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Alternating Bid Bargaining with a Smallest Money Unit

by

Eric van Damme, Reinhard Selten and Eyal Winter

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Alternating Bid Bargaining with a Smallest Money Unit

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In a seminal paper, Ariel Rubinstein has shown that impatience implies determinateness of the two-person bargaining problem. In this note we show that this result depends also on the assumption that the set of alternatives is a continuum. If the pie can be divided only in finitely many different ways (for example, because the pie is an amount of money and there is a smallest money unit), any partition can be obtained as the result of a subgame perfect equilibrium if the time interval between successive offers is sufficiently small. We also show that, for a fixed discount rate, all subgame perfect equilibrium payoffs of the discrete game converge to the solution obtained by Rubinstein if the smallest money unit tends to zero. Journal of Economic Literature Classification Number: 026.

1. INTRODUCTION

A natural way of modeling two-person bargaining as an extensive game makes use of a game structure in which two players take turns in making bids. In each round one player makes a bid; then the other player either accepts or rejects this offer; in case of rejection the rejector makes the
next bid, etc. We use the term “alternating bid models” for bargaining games of this structure. A pioneering investigation of alternating bid models is due to Ingolf Stáhl (1972). Later Ariel Rubinstein (1982) created a very influential bargaining theory based on an alternating bid model.

The bargaining problem considered here is the division of a fixed amount of money. Rubinstein’s model permits arbitrary divisions and thereby provides a continuum of possible agreements. An alternative to the assumption of infinite divisibility of money is the introduction of a smallest money unit. It is our aim to explore the consequences of this assumption for alternating bid bargaining. As far as other assumptions are concerned our analysis is based on Rubinstein’s framework.

Rubinstein’s theory specifies a unique solution, the uniquely determined subgame perfect equilibrium of his model. It will be shown that the introduction of a smallest money unit destroys Rubinstein’s uniqueness result. If both players are risk neutral, the amount of money to be distributed is $50,000, the smallest money unit is 1 cent, the yearly interest rate is 10%, and one bargaining round takes 1 min, then all divisions of the $50,000 are supported by subgame perfect equilibria of the modified model (see Proposition 1 and the explanation of Table 1 in Section 4).

Ingolf Stáhl already investigated alternating bid models with a finite number of alternatives (Stáhl, 1972; also see Stáhl (1988) in which Stáhl’s original model is compared to that of Rubinstein). He examined models of finite and of infinite length. In most of his work, time preferences are nonstationary (hence, preferences in his model are different than in Rubinstein’s framework). Stáhl also considers a different extensive game; his assumption of “good-faith bargaining” prevents players from increasing their demands during the game. Stáhl aimed at conditions that are sufficient for the solution to be unique and to be independent of who starts the bargaining. We want to find out how the smallest money unit and discounting interact and we are particularly interested in nonuniqueness as a consequence of the presence of a smallest money unit. Hence, Stáhl’s research questions were completely different from ours.

It is impossible to construct an absolutely realistic bargaining model. Every real bargaining situation has many special features which are minor influences on the bargaining result. Idealization is an unavoidable ingredient of model construction. Is it really necessary to model a relatively inconspicuous institutional detail like the presence of a smallest money unit? Maybe the correct answer is no. However, Rubinstein’s theory heavily relies on very small time costs due to discounting.1 It should not

1 In this paper, whenever we speak of Rubinstein’s bargaining theory, we mean that part of his theory in which players have preferences with constant discount rates that are arbitrarily close to 1. It is this part of the theory that has been frequently applied in economics. (See Osborne and Rubinstein (1989) for an overview.)
take more than 1 min to make a bid. The bidder does not have to do more than pronounce a number. Even for quite sizable amounts of money the interest for 1 min at a reasonable yearly rate is very small. Why should very small interest losses be modeled explicitly? Maybe also here the answer is no.

The smallest money unit and the time discount rate are both minor strategic influences, but these forces interact. Therefore, either both should be considered or both should be neglected. In this sense Rubinstein's model is an imbalanced idealization. His theory relies on an explicitly modeled weak influence and ignores a weak counteracting force.

Assume that both bargainers are risk neutral, let $A$ be the amount of money to be distributed and let $\delta$ be the discount rate for one bargaining round. We assume that both players have the same discount rate. In Rubinstein's theory the solution is of the following type: A player who makes a bid asks for $x$. A player accepts every offer which gives him at least $A - x$ and rejects everything else. The solution requires indifference between $A - x$ and $\delta x$. This yields $x = A/(1 + \delta)$.

Rubinstein's theory excludes strategy combinations of the following type as possible solutions: Player 1 always asks for $y$ and player 2 always asks for $A - y$. Both accept any offer which gives them at least what they ask for and reject all others. If player 1 asks for a little more, say $y + \varepsilon$, then player 2 faces a choice between $A - y - \varepsilon$ for acceptance and $\delta(A - y)$ for rejection. Obviously he must accept, if we have $\varepsilon < (1 - \delta)(A - y)$. This contradicts the assumption of a subgame perfectness.

Now consider the consequences of the introduction of a smallest money unit $g$. Agreement payoffs must be integer multiples of $g$. If now player 1 wants to ask for more than $y$, he has to demand at least $y + g$. If $g$ is greater than $(1 - \delta)A$, player 1 cannot give an incentive to player 2 to accept less than $A - y$. The smallest money unit prevents him from increasing his demand by an amount $\varepsilon$ which is smaller than the interest loss $(1 - \delta)(A - y)$. This heuristic argument indicates why Rubinstein's uniqueness result is destroyed by the introduction of a smallest money unit. The size of the smallest money unit puts a lower bound on exploitable interest losses on the other side. Note that decreasing the time between offers corresponds to increasing $\delta$ and that for $\delta$ sufficiently large always $g > (1 - \delta)A$. Hence, for short intervals between offers it may be expected that indeed any distribution can be obtained by some subgame perfect equilibrium (see Proposition 1).  

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2 A referee remarked that Proposition 1 is a "folk theorem" in both of the standard usages of the term, i.e., that proposition was known to a number of researchers in the field. (The referee mentioned Hugo Sonnenschein and Roger Myerson.) After having submitted the paper, we became aware of independent work of Muthoo (1989) who also proves a result similar to our Proposition 1.
In Rubinstein's bargaining solution agreement is reached at once. More than one bargaining round cannot be played, unless mistakes are made. Contrary to this, in the presence of a smallest money unit subgame perfect equilibrium may involve many bargaining rounds, before an agreement is reached (Proposition 2).  

While Propositions 1 and 2 deal with the case of a fixed smallest money unit and a vanishing reaction time, one may also be interested in the case of a fixed reaction time and investigate what happens when the smallest money unit tends to zero. For the special case of risk neutral bargainers, we study this question in Section 4, where a continuity result is established: As $g$ vanishes, all subgame perfect equilibrium payoffs of the discrete game approximate the unique subgame perfect equilibrium payoff of the continuum game (Proposition 3). Furthermore, in equilibrium there is no delay if $g$ is sufficiently small. We also show that, despite these properties, the gap between the largest and the smallest subgame perfect equilibrium payoff, measured in units of $g$, may remain large. In Section 4 we also briefly study the finite horizon model (bargaining ends after $T$ rounds) and show that in this case the introduction of a smallest money unit may have drastic consequences as well.

Undoubtedly Rubinstein's ingenious bargaining theory merits our admiration, but we cannot avoid the conclusion, that his model does not provide a balanced idealization of real bargaining situations. The driving force behind his uniqueness result is provided by the exploitability of small interest losses by even smaller increases of a bidder's demand. In the presence of a smallest money unit such destabilization possibilities are easily lost, since it becomes impossible to deviate sufficiently little.

Our analysis is based on the same game theoretic rationality assumptions as Rubinstein's theory. Presumably real bargaining behavior would not be influenced by a smallest money unit of insignificant size and equally insignificant time discounts. Nevertheless, it is worthwhile to use the concept of a subgame perfect equilibrium point in order to explore the interaction and the comparative importance of both influences.

2. The Bargaining Model

Two players, denoted by 1 and 2, have to divide an amount of money (normalized to) 1. Let $g > 0$ denote the smallest money unit. The set of possible agreements is

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1 Osborne and Rubinstein (1989, Sect. 3.9.1.) give an example to show that finitely many alternatives may lead to multiple equilibria and to delay. They do not investigate the general consequences of shrinking the time between offers.
and $X^e$ denotes the set of efficient agreements $((k_1 + k_2)g = 1)$.

Bargaining takes place over time, starts at $t = 0$, and proceeds according to the following rules:

**Round $t$ ($t \in \mathbb{N}$, $t$ even):** Player 1 proposes $x \in X$; after hearing 1's proposal, player 2 either accepts or rejects. If 2 accepts, the game terminates with agreement $x$, otherwise the game moves to round $t + 1$.

**Round $t$ ($t \in \mathbb{N}$, $t$ odd):** Player 2 proposes $x \in X$, after hearing 2's proposal, player 1 either accepts or rejects. If 1 accepts, the game ends with agreement $x$, otherwise the game reaches round $t + 1$.

Denote by $(x, t)$ the outcome where agreement on $x$ is reached in round $t$ and let $D$ denote perpetual disagreement. Let $\Delta$ be the length of a single bargaining round. We will assume that there exist constants $r_1, r_2, > 0$, and strictly increasing concave functions $U_1, U_2$ (having domain $[0, 1]$) with $U_i(0) = 0$ such that the preferences of the players can be represented by the utility functions $V_i$ given by

$$V_i((x, t)) = e^{-r_i\Delta}U_i(x) \quad \text{and} \quad V_i(D) = 0. \quad (2)$$

For justification of this assumption, we refer to Fishburn and Rubinstein (1982). The above fully describes the game to be denoted $\Gamma(\Delta)$. Strategies, Nash equilibria, and subgame perfect equilibria (SPE) are defined in the standard way, hence, these definitions will not be repeated here (see Rubinstein, 1982). Rather, we directly turn to our main results.

### 3. A Folk Theorem

**Proposition 1.** If $\Delta$ is sufficiently small, specifically if $\Delta$ is such that for $i = 1, 2$

$$U_i(1 - g)/U_i(1) \leq e^{-r_i\Delta}. \quad (3)$$

then for any efficient agreement $x \in X^e$ there exists a subgame perfect equilibrium of $\Gamma(\Delta)$ that results in the outcome $(x, 0)$.

**Proof.** Note that (3) says that player $i$ prefers getting the full amount 1 one period later to receiving $1 - g$ now. Since $U_i$ is concave this condition implies that for all $x_i \in [g, 1]$

$$U_i(x_i - g)/U_i(x_i) \leq e^{-r_i\Delta}. \quad (4)$$
Let $x \in X^e$ and write $x = (x_1, x_2)$. Consider the pair of stationary strategies $\sigma^x = (\sigma^1, \sigma^2)$ defined by

\begin{align*}
\sigma^1: & \quad \text{Always propose } x; \\
& \text{Accept any proposal } y \text{ with } y_i \geq x_i, \quad (5) \\
& \text{Reject any other proposal.}
\end{align*}

If $\sigma^x$ is played, the outcome $(x, 0)$ results. We claim that $\sigma^x$ is an SPE if (3) is satisfied. Because of stationarity, it suffices to show that one-period deviations are not profitable. Hence, we must verify that it does not pay to deviate in round $t$ when from round $t + 1$ on play will always be in accordance to $\sigma^x$. Obviously, given the acceptance/rejection decision of the player, the best one can propose is $x$ since $x$ is efficient in $X$. Clearly, it is also optimal for player $i$ to accept any offer that yields him at least $x_i$. The crucial step is to verify that, if $x_i > 0$, it is optimal for player $i$ to reject any offer $y$ with $y_i < x_i$. However, this is guaranteed by (4).

All equilibria constructed thus far result in an immediate efficient agreement. However, since there is a multiplicity of such equilibria, it is easy to construct alternative equilibria that do not have these nice properties. In fact, if inequality (3) holds, any outcome, and even perpetual disagreement, can occur in an SPE. The idea is to sustain a path $\pi$ with the threat to continue with the equilibrium from Proposition 1 that yields player $i$ the payoff zero if $i$ deviates from $\pi$. Formally, this construction is carried out in Proposition 2. It is convenient to introduce the following notation: $x^1 = (1, 0)$, $x^2 = (0, 1)$, and $\sigma^i = \sigma^v$. Finally, $i_T$ denotes the player who proposes in round $T$ (hence $i_T = T \mod 2 + 1$).

**Proposition 2.** Let $x \in X$ and $T \in \mathbb{N}$. If $\Delta$ satisfies (3), there exists a subgame perfect equilibrium of $\Gamma(\Delta)$ that results in the outcome $(x, T)$. In this case, there also exists a subgame perfect equilibrium that results in perpetual disagreement.

**Proof.** Let $x \in X$ and $T \in \mathbb{N}$. Consider the strategy pair $\sigma = (\sigma_1, \sigma_2)$ defined by

\begin{align*}
\sigma_1: & \quad \text{In round } t (t < T): \text{Propose } x^t; \\
& \quad \text{accept } x^t \text{ but reject any other proposal.} \\
& \quad \text{In round } T: \text{Propose } x; \\
& \quad \text{accept } x \text{ and } x^t, \text{ but reject any other proposal.} \\
& \quad \text{In round } t (t > T): \text{Play according to } \sigma^i. \\
\end{align*}
Let \( \pi(\sigma_1, \sigma_2) \) be the path induced by \((\sigma_1, \sigma_2)\). The strategy pair \( \tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2) \) is constructed from \( \sigma \) by means of

\[ \tilde{\sigma}_i : \text{Play according to } \sigma_i, \text{ however, if in any round } t \leq T \text{ there is a deviation from } \pi(\sigma_1, \sigma_2) \text{ and if the first deviation is by player } k, \text{ then immediately after this deviation switch to playing } \sigma_i^{k-1} \text{ for the remainder of the game.} \]

We claim that \( \tilde{\sigma} \) is an SPE of \( \Gamma(\Delta) \) whenever (3) is satisfied. Note that \( \tilde{\sigma} \) results in the outcome \((x, T)\). To prove the claim, it suffices (because of Proposition 1) to show that deviating in some round \( t \leq T \) is not profitable. However, this is easily verified: If a proposing player deviates he ends up with zero, hence, deviating is not profitable for him. Deviating is clearly not attractive for the responding player as long as the proposal is on the equilibrium path (it will yield payoff zero). If the proposer has deviated, rejection yields the responder 1 in the next period, hence, if (3) is satisfied, he should reject anything less than 1, exactly as \( \tilde{\sigma} \) says that he should do. Consequently, \( \tilde{\sigma} \) is an SPE if \( \Delta \) is small.

To sustain perpetual disagreement, consider the strategy \( \tilde{\sigma}_i \) defined by "always propose \( x' \), accept \( x' \) but reject any other proposal." If \( \tilde{\sigma}_i \) is constructed from \( \sigma_i \) as above then \((\tilde{\sigma}_1, \tilde{\sigma}_2)\) is an SPE resulting in perpetual disagreement.

4. The Risk Neutral Case

In this section we confine ourselves to the case where players are risk neutral, i.e., \( U_i(x) = x \). This case is most favorable for the point we wish to make. If both players are (equally) risk averse, the range of equilibrium payoffs will be smaller: If the utility function \( W \) displays greater risk aversion than \( U \), then \( W(x - g)/W(x) \geq U(x - g)/U(x) \) so that is becomes more difficult to obtain \( x \) in a subgame equilibrium (cf. Eq. (4)).

Let us first illustrate the bound on \( \Delta \) given in (3) by performing some numerical calculations. Take \( r_1 = r_2 = 10\% \) per year, and let the smallest money unit be 1 cent (\$0.01). Proposition 1 implies that, if the time between offers is \( \Delta \), any efficient division of an amount up to \( A(\Delta) \) (as given in Table I) can be obtained in an SPE.

If the amount of money to be divided is larger than \( A(\Delta) \), then the simple strategies from Proposition 1 are no longer in equilibrium if \( x \) is "sufficiently asymmetric." but, of course, there may be more sophisti-

\[ ^4 \text{Note that, if } i \text{ is the proposer in round } t \leq T, \text{ then player } j \text{ will switch to } \sigma_i^{k-1} \text{ already in round } t. \]
cated equilibria that still result in such \( x \). These remarks immediately raise the question of how, for fixed \( U_i, \Delta, \) and \( g \), the gap between the largest and the smallest equilibrium payoff behaves as \( A \) tends to infinity. Below it is shown that this gap becomes negligible relative to \( A \), but that measured in units of \( g \) the gap may remain large. We first prove the first statement. By rescaling we may assume that \( A = 1 \) and then let \( g \) tend to zero. Also assume \( r_1 = r_2 \), write \( \delta = e^{-r_1 \Delta} \), denote by \( \Gamma(g, \delta) \) the game with smallest money unit \( g \) and discount rate \( \delta \), and let \( \Gamma(\delta) \) be the Rubinstein continuum game with discount rate \( \delta \).

Denote by \( M \) (resp. \( m \)) the supremum (resp. infimum) of the SPE payoffs of player 1 in \( \Gamma(\Delta) \) and for \( x \in X \), write \( [Sx] \) for the smallest integer multiple of \( g \) that is at least equal to \( S \cdot x \). The argument outlined in Shaked and Sutton (1984) shows that in the generic case where \( SM \) is not an integer multiple of \( g \), \( M \) and \( m \) must satisfy the equations

\[
M = 1 - [\delta m], \quad m = 1 - [\delta M].
\]

Equations (6) and (7) imply

\[
M - m = [\delta M] - [\delta m].
\]

Now

\[
[\delta M] \leq \delta M + g \quad \text{and} \quad [\delta m] \geq \delta m
\]

and substituting the latter inequalities into (7) yields

\[
M - m \leq \delta M - \delta m + g.
\]

\* If \( \delta M \) is an integer multiple of \( g \), (7) has to be replaced by \( m = 1 - \delta M - g \). In this case (9) can be sharpened and Eqs. (11) and (13) remain valid.

---

**Table I**

<table>
<thead>
<tr>
<th>( \Delta )</th>
<th>( A(\Delta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 day</td>
<td>$36.50</td>
</tr>
<tr>
<td>1 hr</td>
<td>$876</td>
</tr>
<tr>
<td>1 min</td>
<td>$52.576</td>
</tr>
<tr>
<td>1 sec</td>
<td>$3,225.806</td>
</tr>
</tbody>
</table>
which implies

\[ M - m \leq g/(1 - \delta). \]  

(11)

We see that \( M - m \) tends to zero if \( g \) tends to zero. Furthermore, substituting (9) (together with \( m \leq M \)) into (6) and (7) yields

\[ m \leq \frac{1}{1 + \delta} \quad \text{and} \quad M \geq \frac{1 - g}{1 + \delta} \]  

(12)

so that

\[ \lim_{g \to 0} m = \lim_{g \to 0} M = \frac{1}{1 + \delta}. \]  

(13)

Hence, if \( g \) tends to zero, the payoffs of player 1 in any SPE of \( \Gamma(g, \delta) \) converge to this player’s payoffs in the unique SPE of \( \Gamma(\delta) \). Furthermore, in any SPE, the payoff to player 2 must lie in the interval \([\delta m, 1 - m]\) so that (13) shows that also this player’s payoffs converge. Finally, in any SPE the sum of the players’ payoffs is at least equal to \( m + \delta m \) and (13) allows us to conclude that if \( g \) is sufficiently small

\[ m + \delta m > \delta. \]  

(14)

If agreement would not be reached immediately, the sum of the players’ payoffs could be at most \( \delta \), hence (14) shows that, if \( g \) is sufficiently small, in any SPE of the discrete game agreement is reached immediately.

**Proposition 3.** If \( g \) tends to zero, the payoff vector associated to any SPE of the game \( \Gamma(g, \delta) \) converges to \((1/(1 + \delta), \delta/(1 + \delta))\), that is, to the payoffs of the unique equilibrium of \( \Gamma(\delta) \). Furthermore, if \( g \) is sufficiently small, then in any SPE of \( \Gamma(g, \delta) \) agreement is reached immediately.

The next proposition shows that, for some parameter values, the SPE may be unique also if the smallest money unit is positive. However, the SPE of \( \Gamma(\delta) \) being feasible in \( \Gamma(g, \delta) \) is not a sufficient condition for uniqueness. In this case there exist two SPE payoff vectors in \( \Gamma(g, \delta) \) if \( g \) is small (Remark 1). We also show (Remark 2) that there exist parameter values for which the gap between the largest and the smallest equilibrium payoff, measured in units of \( g \), is large, that is, the bounds in (11) are sharp.
PROPOSITION 4. If condition (15) is satisfied, then the game \((\Gamma(g, \delta))\) has a unique SPE

\[
\frac{1}{(1 + \delta)} - \frac{1}{(1 + \delta)} > 2\delta g/(1 + \delta).
\]

Proof. Write \(z = \frac{1}{(1 + \delta)} - g\). Condition (15) is equivalent to

\[
\frac{1}{(1 + \delta)} = z + \alpha g \quad \text{with } 0 < \alpha < (1 - \delta)/(1 + \delta).
\]

Assume (16) holds. It is easily verified that the pair of stationary strategies \(\sigma = (\sigma_1, \sigma_2)\) defined by

\[
\sigma_i: \quad \text{propose } (z, 1 - z) \\
\text{accept } x \text{ if and only if } x_i \geq \delta z
\]

constitutes an SPE. Next, assume there exists an SPE that yields the player making the first proposal more than \(z\), say it yields \(z + kg\), where \(k \in \mathbb{N}, k \geq 1\). The proposal \((z + kg, 1 - z - kg)\) is acceptable to the responder only if there exists an SPE of the game that yields the initiator \(z - lg\), where

\[
1 - z - kg \geq \delta(z - lg).
\]

This condition is equivalent to

\[
\delta l \geq k - \alpha(1 + \delta),
\]

which, if (16) is satisfied, implies that

\[
l \geq k + 1.
\]

An SPE where the payoff to the initiator is \(z - lg\) is possible only if the responder can rationally reject the proposal \((z - (l - 1)g, 1 - z + (l - 1)g)\) and this is the case only if there exists an SPE that yields the initiator \(z + ng\), where

\[
1 - z + (l - 1)g \leq \delta(z + ng).
\]

The latter condition is equivalent to

\[
\delta n \geq l - 1 + \alpha(1 + \delta).
\]
which by substituting (16) and (17) yields
\[ \delta n \geq k \]

hence
\[ n \geq k + 1. \]  

(20)

By repeating the process we see that (11) is violated if there exists an SPE which yields the initiator a payoff \( z + kg \) with \( k \geq 1 \). Equation (19) shows that, if there exists an SPE that gives the initiator less than \( z \), then there also exists an SPE that gives this player more than \( z \) (recall that \( \alpha > 0 \)), and we have just shown that the latter is impossible. Hence, if (15) holds, every SPE yields the initiator \( z \). Consequently, the responder will accept a proposal if and only if it yields him at least \( 1 - z \), implying that the payoff vector is \((z, 1 - z)\) in any SPE. From this it easily follows that an SPE has to be as in (17), hence, that the SPE is unique if (15) holds.

Remark 1. Assume that \( z = 1/(1 + \delta) \) is an integer multiple of \( g \), so that the SPE from \( \Gamma(\delta) \) (i.e., the strategy combination from (17) with \( z = \bar{z} \)) is feasible in \( \Gamma(g, \delta) \). Clearly this profile constitutes an SPE of \( \Gamma(g, \delta) \) as well. The proof of Proposition 4 shows that besides \((\bar{z}, 1 - \bar{z})\) only \((\bar{z} - g, 1 - \bar{z} + g)\) is a possible SPE payoff vector. If \( \delta + g \leq 1 \), this payoff vector can indeed be sustained by an SPE. Namely, if after a responder’s rejection, players continue with the equilibrium with payoffs \((\bar{z}, 1 - \bar{z})\), the responder may rationally reject \((\bar{z}, 1 - \bar{z})\) and the proposer finds it optimal to propose \((\bar{z} - g, 1 - \bar{z} + g)\). Hence, if \( 1/(1 + \delta) \) is an integer multiple of \( g \) and \( g \) is small, then there exist two SPE payoffs.

Remark 2. Let \( z = [1/(1 + \delta)] - g \) and let \( \alpha \) be such that \( 1/(1 + \delta) = z + \alpha g \). For \( k \in \mathbb{N} \) and \( i \in \{1, 2\} \), write \( z_i(k) = z + (-1)^i kg \) and let \( \sigma_i \) be as in (17) but with \( z \) replaced with \( z_i(k) \). Then \( \sigma = (\sigma_1, \sigma_2) \) is an SPE if and only if player 1 finds it optimal to accept \( 1 - z - kg \), that is
\[ 1 - z - kg \geq \delta(z - kg) \]

and player 2 finds it optimal to reject \( 1 - (z - kg + g) \), that is
\[ 1 - z + kg - g \leq \delta(z + kg). \]

If \( \delta \) and \( g \) are such that \( \alpha = 1/2(1 + \delta)(1 - \delta) \), then \( k \) can be chosen approximately equal to \( k = 1/2(1 - \delta) \) without violating these inequalities. This shows that for some parameter values the gap between the smallest and the largest SPE payoff is indeed of the order of magnitude of the RHS of (11).
To conclude this section, we briefly study the finite horizon version of the model. Let $\Gamma(g, \delta, T)$ (resp. $\Gamma(\delta, T)$) be the game in which $\Gamma(g, \delta)$ (resp. $\Gamma(\delta)$) is truncated after $T$ rounds with both players receiving zero if no agreement is reached by then. The game $\Gamma(g, \delta, T)$ is easy to analyze if inequality (3) is satisfied, that is, if $g + \delta > 1$. Let player 2 make the final proposal. In the last round, this player obtains at least $1 - g$, so that in the second to last round he rejects any proposal that yields him less than $1 - g$. Consequently, the equilibrium payoffs of player 1 in the second to last round are bounded above by $g$. Therefore, in the third to last round player 2’s equilibrium payoff is again at least $1 - g$, and the argument can be continued to the beginning of the game: The equilibrium payoffs of the player who makes the last proposal are bounded below by $\delta(1 - g)$. Comparing this result with Proposition 2 we see that if $\delta + g > 1$, then there is a discontinuity at $T = \infty$. This discontinuity is not present in Rubinstein’s continuous specification. In that case, also the finite horizon model has a unique subgame perfect equilibrium and, as $T$ tends to infinity, the payoffs associated with this equilibrium converge to the equilibrium payoffs of the infinite horizon game.

Finally, we investigate what happens when $g$ tends to zero while $\delta$ remains fixed. Let $m(T)$ (resp. $M(T)$) be the smallest (largest) SPE payoff to the first moving player in $\Gamma(g, \delta, T)$. Then $m(1) = 1 - g$ and $M(1) = 1$. Furthermore, we have

$$m(T + 1) \geq 1 - [\delta M(T)] \geq 1 - \delta M(T) - g, \quad (21)$$

$$M(T + 1) \leq 1 - \delta m(T), \quad (22)$$

so that

$$M(T + 1) - m(T + 1) \leq g \sum_{t=0}^{T} \delta^t.$$

Hence, the gap vanishes as $g$ tends to zero. Furthermore, if $V(T)$ denotes the unique SPE payoff to the first moving player in $\Gamma(\delta, T)$, then with the bounds determined in (21), (22) we have

$$m(T) \leq V(T) \leq M(T) \quad \text{for all } T,$$

This bound is sharp: There exist equilibria in which, in the second to last round, player 1 makes the proposal $(1, 0)$ that is rejected and upon which player 2 continues with $(g, 1 - g)$. It is not optimal for player 1 to make an alternative proposal since player 2 would interpret this as a signal to continue with $(0, 1)$ in the final round (and hence, would reject it unless the proposal itself was $(0, 1)$).
so that for each $T$ the SPE payoffs of $\Gamma(g, \delta, T)$ converge to those of $\Gamma(\delta, T)$ as $g$ tends to zero. We have shown

**Proposition 5.** If $g + \delta > 1$, then any SPE of $\Gamma(g, \delta, T)$ yields the player who can make the last proposal at least $\delta(1 - g)$. If $g$ tends to zero, while $\delta$ and $T$ remain fixed, then any SPE payoff vector of $\Gamma(g, \delta, T)$ converges to the unique SPE payoff vector of $\Gamma(\delta, T)$.

5. **Concluding Remarks**

We have shown that the SPE outcomes of alternating bid bargaining games depend crucially on how small the smallest money unit, $g$, is relative to the time between offers, $\Delta$. If $\Delta$ is much smaller than $g$, we have a folk theorem or an extreme solution (Propositions 2 and 5), while when $g$ is small relative to $\Delta$ we approximate Rubinstein's solution (Propositions 3 and 5). These observations raise the question of the relative importance of the two cases. A referee conjectured that, if $\Delta$ and $g$ were such that (3) would hold, there would be reason to create a smaller money unit. We do not agree: the money system is not designed solely to make the solution of the bargaining problem easy, other economic considerations play a role. Making $g$ smaller may be prohibitively costly: The smallest coin must have a certain size, computer time (hence, cost) increases with precision. Furthermore, if one wants to empirically test the model, one is forced to work with a positive smallest money unit. (In experiments one also has to work with a finite time horizon, since infinite plays cannot be carried out.) Let us also remark that in real life bargaining, prominence levels play an important role. If bargaining is about $1$ million the prominence level (i.e., the smallest money unit used in actual bargaining) is probably not smaller than $500$. If the proposer asks for $501,297.34$ it is not excluded that the responder concludes that the proposer is insane, and we do not know how bargaining would continue in this case. (See Albers and Albers (1983) for more on the role of prominence in bargaining.) Hence, we consider the case of a fixed positive smallest money unit to be most relevant.

We are aware of the fact that, by increasing the contract space, the multiplicity of SPE payoffs from Proposition 2 may vanish. For example, the game could allow one to propose contracts that not only specify a division but also a point in time at which the division should take place. If transactions can take place at any point in time, the space of agreements is a continuum and the Rubinstein payoffs ensue. Similarly, uniqueness of SPE payoffs results if the game allows one to propose lotteries over divisions instead of just divisions. In view of transaction costs, we consider our original model to be more relevant than these more elaborate games. It is beyond the scope of the present paper to investigate whether
players would have an incentive to endogenously change the game from one in which they can propose divisions to one in which they can propose lotteries.

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