilibrium in coordination games. Models with cheap talk as well as models in which communication is costly are considered. In Section 3 we review Aumann's (1989) argument of why an announcement to play a strict Nash equilibrium is not necessarily credible and introduce the Harsanyi/Selten concept of risk dominance. The Sections 4 and 5 constitute the heart of the paper. Section 4 describes a reasoning process that leads boundedly Bayesian rational players, who truncate the hierarchy of beliefs at the second level, to play the risk dominant equilibrium of each 2 x 2 game in which there is a conflict of interest. This section builds on ideas developed by Harsanyi and Selten, specifically the reasoning process is motivated by their definition of the bicentric prior which plays a
role in the tracing procedure (cf. Harsanyi and Selten (1988, pp. 207-208), Harsanyi (1976, p. 221)). Section 5 briefly discusses the approach to equilibrium selection that was proposed in Carlsson and Van Damme (1990). In that approach it is assumed that players are slightly uncertain about which game they are playing and it is shown that such uncertainty forces the players to coordinate on the risk dominant equilibrium. Finally, it is indicated in Section 6 that the risk dominant equilibrium is also likely to emerge in a context in which the game is played repeatedly by naive players who make mistakes with a small probability. This section is based on Kandori et al. (1991).

2. COMMUNICATION

One argument that one finds in the literature for why players would choose \( R \) in Figure 1 even in the absence of communication is based on Schelling's (1960) principle of tacit bargaining. It is argued that players will compare the situation without communication with that in which communication is possible and that, since players know that, if communication would be possible, they would talk themselves into equilibrium \( R \), they do not need to actually communicate to reach this same conclusion. The question, of course, is why communication necessarily leads to \( R \). In this section several arguments in favor of this conclusion are presented. Attention will be confined to games of «common interest». (For experimental evidence of the effect of «cheap talk», the reader is referred to Cooper et al. (1989, 1991)).

2.1. Cheap Talk

Given a game \( G \), let the game \( T_i(G) \) be defined as follows: First player \( i \) makes a suggestion of which strategy vector to play in game \( G \) (player \( i \) may also choose to remain silent), thereafter, the players play \( G \). Payoffs in \( T_i(G) \) only depend on what happens in \( G \), they do not depend directly on the first stage talk. Hence, talk is cheap but it may influence the actions chosen in \( G \). Clearly, if \( G_1 \) is in Figure 1, then \( T_i(G_1) \) has a subgame perfect equilibrium in which the players simply ignore whatever happened in the communication stage and play the «wrong» equilibrium \( L \). As noted in Farrell (1988), the problem is that we have no link between words and actions. To overcome this problem Farrel suggested to assume that statements be interpreted according to their literal meanings unless there is strong logical reason to distrust the statements. (Also see Myerson (1989)). Specifically, Farrell (1988) assumes that any announcement to play a Nash equilibrium will be believed and honored. Clearly, under this assumption, player \( i \) will announce \( R \) in \( T_i(G_1) \) and players will play \( R \) in \( G_1 \).

The above conclusion has to be modified if both players can make announcements before the play of the game. For a 2-person game \( G \) Farrell (1987) investigates the game \( T(G) \) in which the players simultaneously announce which strategies they will choose in \( G \). Farrell assumes that the announced strategies will be played if they constitute an equilibrium. He also assumes that, if \( G \) is symmetric and the announcements are not in equilibrium, players will play a symmetric equilibrium of \( G \). Under these assumptions, the game \( T(G_1) \) can be represented as in Figure 2 with \( \nu=2/3 \), hence, \( T(G_1)=G_2(2/3) \).
It is clear that there are equilibria of $T(G_1)$ in which players do not announce $R$. Furthermore, even multiple rounds of announcements do not necessarily lead to $R$. Namely, it is easily seen that $T(G_2(v)) = G_2((2-v^2)/(3-2v))$, hence $T^n(G_1) \rightarrow G_2(1)$ ($n \rightarrow \infty$). Hence, even when there are many rounds of communication, $L$ remains as a strict equilibrium.

A similar symmetric procedure of repeated strategy announcements has been proposed in Kalai and Samet (1985). That paper uses persistent equilibrium (Kalai and Samet (1984)) as the basic solution concept. For our purposes it is sufficient to note that in the game $G_1$ of Figure 1 only the pure equilibria $L$ and $R$ are persistent and that in a $2 \times 2$ game each persistent equilibrium is perfect (i.e. involves undominated strategies). Kalai and Samet assume that the announced strategies will be played if they constitute a persistent equilibrium, and that, if the announcements do not combine to a persistent equilibrium, and that, if the announcements do not combine to a persistent equilibrium, players will continue with some persistent equilibrium that is independent of the announcements (subgame symmetry). Clearly, under these assumptions, the cheap talk game is either equivalent to $G_2(1)$ or to $G_2(2)$ and in both cases only $R$ is the persistent equilibrium. Hence, Kalai and Samet conclude that 1 round of preplay communication is enough to obtain $R$ in game $G_1$.

2.2. An Evolutionary Approach

The approach from the previous subsection may be criticized for the fact that it assumes that certain announcements are interpreted according to their literal meanings rather than that it derives these focal meanings as a conclusion. One may argue that, in the model from the previous subsection, there is actually no room for real communication since the game and the player’s rationality and beliefs are common knowledge. Anderlini (1990) studies a model in which there is room for real communication. He considers the situation in which the game $G_1$ from Figure 1 is played between Turing machines. First two Turing machines are drawn from the pool of all such machines by «metaplayers», these selected machines then exchange their Gödel numbers and finally they play $G_1$. Anderlini shows that all trembling hand perfect equilibria of the game between the metaplayers result in the Pareto dominant outcome $R$. As Anderlini remarks, the metaplayers can best be viewed as a metaphor for an evolutionary process.

The logic supporting Anderlini's result may informally be explained as follows (cf. Robson (1989)). Assume that in game $G_1$ initially there is no communication and the
equilibrium $L$ is played. Consider a mutant that carries a signal to play $R$; the mutant will play $L$ if it does not receive a signal that the other is going to play $R$ while it will play $R$ if such a signal is indeed received. Note that the mutant does equally well against a strategy in the original population as such a strategy does itself, but that, when a mutant meets himself, it fares better than a strategy from the original population. Consequently, the mutant will thrive in the population and it will gradually drive out the original strategy. Formally, the original strategy is not an evolutionarily stable strategy or ESS (Maynard Smith (1982)).

Maynard Smith (1982) defined ESS only for static games, but Selten (1983) extended the definition to general extensive form games. It can be shown that an ESS of a game in extensive form induces an ESS in each subgame and that, if subgames are replaced by their ESS-values, an ESS must result in the truncated game as well (cf. Van Damme (1987, Corollary 9.8.8)). Since $G_1$ has two ESS, viz. $L$ and $R$ it follows that an ESS of the preplay game $T(G_1)$ must be an ESS of some game $G_3(\alpha, \beta, \gamma)$ as in Figure 3 with $\alpha, \beta, \gamma \in \{1, 2\}$. (The entry $(\beta, \beta)$ occurs twice because of the symmetry requirement in ESS).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure3}
\caption{GAME $G_3(\alpha, \beta, \gamma)$}
\end{figure}

If $\alpha = \beta = \gamma$, then $G_3(\alpha, \beta, \gamma)$ does not admit ESS, hence, at least one of the entries must be $(2, 2)$. But then it follows (from the fact that ESS must be are in undominated strategies) that all ESS of the preplay communication game $T(G_1)$ result in the Pareto-efficient equilibrium $R$. (It should be noted that the above analysis is preliminary; we glossed over such issues as the difference between limit ESS and ESS, nor did we discuss existence. However, evolutionary analysis of cheap talk games clearly seems a promising area for further research).

2.3. Costly Communication

Next let us assume that talking is not free, that speaking a word costs a small amount $\varepsilon$. Consider first the case in which only player 1 can talk and in which player 2 exactly hears what player 1 says. This model was analysed in Ben-Porath and Dekel (1987). Van Damme (1989) considered costly communication in the Battle of The Sexes Game. Formally, before playing the normal form game $G = (S, \{u_i\}_i, 2)$, player 1 chooses to speak a certain number, $m$, of words and if $m$ is chosen and $s$ is played in $G$, the payoffs are $u_1(m, s) = u_1(s) - m\varepsilon$ and $u_2(m, s) = u_2(s)$. Denote this modified game by $T_1^\varepsilon(G)$. Note that
we assume that listening and counting are costless, however, such costs can be incorporated in the model without changing the results. We claim that, if player 1 can talk long enough (i.e. if the maximal number of words \( M \) that he is allowed to speak is large enough), then the game \( T_1^e (G_1) \) is dominance solvable with solution \((0, R)\). Hence, by iterative elimination of (weakly) dominated strategies \( T_1^e (G_1) \) can be reduced to the outcome in which there is no talking and players play \( R \). The argument is simple. Let \( M \) be such that \( M \gg 1 \) and consider \( m \) with \( m \gg 1 \) but \( m \ll 1 \). Then \((m, L_1)\) (i.e. speaking \( m \) words and then playing \( L_1 \)) is dominated for player 1 and after having eliminated \((m, L_1)\) responding to \( m \) by \( L_2 \) becomes dominated for player 2. If \( L_2 \) after \( m \) is eliminated, then \((m', L_1)\) becomes dominated by \((m, R_1)\) if \( m' \ll 1 - m \). Next, responding to \( m' \) with \( L_2 \) is dominated for player 2, so that \((m', R_1)\) «guarantees» player 1 the payoff \( 2 - m' \). Hence, if \( m' \) is sufficiently small, \((m', R_1)\) dominates \((0, L_1)\). However, if \((0, L_1)\) is eliminated, then player 2 must respond to 0 with \( R_2 \). Consequently, only the outcome is which there is no talking and players play \( R \) remains. (Hence, this argument formalizes the principle of tacit bargaining: Players do not talk but the fact that it is common knowledge that they could talk fully determines the outcome).

The reader can verify that the game \( T^e(G_1) \) in which both players can talk simultaneously (i.e. the payoff functions in \( T^e(G) \) are \( u_1(m, s) = u_1(s) - m \epsilon \), \( u_2(m, s) = u_2(s) - m \epsilon \), where \( m_i \) is the number of words spoken by player \( i \)) is no longer dominance solvable. However, application of the Kohlberg and Mertens (1986) notion of strategic stability implies that players will still succeed to play \( R \) in \( T^e(G_1) \) without actually having to communicate. Namely, consider an equilibrium outcome in which the players play \( L \) then by talking a little bit more as in this outcome, player \( i \) can signal that he wants to switch to \( R_i \), but then his opponent should switch to \( R_j \), and player \( i \) improves his payoff: he spends \( \epsilon \) more on communication to increase his payoff from 1 to 2. (The formal argument uses the fact that the strategies involved are not best responses against the equilibrium outcome, hence, that it should be possible to eliminate them without affecting the outcome). Hence, playing \( L \) is not stable in \( T^e(G_1) \). Similarly, any outcome in which at least one player communicates and the players choose \( R \) cannot be stable. Stability would force the opponent to interpret a slightly shorter speech as a signal that the player still wants to play \( R \), but then obviously, the player will make this shorter speech. Consequently, only playing \( R \) without any communication taking place is stable. (Van Damme (1989) shows that, in the Battle of the Sexes, stability requires that there is some communication if players can talk simultaneously. Hence, it this game, simultaneous talk necessarily leads to inefficient outcomes).

3. RISK DOMINANCE

Aumann (1989) has challenged the point of view that an announcement to play a strict Nash equilibrium is credible. Aumann used the game from Figure 4 to show that a preplay agreement to play a strict equilibrium might not be kept.

The game from Figure 4 has two strict equilibria, viz. \( L \) and \( R \). Each equilibrium has something going for it: \( L \) is Pareto dominant, but \( R \) is much safer. In the absence of preplay communication each player \( i \) might decide to play \( R_i \) since playing \( R_i \) guarantees the equilibrium payoff of 7 (in fact, \( R_i \) might yield 8) while his payoff might drop from 9 to 0 if he chooses \( L_i \) but is so unlucky to be matched with a cautious opponent choosing \( R_j \).
Hence, if player $i$ fears that his opponent is a cautious player, then he might prefer $R_i$. Of course, the reasoning does not stop here, player $i$ realizes that player $j$ reasons in the same way, hence, player $i$ might prefer to play $R_i$ if he believes that $j$ believes that $i$ is a cautious player. In fact, player $i$ might prefer to play $R_i$, if he believes that $j$ believes that $i$ believes that $j$ is a cautious player, etc.

Given the above arguments in favor of $R$, let us assume that we are in a world in which players are convinced that, without preplay communication, it only makes sense to play $R$ and let us investigate whether players in this world can talk each other into $L$. Given the above each player $i$ needs considerable reassurance from his opponent $j$ that $j$ will indeed play $L_j$ after an announcement to do so. But what reassurance can player $i$ get? Actually there is no reassurance whatsoever since also a player $j$ who intends to play $R_j$ will announce that he intends to play $L_j$ trying thereby to induce player $i$ to choose $L_i$, and hence, to increase his payoff from 7 to 8. Consequently, player $i$ realizes that the opponent will always announce $L_j$ irrespective of what he intends to do. Hence, no new information can be transmitted during the preplay communication phase, and it seems that with communication the game will be played in exactly the same way as when there is no communication. Therefore, in the world that we postulated, players will play $R$ also when communication is possible, and announcements to play $L$ are not credible.

Harsanyi and Selten (1988) formalized the idea that one equilibrium is safer than another by means of the notion of risk dominance that will be introduced now. Let $G$ be a 2 x 2 game (with strategies $L_i$ and $R_i$ for player $i$) and write $B_i(l_j)$ for the set of best responses of player $i$ against the mixed strategy $l_jL_j + (1-l_j)R_j$ of player $j$. The correspondence $B=B_G=(B_1,B_2)$ is called the best reply correspondence of $G$. One might take the point of view that this correspondence contains all essential information about a game: After the preplay communication is over and player $i$ has formed his beliefs about what player $j$ is going to do, $B_G$ is all that player $i$ needs to know to compute which action is optimal. Taking this point of view implies that whether equilibrium $L$ or $R$ is safer in $G$ can only depend on $B_G$. The games $G$ and $G'$ will be called best reply equivalent if $B_G=B_{G'}$.

It is easily seen that the best reply correspondence is not changed by multiplying a player's payoff with a positive constant nor if we add to player $i$'s payoff a constant that only depends on player $j$'s strategy (i.e. if $u'_i(s)=ku_i(s)+f(s_j)$ and $k>0$, then the game with payoffs $(u'_1,u'_2)$ is best reply equivalent to the game payoffs $(u_1,u_2)$. By applying such transformations one sees that the game from Figure 4 is best reply equivalent to that of Figure 5. Obviously, in the latter $R$ is the most attractive equilibrium. We see that the Pareto optimality concept is not best reply invariant.
The reader can easily verify that, more generally, each 2 x 2 game \( G \) with two strict Nash equilibria (say \( L \) and \( R \)) is best reply equivalent to a pure coordination game, i.e. to a game in which the off-diagonal payoffs are zero and the diagonal payoffs are strictly positive. Following Harsanyi and Selten (1988) we say that \( L \) risk dominates \( R \) in \( G \) if in the (or, equivalently, any) pure coordination game \( G' \) that is best reply equivalent to \( G \), \( L \) Pareto dominates \( R \). Hence, the notion of risk dominance is best reply invariant and coincides, for coordination games, with our intuition that in such games the Pareto dominant equilibrium is safest. One can show that, equivalently, \( L \) risk dominates \( R \) if the product of the deviation losses associated with \( L \) is larger than the product of deviation losses at \( R \). (One takes the product of the two players' deviation losses where player i's deviation loss at \( L \) is defined as how much this player loses if he deviates unilaterally from \( L \)). Yet another equivalent characterization is that \( L \) risk dominates \( R \) if and only if the best reply region of \( L \) (i.e. the strategy vectors \( s \) for which \( L_s \) is a best response against \( s_i \) (i.e.)) has a larger area than the best reply region of \( R \).

By going over a couple of examples the reader will quickly see that the risk dominance concept corresponds very well with our intuition about which equilibrium is less risky. Obviously, in Figure 4, \( R \) risk dominates \( L \). As has probably become clear from the heuristic argument from the beginning of this section, risk dominance is related to hierarchies of beliefs, hence, to the concept of common knowledge (Aumann (1976)). Intuitively speaking, an event is common knowledge if all players know that all players know that all players know it, and so on. In the following sections, risk dominance will figure prominently and the link between risk dominance and common knowledge will be made more explicit. (For a popular introduction to the subject of common knowledge and for a discussion of the importance of this concept in game theoretic analysis, the reader is referred to Binmore and Brandenburger (1988)).

4. BOUNDEDLY BAYESIAN PLAYERS

Consider the game \( G(u_1, u_2) \) from Figure 6 in which there are two pure strategy Nash equilibria ((\( s,w \)) and \((w,s)) and a mixed one (in the latter each player i chooses \( s \) with probability \( v_i := 1/(1+u_j) \) where \( j \neq i \)). Note that each 2 x 2 game with two strict equilibria is best reply equivalent to a game \( G(u_1, u_2) \) with \( 0 < u_i \) for some \((u_1, u_2)\). In this section attention will be restricted to the case where \( 0 < u_i < 1 \), hence, there is a conflict of interest: player 1 prefers the equilibrium \((s, w)\) to the equilibrium \((w, s)\) and player 2's preferences go in the opposite direction. The mixed strategy equilibrium yields player 1 the expected
payoff \( u_i / (1 + u_i) \), hence, this equilibrium is Pareto dominated by both pure equilibria. The game may be interpreted as a simple bargaining game: Two offers are on the table and at least one of the players has to give in: If neither gives in (or both yield) then a conflict results.

Since there are multiple equilibria, there is strategic uncertainty and it is not clear which equilibrium to play. Let us assume that our players are «Bayesian» so that they are able to associate probabilities to any event about which they are uncertain. Specifically let \( s_j \) be the probability that player \( i \) assigns to the event that player \( j \) plays \( s \) (where \( i \neq j \)). Hence, player \( i \) believes that player \( j \) plays the mixed strategy \( s_j s + (1 - s_j)w \).

If player \( i \) is «Bayesian rational» he will choose that action that maximizes his expected payoff, hence, player \( i \)'s optimal strategy is:

\[
\varphi_i (s_j) = \begin{cases} 
  s & \text{if } s_j < u_i \\
  w & \text{if } s_j < u_i
\end{cases}
\]

where \( u_i := 1 / (1 + u_i) \). Player \( j \) initially does not know player \( i \)'s beliefs, i.e. he does not know \( s_j \). Being Bayesian he can assign probabilities to the possible values of \( s_j \). Let \( F_j \) denote the distribution function of these «second order» beliefs of player \( j \), hence \( F_j(x_j) \) is the probability that player \( j \) assigns to «player \( i \) believes that \( s_j \) is less than \( x_j \)». From these second order beliefs and the fact that player \( i \) is Bayesian rational (i.e. that player \( i \) plays as in [4.1]) player \( j \) can compute the probability that player \( i \) will choose \( s \) as \( F_j(u_i) \). Now we started from the assumption that player \( j \) assigned a probability \( s_i \) to this event, hence, consistency of the first and second order beliefs requires that:

\[
s_i = F_j(u_i) \quad (i \neq j)
\]

so that player \( j \)'s second order beliefs fully determine this player's first order beliefs. In a fully specified Bayesian model one would also have to consider higher orders of beliefs, i.e. player \( i \) would be required to assign probabilities to the various second order beliefs \( F_j \) that player \( j \) might have. Since this approach does not seem very tractable analytically,
let us assume that the players are bounded in their capabilities and let us assume that they truncate the hierarchy of beliefs at the second level. Let us further assume that the second order beliefs $F_1$ and $F_2$ are common knowledge. Since in this case player $i$ knows the beliefs $F_j$ of player $j$ he has to update his own beliefs $F_i$. Namely, $F_i$ represents player $i$’s original beliefs on $s$, but, if $F_j$ is known, player $i$ can compute $s_i$ from (4.2). Let us assume that updating takes place by assigning a small probability $\varepsilon$ to the computed value of $s_i$. Hence, denoting by $\delta(x)$ the Dirac measure that assigns all weight to $x$, the following updating scheme is suggested:

$$F^*_i = (1-\varepsilon)F^*_i + \varepsilon\delta(s^*_i)$$  \hspace{1cm} [4.3.a]$$

$$s^*_i = F^*_j(u_j)$$  \hspace{1cm} [4.3.b]$$

To investigate where this reasoning process leads to we have to specify the starting point $(F^*_1, F^*_2)$. Let us assume that initially player $i$ does not assign specific weight to any particular value of $s_i$, hence,

$F^*_i$ allows a density that is positive on $[0,1]$  \hspace{1cm} [4.3.c]$$

In order not to bias the final result, we propose to start from a symmetric distribution, i.e.

$$F^*_1 = F^*_2$$  \hspace{1cm} [4.3.d]$$

We can now show:

**Proposition 1:** The reasoning process [4.3.a]–[4.3.d] results in the risk dominant equilibrium of $G(u_1, u_2)$ if $u_1 \neq u_2$ and if $\varepsilon$ is small enough.

**Proof:** Assume $u_1 < u_2$ so that $(s, w)$ risk dominates $(w, s)$ in $G(u_1, u_2)$. Since in this case $u_1 > u_2$ the conditions [4.3.c], [4.3.d] imply:

$$s^*_1 > s^*_2$$  \hspace{1cm} [4.4]$$

Next, it is easily seen that [4.3.a]–[4.3.b] imply that:

$$s^*_j < s^*_j$$ if and only if $u_1 < s^*_i$  \hspace{1cm} [4.5]$$

For $\varepsilon$ sufficiently small the phase diagram of [4.5] is, in the region determined by [4.4], given as in Figure 7. Clearly, we always converge to $(s_1, s_2) = (1, 0)$, hence, to the risk dominant equilibrium $(s, w)$.
5. GLOBAL PAYOFF UNCERTAINTY

In this section we briefly discuss the justification of risk dominance that is proposed in Carlsson and Van Damme (1990) (henceforth CD). Roughly, the answer that CD give to the question of how players can know that only R makes sense in the game of Figure 1 (and also in the game of Figure 4) is that players know this from experience with (or from reasoning through) similar games. CD argue that these games should not be analyzed in isolation: Players know what to do in the game from Figure 1 since they know that it is optimal to play the Pareto best equilibrium in each coordination game with Pareto ranked payoffs. CD suggest to analyze classes of games with the same structure simultaneously. They picture players in the context in which the payoffs of the game are only «almost common knowledge» and they show that when a 2 x 2 game is played in this context, players are forced to analyze all games simultaneously and thereby reason themselves to the risk dominant equilibrium. (Of course the idea that a solution of a game should be part of a plan that is consistent across a larger domain occurs already in the seminal work of Nash (1950) on bargaining and that of Schelling (1960) on focal points). Hence, CD do not work explicitly with the hierarchy of beliefs as we did in the previous section. However, an equilibrium of the CD-perturbed game induces such an infinite hierarchy of beliefs.

The CD approach will now be illustrated by means of the game Γ(θ) from Figure 8.
Note that $\Gamma(7)$ is Aumann's game from Figure 4. Imagine that the players are in the context in which they know that they have to play a game $\Gamma(\theta)$ as in Figure 8 but they do not yet know which one. Hence, they know that they have to play a game in which the conflict between risk dominance and payoff dominance exists. (The reader may argue that the parametrization from Figure 8 is not natural; However, it is just chosen to simplify the argument. The other assumptions that are to be discussed next should also be viewed in this spirit; the results from Carlsson and Van Damme (1990) are more general). The reader will probably agree that as $\theta$ increases playing $L$ becomes less and less attractive and that a natural way to play this game is by specifying a cutoff value $\delta$ and play $L$ if and only if $\theta$ is less than $\delta$. CD show that, if the players can observe the actual parameter value $\theta$ only with some slight noise, then the value of $\delta$ is uniquely determined in equilibrium. In fact, CD show that $\delta = 4$, hence, the players always choose the risk dominant equilibrium. (Note that some noise is essential to derive uniqueness: if $\theta$ could be perfectly observed, then each game $\Gamma(\theta)$ would occur as a simple subgame an the cutoff may lie anywhere, in fact, in this case the equilibrium strategies need not be stepfunctions).

To formally derive the above result let us assume that the set $\Theta$ of all possible parameter values is finite, that initially all values of $\theta$ are equally likely and that $\Theta$ includes values $\theta$ with $\theta < 0$ (which makes $L_1$ strictly dominant) as well as values with $\theta > 0$ (such that $R_1$ is strictly dominant). Furthermore, assume that, if the actual parameter value is $\theta$, then one player receives the signal $\theta^+$ (i.e. the smallest value in $\Theta$ that is larger than $\theta$) while the other gets to hear $\theta^-$ (i.e. the largest value in $\Theta$ that is smaller than $\theta$) with both possibilities being equally likely (with the appropriate modifications at the endpoints of $\Theta$). Since the observations are noisy no player knows exactly which «game» he is playing, however, if the grid of $\Theta$ is fine then each player has fairly accurate information about the payoffs in the game. Furthermore, each player also has good knowledge about the information of his opponent and the players know that their perceptions of what the payoffs are do not differ too much. Hence, if the grid of $\Theta$ is fine, the game with noisy observations, which will be denoted by $\Gamma(\Theta)$, may be viewed as a small perturbation of the game in which observations are perfect and in the latter $\Gamma(\theta)$ occurs as a subgame for each value of $\theta$. However, from the point of view of common knowledge (Aumann (1976)), the games are completely different. Namely, if a player receives the signal $\theta$, then he knows that the payoffs either are as in $\Gamma(\theta^-)$ of as in $\Gamma(\theta^+)$ and he knows that his opponent either received the signal $\theta^-$ or $\theta^+$. Hence, he also knows that the opponent believes that the game is either $\Gamma(\theta^-)$ or $\Gamma(\theta^+)$ or $\Gamma(\theta^+)$ or $\Gamma(\theta^{++})$, with all probabilities being equally likely, and that the opponent believes that his signal is either $\theta^-$, or $\theta^{++}$ or $\theta$ with the latter having probability $1/2$. Continuing inductively it is therefore seen that no matter how fine the grid size of $\Theta$ is, basically the only information that is common knowledge is that some game $\Gamma(\theta)$ with $\theta$ in $\Theta$ has to be played. This lack of common knowledge forces the players to take a global perspective in order to solve the perturbed game: To know what to do if one receives the signal $\theta$ one should also investigate what to do at parameter values $\theta'$ that are far away from $\theta$. It is this phenomenon that drives the CD results. It is easy to derive the following result.

**Proposition 2:** In the game $\Gamma(\Theta)$ choosing $L_i$ is iteratively dominant for each observation $\theta_i < 4^-$, while choosing $R_i$ is iteratively dominant if $\theta_i > 4^+$.
Proof: Let $\theta_i$ be the observation of player $i$. If $\theta_i^+ < 0$ (resp. $\theta_i^- > 8$) then player $i$ chooses $L_i$ (resp. $R_i$) since he knows that this action is strictly dominant. Assume that it has already been shown by iterative elimination of strictly dominated strategies that $L_1$ and $L_1$ (resp. $R_1$ and $R_1$) are strictly dominant at each observation $\theta$ with $\theta > \alpha$ (resp. $\theta \geq \beta$). Hence, the iterative procedure starts with $\alpha = 0^-$ and $\beta = 8^+$. Consider $\theta = \alpha^+$, so that player $i$ knows that either $\theta = \alpha^-$ or $\theta = \alpha^+$, hence, player $i$ knows that player $j$ will choose $L_i$ with a probability $p$ that is at least $1/2$. Choosing $L_i$ yields an expected payoff of $9p$ while $R_i$ yields at most $\alpha^++p$, so that player $i$ will find it strictly dominant to choose $L_i$ if $\alpha^++<4$. Consequently, $L_i$ is iteratively dominant for player $i$ at $\theta_i$ if $\theta_i<4^-$ and similarly $R_i$ is iteratively dominant at $\theta_i$ if $\theta_i>4^+$. □

Proposition 2 shows that the perturbed game $\Gamma(\Theta)$ is almost dominance solvable: For all but a small set of parameter values (viz. the interval $[4^-,4^+]$) unique iteratively dominant actions exist. By playing these dominant strategies players coordinate on the risk dominant equilibrium of the actual game that was selected by chance, hence, by just relying on rationalizability (Bernheim (1984), Pearce (1984)) in the perturbed game we obtain equilibrium selection according to the risk dominance criterion for every game $\Gamma(\theta)$ with $\theta \in [4^-,4^+]$. In the limit, as the grid size of $\Theta$ tends to zero and, hence, the information about the payoffs in the game becomes «perfect», players play the risk dominant equilibrium of $\Gamma(\Theta)$ for every value of $\theta$. (Carlsson and Van Damme (1990) prove a result similar to, but considerably more general than Proposition 2; see their Theorem 2).

As the proof of Proposition 2 makes clear, the lack of common knowledge in $\Gamma(\Theta)$ leads the regions in which $L$ (resp. $R$) is dominant to exert a remote influence on other parameter values. A similar action from a distance also drives the results in Rubinstein’s (1989) electronic mail game. Rubinstein considers the following scenario in which two players are involved that have to play one of two games, $\Gamma_a$ or $\Gamma_b$, depending upon whether the state of nature is $a$ or $b$. Initially the players consider the two states of nature to be equally likely. For concreteness assume that game $\Gamma_a$ is Aumann’s game from Figure 4 (which is reprinted below as Figure 9.a) while $\Gamma_b$ is the game from Figure 9.b. Note that in the latter game $R_1$ is strictly dominant for each player $i$.

![Figure 9.a](image)

\textbf{GAME }$\Gamma_a$

<table>
<thead>
<tr>
<th></th>
<th>$L_2$</th>
<th>$R_2$</th>
</tr>
</thead>
<tbody>
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<td>$L_1$</td>
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<td>0,8</td>
</tr>
<tr>
<td>$R_1$</td>
<td>8,0</td>
<td>7,7</td>
</tr>
</tbody>
</table>

![Figure 9.b](image)

\textbf{GAME }$\Gamma_b$

<table>
<thead>
<tr>
<th></th>
<th>$L_2$</th>
<th>$R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>0,0</td>
<td>0,1</td>
</tr>
<tr>
<td>$R_1$</td>
<td>1,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Clearly, the players would like to coordinate on $L$ in state $a$ and on $R$ in state $b$. Rubinstein assumes that the information which state of nature prevails is initially only known to player 1 but that this player can communicate to player 2 by electronic mail.
Unfortunately the mail system is not perfect and the message «the state is a» is lost with probability \( \varepsilon \) (Rubinstein assumes that no messages are sent in state \( b \)). In fact, every message that is sent is lost with probability \( \varepsilon \) along the way. To make up for this deficiency, the mail system is set up such that, upon receipt of any message, a confirmation of receipt is sent automatically. (Hence, if player 2 receives player 1's initial message then he confirms and this confirmation, when received by player 1, is reconfirmed, etc.).

The analysis of this communication game is easy. Player 1 has \( R_1 \) as a dominant strategy if he hears that the game is \( T \). If player 2 does not receive any messages, he concludes that with probability \( 1 - \varepsilon \) the game is \( T \), hence, that \( R_1 \) will be chosen with a probability of at least \( 1 - \varepsilon \). Therefore, he concludes that it is dominant to choose \( R_2 \). Next, if in state \( a \) a player 1 does not receive a confirmation of player 2, he thinks that it is (roughly) just as likely that his first message was lost as that the confirmation was lost, hence, he assigns a probability of at least \( \frac{1}{2} \) to player 2 playing \( R_2 \) and he finds it optimal (dominant) to play \( R_1 \). Continuing inductively, one finds that each player \( i \) finds it iteratively dominant to play \( R_i \) no matter how many messages he receives. Again one sees that the situation with zero messages exerts a remote influence on the case with \( N \) (\( N \) large) messages. Even though it is the case, if \( N \) is large, that the players know that the game is \( T \), and that they know that they know this, this fact is not common knowledge, and this prevents them from choosing \( L \). (Note that this result would not be changed if also in state \( b \) messages were exchanged).

6. LEARNING AND MUTATION

Up to now we have mainly considered equilibrium selection in the situation where the game is played only once and players have no direct evidence with the game (the only exception has been Subsection 2.2.). To use Binmore's (1987) terminology: we have remained in the eductive context. We have assumed that our players are superrational and that they can reason themselves towards equilibria. We have presented two reasoning processes that, when adopted by the players, lead them to the risk dominant equilibrium in 2 \( \times \) 2 games (Propositions 1 and 2).

In this section we briefly discuss the very interesting recent paper Kandori et al. (1991) in which equilibrium selection according to risk dominance is derived in the context of repeated play by myopic players. Specifically, Kandori et al. consider the following situation in which a 2-person 2 \( \times \) 2 game has to be played by members of two (equally) large but finite populations. In each round, the members of the populations are randomly matched, each player chooses an action and the average payoff resulting from each action is made public. (Van Huyck et al. (1990) report on experimental results obtained in a similar context, and they find that play converges to the risk dominant equilibrium quite rapidly. Also see Crawford (1991)). Kandori et al. assume that players are myopic, i.e. that in each period \( t \) each player will choose that action that was most successful in the previous period. Furthermore, they assume that players independently make mistakes (i.e. choose an action different from the intended one) with a small probability \( \varepsilon \).

Kandori et al. (1991) are interested in the long run behavior of this system and they find that, given the learning process postulated, the system will find itself at the risk
dominant equilibrium most of the time. To illustrate why this is the case, let us consider the game $G_1$ from Figure 1.

Given that behavior is myopic, the state of the system at time $t$ is completely characterized by the pair of numbers $r = (r_1, r_2)$ where $r_i (r_i \in \{0,1,\ldots,N\}$ with $N$ being the size of each population) is the number of individuals in population $i$ that chooses $R_i$, and the system evolves according to a Markov chain. (Note that two games that are best reply equivalent induce the same Markov chain). Since players make mistakes, all states are connected, hence, there is a unique limit distribution. Note that one needs (roughly) at least $2N/3$ mistakes to move the system away from the state $r = (N,N)$ where everybody plays $R$. (With less mistakes, action $R_i$ remains optimal). On the other hand, if there are $N/3$ mistakes in one population (and at least one in the other so that the payoffs to both strategies can be observed), the system moves from the state $r = (0,0)$ in which all players play $L$. Since mistakes are assumed to be independent, the system remains much longer at $r = (N,N)$ than at $r = (0,0)$ and if the mistake probability $\varepsilon$ tends to zero, the probability that the system is found in the state where everybody plays $R$ tends to 1.

REFERENCES


Reprint Series, CentER, Tilburg University, The Netherlands:


No. 5 Th. ten Raa and F. van der Ploeg, A statistical approach to the problem of negatives in input-output analysis, *Economic Modelling*, vol. 6, no. 1, 1989, pp. 2 - 19.


No. 8 Th. van de Klundert and F. van der Ploeg, Wage rigidity and capital mobility in an optimizing model of a small open economy, *De Economist* 137, nr. 1, 1989, pp. 47 - 75.


