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Asymptotic normality of least squares estimators in autoregressive linear regression models.

B.B. van der Genugten
For the linear regression model \( y_t = \beta' x_t + \epsilon_t \) \((t = 1, \ldots, n)\)
asymptotic normality of the least squares estimator of \( \beta \) is proved in the case that the \( \epsilon_t \) are mutually independent with finite second moments and that \( \epsilon_t \) is independent of \( x_1', \ldots, x_t \) for each \( t \). The results obtained are applied to autoregressive models with nonstochastic, possibly unbounded regressors.
1. Introduction.

Let $x_t$ and $\varepsilon_t$ ($t \in \mathbb{N}$) be a sequence of random $k \times 1$ vectors and random variables, respectively, defined on some probability space $(\Omega, \mathcal{F}, P)$. We shall be concerned with the linear regression model

$$y_t = \beta' x_t + \varepsilon_t, \quad t \in \mathbb{N},$$

where $\beta$ is a nonstochastic $k \times 1$ parameter vector of regression coefficients. A least squares (LS) estimator $b_n$ of $\beta$, based on the first $n$ observations, is defined by

$$b_n = S_n^{-1} \sum_{t=1}^{n} x_t y_t,$$

where

$$S_n = \sum_{t=1}^{n} x_t x_t'$$

and $S_n^{-1}$ is some pseudo-inverse of $S_n$.

Theorem 2.1 below gives conditions for the $x_t$ and $\varepsilon_t$ under which $b_n$ is asymptotically normal, i.e. there exists a sequence $C_n$ of nonstochastic positive definite $k \times k$ matrices, not depending on $\beta$, such that

$$\sqrt{n}(b_n - \beta) \xrightarrow{d} N_k(0, I), \quad n \to \infty.$$

Here $N_k(0, I)$ denotes the $k$-dimensional standard normal distribution. This theorem generalizes the results for nonstochastic $x_t$ of Eicker [4]; theorem 3.1 or [5], section 2, theorem. Furthermore, the conditions are such that they apply to autoregressive models of the form

$$y_t = \beta' x_t + \varepsilon_t, \quad t \in \mathbb{N}, \quad (\alpha_0 = 1).$$

Here, $a = (a_1, \ldots, a_p)'$ and $\gamma$ are nonstochastic $p \times 1$ and $q \times 1$ vectors, respectively, of regression coefficients, $w_t$ is a nonstochastic $q \times 1$ vector, and the initial values $y_{1-p}, \ldots, y_0$ are random variables. By taking $\beta' = (\gamma', -a')$ and $x_t' = (w_t', y_{t-1}, \ldots, y_{t-p})$ we see that (1.2) is a special case of (1.1).

Theorem 2.2 below states that for this autoregressive model the conditions for the $x_t$ in theorem 2.1 are implied by certain conditions for
the $w_t$ and the $\alpha, \gamma$.

Finally, theorem 2.3 below is added to simplify the verification of the conditions of theorem 2.2. The examples given after this theorem show that these conditions are much weaker than those appearing in Anderson [1], theorem 5.5.14 or Schöpf [9]. Here the $w_t$ have to be bounded and certain matrix functions of the $w_t$ have to converge to a nonsingular matrix. Our conditions admit unbounded sequences $(w_t)$, and convergence to nontrivial values is not imposed. Finally, we remark that no conditions on moments of $\varepsilon_t$ of order higher than 2 appear.
2. Statement of the results.

In the following we suppose that the \( \varepsilon_t \) are mutually independent with 
\( \mathbb{E}(\varepsilon_t) = 0, \sigma_t^2 = \mathbb{E}(\varepsilon_t^2) < \infty, t \in \mathbb{N} \). Furthermore, we assume that they satisfy the Eicker conditions (see e.g. Eicker [4], theorem 3.1):

\[
\inf_{t \in \mathbb{N}} \sigma_t^2 > 0, \quad \sup_{t \in \mathbb{N}} \mathbb{E}(\varepsilon_t^2 I(|\varepsilon_t| \geq \delta)) \to 0, \quad \delta \to \infty.
\]

Define \( S_n = \mathbb{E}(S_n^2) \). We have:

**Theorem 2.1** (general model (1.1)). If

\[ \varepsilon_t \text{ is independent of } x_1, \ldots, x_t \text{ and } \mathbb{E}|x_t|^2 < \infty \text{ for each } t \in \mathbb{N}, \]

\[ S_n = \mathbb{E}(S_n^2), \quad \text{for } \lambda_t = 1 \text{ and } \lambda_t = \sigma_t^2, \]

then the LS-estimator \( b_n \) is asymptotically normal.

**Example 2.1** (nonstochastic \( x_t \)). For such \( x_t \) the conditions (2.2), (2.3) are fulfilled in a trivial way and (2.4) reduces to

\[ \max_{1 \leq t \leq n} x_t^t S_n^{-1} x_t \to 0, \]

or, equivalently (lemma 3.5 below),

\[ S_n^{-1} \to 0, \quad x_n^t S_n^{-1} x_n \to 0. \]

This implies that \( x_n \) is non-exponentially increasing (i.e. \( \rho^n x_n \to 0 \) for any \( \rho \) with \( |\rho| < 1 \)). However, polynomial trends are included. For some special cases see Eicker [5], p. 64-66.
For the model (1.2) we assume that the starting values $y_{1-p} \ldots, y_0$ are independent of $(\varepsilon_t)_{t \in \mathbb{N}}$ and have finite second moments. Let the $(p+q) \times 1$ vector $v_t$ be defined by

$$v_t' = (\sum_{g=0}^{p} \alpha_g w_{t-g}', \gamma_{t-1}', \ldots, \gamma_{t-p}'), t = p + 1, p + 2, \ldots$$

and the $(p+q) \times (p+q)$ matrix $Z_n$ by

$$Z_n = \sum_{t=p+1}^{n} v_t v_t' + n I_0.$$

Here $I_0$ is a $(p+q) \times (p+q)$ matrix with all elements equal to zero except the last $p$ elements of its diagonal which are equal to one. Finally, let

$$A(z) = \sum_{g=0}^{p} \alpha_g z^g, \quad z \in \mathbb{C}.$$  

We have:

**Theorem 2.2 (autoregressive model (1.2)).** If

(2.5) $A(z) \neq 0$, $|z| < 1$,

(2.6) $\|Z_n^{-1}\| = O(1/n)$,

(2.7) $v_n' Z_n^{-1} v_n \to 0$,

then the conditions (2.2) - (2.4) of theorem 2.1 are satisfied.

For many interesting sequences $(w_t)$ the verification of the conditions (2.6), (2.7) of theorem 2.2 is not easy. Therefore we add theorem 2.3 below to simplify this verification.

Suppose for some integer $s \geq q$ there exists a sequence $(q_t)_{t \geq p+1}$ of nonstochastic $s \times 1$ vectors such that

$$v_t = \phi q_t, \quad t = p + 1, p + 2, \ldots$$

for some nonstochastic $(p+q) \times s$ matrix $\phi$. Let $(D_n)_{n \geq p+1}$ be a sequence of $s \times s$ nonstochastic positive definite diagonal matrices. Consider the normed $s \times 1$ vectors
\( \tilde{q}_t(n) = D_n^{-1} q_t \), \( t = p + 1, \ldots, n \).

For some integer \( r \) such that \( q \leq r \leq \min(s, p + q) \) let \( \phi_{00} \) be the \( r \times r \) submatrix of \( \phi \) formed by the first \( r \) rows and \( r \) columns of \( \phi \), let \( D_{n0} \) be a similar \( r \times r \) submatrix of \( D_n \), and for the case that \( r < s \) let \( D_{n1} \) be the \( (s-r) \times (s-r) \) submatrix of \( D_n \) formed by the last \( s - r \) rows and \( s - r \) columns of \( D_n \). We have:

**Theorem 2.3.** Suppose

\( \det(\phi_{00}) \neq 0 \),

\( \|D_{n0}^{-1}\| = O(1/n) \),

if \( r < s \) then \( \|D_{n1}\| = O(1/\|D_{n0}\|) \),

\( \| (\sum_{p+1}^{n} \tilde{q}_t(n)\tilde{q}_t(n)^t)^{-1} \| = O(1) \),

\( \tilde{q}_n(n) \rightarrow 0 \),

then the conditions (2.6), (2.7) of theorem 2.2 are satisfied.

**Example 2.2.** We apply theorem 2.3 for the simple case \( s = (p+1)q \) and \( q_t = (w_t', \ldots, w_{t-p})' \). Then \( v_t = \phi q_t \) with

\[
\phi = \begin{bmatrix}
    a_0 & a_1 & \cdots & a_{p-1} \\
    0 & \gamma' & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \gamma'
\end{bmatrix}
\]

Take \( D_n = nI \). Then the conditions (2.11), (2.12) are satisfied if

\( \| (\sum_{p+1}^{n} q_t q_t^t)^{-1} \| = O(1/n) \), \( q_n = o(\sqrt{n}) \)

Take \( r = q \). Then (2.9), (2.10) are fulfilled. Since \( \phi_{00} = a_0 I \) the condition (2.8) holds as well. Therefore, with theorem 2.3 we see that (2.13) leads to the asymptotic normality of the LS-estimator in the autoregressive model (1.2) for all \( \gamma \) and all \( a \) satisfying (2.5). The condition (2.13) is implied by
for some $Q > 0$.

Note that (2.13) will not be satisfied if the model contains a constant term. However, a slight modification of the derivation above leads again to a condition of the type (2.13). We write $w_t' = (1, \tilde{w}_t'), \gamma' = (\gamma_0', \gamma')$. Choose $s = pq + q - p$ and $q_t' = (1, \tilde{w}_t', \ldots, \tilde{w}_{t-p})$. Then $v_t = \phi q_t$ with

\[
\phi = \begin{bmatrix}
    a_p & 0 & 0 & \cdots & 0 \\
    0 & a_{p-1} & a_{p-2} & \cdots & a_1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \gamma_0' & \cdots & 0 \\
    \gamma_0 & 0 & 0 & \cdots & \gamma_1'
\end{bmatrix}
\]

Take $r = q$ then again $\det (\phi_{00}) \neq 0$ since $\sum a_j \neq 0$ because of (2.5).

Example 2.3. Let $w_t' = (t_1, \ldots, t_q)$ for integers $a_1 > a_2 > \ldots > a_q > 0$. If $a_1 > 0$ then (2.13) of example 2.2 cannot be applied. For $q \leq r \leq \min (a_1 + 1, p + q)$ we can write

\[
\begin{pmatrix}
    w_t \\
    w_{t-1} \\
    \vdots \\
    w_{t-p}
\end{pmatrix} = \begin{bmatrix}
    \psi_0 & \star & \star & \cdots & \star \\
    \psi_1 & \star & \star & \cdots & \star \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \psi_p & \star & \star & \cdots & \star
\end{bmatrix} \begin{pmatrix}
    a_1 \\
    a_1 - 1 \\
    \vdots \\
    1
\end{pmatrix}
\]

Here $\psi_0, \ldots, \psi_p$ are $q \times r$ upper triangular matrices, not interesting matrices are denoted by stars. Choose $s = a_1 + 1$ and $q_t' = (t_1, t_2, \ldots, 1)$. Then $v_t = \phi q_t$ with
\[ \phi = \begin{bmatrix} a_0 & a_1 & \ldots & a_p \\ 0 & \gamma' & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \gamma' \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_p \end{bmatrix} \Rightarrow \phi_{00} = \begin{bmatrix} \Sigma a_j \psi_j \\ 0 \\ \gamma' \psi_1 \\ \gamma' \psi_{r-q} \end{bmatrix} \]

Take \( D_n = \text{diag}(n, n, \ldots, n) \). Then

\[ \sum_{t=1}^n q_t(n)q_t'(n) = H \]

where \( H = \{(2a_1+3-i-j)^{-1}; i, j = 1, \ldots, a_1 + 1\} \) is the so-called Hilbert matrix. Note that \( H > 0 \) and \( q_t(n) \to 0 \) for fixed \( t \). Therefore, the condition (2.11) is satisfied. Since

\[ q_n(n) = D_n^{-1} q_n = n^{-b}(1,1,\ldots,1) \]

we see that (2.12) holds. Furthermore,

\[ D_{n0} = \text{diag}(n, \ldots, n) \]

\[ D_{n1} = \text{diag}(n, \ldots, n) \text{ if } r < a_1 + 1 \]

and so the conditions (2.9), (2.10) are fulfilled also. Therefore it remains to verify the condition (2.8). In general, this cannot be done for all \( \gamma \) and all \( a \) satisfying (2.5).

1°) Consider the case of successive powers

\[ a_q = a_1 - q + 1 \]

Then all \( \psi_j \) have diagonal elements 1. From (2.5) it follows that \( \Sigma a_j \neq 0 \). Hence, if we take \( r = q \) then \( \phi_{00} = \Sigma a_j \psi_j \) is nonsingular, and consequently the condition (2.8) is fulfilled for all \( \gamma \) and all \( a \) satisfying (2.5).

2°) Consider the particular case of non-successive powers \( p = 1, q = 2, \]

\[ a_1 = 2, a_2 = 0. \] Set \( \gamma' = (\gamma_1, \gamma_2) \). The choice \( r = 3 \) leads to
\[ \phi_{00} = \begin{bmatrix} a_0 + a_1 & -2a_1 & a_1 \\ 0 & 0 & a_0 + a_1 \\ \gamma_1 & -2\gamma_1 & \gamma_1 + \gamma_2 \end{bmatrix} \Rightarrow \text{det}(\phi_{00}) = 2a_0 \gamma_1 (a_0 + a_1). \]

So the condition (2.8) is fulfilled for all \( \gamma \) with \( \gamma_1 \neq 0 \) and for all \( a \) satisfying (2.5). This makes clear that in this example it is difficult to prove the asymptotic normality for all \( \gamma \) and all \( a \) satisfying (2.5).
3. Proofs of the theorems.

We mention the following notations and conventions. For the norm $\| A \|$ of an arbitrary matrix $A$ we take $\lambda_{\max}^2 (A^t A)$. We write $A \succeq 0$ if $A$ is positive semidefinite and $A \succ 0$ if $A$ is positive definite. We write $A \succeq B$ if $A \succeq 0$, $B \succeq 0$ and $A - B \succeq 0$. Note that $A \succeq 0$ implies $\| A \| = \lambda_{\max} (A)$ and $A \succ 0$ that $\| A^{-1} \| = \lambda_{\min}^{-1} (A)$.

The random sequence $(x_n)_{n \in \mathbb{N}}$ is called $P$-bounded if for every $\varepsilon > 0$ there exists a number $M$ such that $P (|x_n| > M) < \varepsilon$ for all $n$. If $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences of random variables such that $(x_n)$ is $P$-bounded and $y_n \xrightarrow{P} 0$ then $x_n y_n \xrightarrow{P} 0$. If $E |x_n|$ is bounded then $(x_n)$ is $P$-bounded.

The proof of theorem 2.1 is based on a central limit theorem for martingale triangular arrays. Let

$$1 = \inf_{(t \in \mathbb{N})} \sigma_t^2, \quad m = \sup_{(t \in \mathbb{N})} \sigma_t^2,$$

then (2.1) implies

$$0 < 1 < m < \infty$$

**Proof of theorem 2.1.** We take

$$C_n = \text{Cov}\{S_{n-1} \sum_{t=1}^{n} x_t \epsilon_t \}.$$

Then $C_n$ does not depend on $\beta$. We have

$$C_n^{-1} (b - \beta) = C_n^{-1} S_{n-1} \sum_{t=1}^{n} x_t \epsilon_t =$$

$$= [ I + C_n^{-1} S_{n-1} \sum_{t=1}^{n} x_t \epsilon_t ] C_n^{-1} S_{n-1} \sum_{t=1}^{n} x_t \epsilon_t.$$

From (2.2) and (3.1) we get

$$1 S_{n-1} \leq C_n = S_{n-1} E \{ \sigma_t^2 x_t x_t^t \} S_{n-1} \leq m S_{n-1}$$

This implies that $\| C_n^{-1} S_{n-1} \|$ and $\| C_n^{-1} S_{n-1} \|$ are bounded. From (2.3) for $\lambda_t = 1$ it follows that $S_{n-1} S_{n-1} = I$. Therefore in the relation above (3.2) the
expression between square brackets tends to I in probability. Thus it remains to prove that

$$\mathcal{L}\{C_n^{-1} S_n^{-1} \sum_{t=1}^{n} x_t \varepsilon_t \} \rightarrow N_k(0, I).$$

Let $a \in \mathbb{R}_k$ with $|a| = 1$ and set $x_{nt} = a'C_n^{-1} S_n^{-1} x_t \varepsilon_t$.

Then it suffices to prove

$$\mathcal{L}\{x_{nt} \} \rightarrow N(0, 1).$$

(3.3) Let $\mathcal{F}_{n0} = \mathcal{F}(x_1), \mathcal{F}_{nt} = \mathcal{F}(x_1, \ldots, x_{t+1}, \varepsilon_1, \ldots, \varepsilon_t)$ for $1 \leq t \leq n$.

Then the $\mathcal{F}_{nt}$ are sub-$\sigma$-fields of $\mathcal{F}$ with $\mathcal{F}_{n, t-1} \subset \mathcal{F}_{nt}$. Furthermore, $x_{nt}$ is $\mathcal{F}_{nt}$-measurable and $\varepsilon_t$ is independent of $\mathcal{F}_{n, t-1}$, as follows from (2.2).

This gives $E\{x_{nt} | \mathcal{F}_{n, t-1} \} = 0$ a.s., and so $(x_{nt})_{1 \leq t \leq n, n \in \mathbb{N}}$ is a martingale triangular array (MTA). Furthermore,

$$E\{x_{nt}^2 | \mathcal{F}_{n, t-1} \} = a'C_n^{-1} S_n^{-1} (E x_{nt}^2 - E(x_{nt}^2 x_t')) S_n^{-1} C_n^{-1} a + 0,$$

using (2.3) for $\lambda_t = \sigma_t^2$ and the boundedness of $\|C_n^{-1} S_n^{-1}\|$.

Then (3.3) follows from a central limit theorem for MTA's (see e.g. Mcleish [8], Corollary (3.8) or Gaenssler [6], theorem 2), provided that we can verify the conditional Lindeberg-Feller condition

$$\sum_{t=1}^{n} E\{x_{nt}^2 I(|x_{nt}| \geq \varepsilon) | \mathcal{F}_{n, t-1} \} \rightarrow 0, \text{ for every } \varepsilon > 0.$$

Set

$$r_{nt} = x_t'^{-1} S_n^{-1} x_t, \quad r_n = \max_{1 \leq t \leq n} r_{nt}^2, \quad Q_t(\delta) = E\{\varepsilon_t^2 I(|\varepsilon_t| \geq \delta)\}, \quad R_n(\delta) = \max_{1 \leq t \leq n} Q_t(\delta)$$

Then (3.2) gives

$$x_{nt}^2 \leq x_t'^{-1} C_n^{-1} S_n^{-1} x_t \varepsilon_t^2 \leq (r_{nt}/m) \varepsilon_t^2.$$

Since $r_{nt}$ is $\mathcal{F}_{n, t-1}$-measurable and $\varepsilon_t$ is independent of $\mathcal{F}_{n, t-1}$ it follows that
From (2.1) we get that \( R_n(d_n) \to 0 \) for any nonstochastic sequence \((d_n)\) with \( d_n \to \infty \), and from (2.4) that \( r_n \to 0 \). By considering the a.s. convergence of subsequences it follows that \( R_n(\varepsilon m_n) \to 0 \). Furthermore, \( E(\sum r_t) = k \) and \( r_{nt} \to 0 \) imply that \( \sum r_{nt} \) is \( P \)-bounded. Together this shows that the right-hand side of (3.5) \( \frac{1}{n} \) tends to 0 in probability, completing the proof.

The proof of theorem 2 is rather tedious. We make some preliminary remarks and formulate some intermediate results as lemma's. Let

\[
(3.6) \quad s_n = E \left( \varepsilon^2 \right) \to 0, \quad n \to \infty.
\]

From (2.1) it follows that \( s_n \geq n \) and that for every \( \varepsilon > 0 \)

\[
(3.6) \quad s_n^{-1} \sum_{t=1}^{n} E(\varepsilon_t^2 I(\varepsilon_t > \varepsilon s_n)) \leq s_n^{-1} \sup_{t \in \mathbb{N}} E(\varepsilon_t^2 I(\varepsilon_t > \varepsilon \sqrt{n})) + 0.
\]

So the Eicker conditions (2.1) imply the well-known Lindeberg-Feller (LF) condition for the sequence \((\varepsilon_t)_{t \in \mathbb{N}}\). This leads to the corollary of lemma 3.1 below. The lemma itself is a slight generalization of a theorem of Raikov (see Gnedenko [7], §28, theorem 4. Its proof is kept short and added for the sake of completeness.

**Lemma 3.1.** Let \((x_{nj})_{1 \leq j \leq k, n \in \mathbb{N}}\) be a triangular array of random variables such that \( x_{nj} \) are mutually independent for each fixed \( n \). Suppose \( E(x_{nj}) = 0, \quad \sigma_{nj}^2 = E(x_{nj}^2) < \infty \) for all \( n, j \).

If the LF condition

\[
\sum_{j=1}^{k_n} E(x_{nj}^2 I(\varepsilon_{nj}^2 > \varepsilon)) \to 0, \text{ for every } \varepsilon > 0,
\]

then \( \sum_{j=1}^{k_n} E(x_{nj}^2 I(\varepsilon_{nj}^2 > \varepsilon)) \to 0, \text{ for every } \varepsilon > 0 \).
holds and if \( \sum_{j=1}^{k} \sigma^2_{nj} \) is bounded in \( n \), then

\[
\max_{1 \leq j \leq k_n} \left| x_{nj} \right| \leq P(0)
\]

\[
\max_{1 \leq j \leq k_n} \sigma^2_{nj} \leq P(0)
\]

and for any bounded nonstochastic array \((\lambda_{nj})\) we have

\[
\sum_{j=1}^{k_n} \lambda_{nj}(x^2_{nj} - \sigma^2_{nj}) \leq 0.
\]

**Proof.** The first relation follows from

\[
P\{\max_{1 \leq j \leq k_n} \left| x_{nj} \right| \geq \epsilon \} \leq \sum_{1}^{k_n} P\{\left| x_{nj} \right| \geq \epsilon \} \leq \epsilon^{-2} \sum_{1}^{k_n} E\{x^2_{nj} I(\left| x_{nj} \right| \geq \epsilon)\}
\]

and the second one from

\[
\sigma^2_{nj} \leq \epsilon^2 + \max_{1 \leq j \leq k_n} E\{x^2_{nj} I(\left| x_{nj} \right| \geq \epsilon^2)\} \leq \epsilon^2 + \sum_{1}^{k_n} E\{x^2_{nj} I(\left| x_{nj} \right| \geq \epsilon)\}.
\]

The last relation will be proved first for \( \lambda_{nj} = 1 \).

Let \( \varphi_{nj}(u) = E\{\exp(\alpha u^2_{nj})\} \) then it suffices to prove that

\[
\sum_{1}^{k_n} (\log \varphi_{nj}(u) - iu \sigma^2_{nj}) \rightarrow 0.
\]

Since \( |\varphi_{nj}(u) - 1| \leq |u| \sigma^2_{nj} \), we have

\[
\max_{1 \leq j \leq k_n} |\varphi_{nj}(u) - 1| \rightarrow 0,
\]

\[
\sum_{1}^{k_n} |\varphi_{nj}(u) - 1|^2 \rightarrow 0
\]

Since \( \log(1+z) - z \leq |z|^2 \), \( |z| \leq 1/2 \), this gives

\[
\sum_{1}^{k_n} |\log \varphi_{nj}(u) + 1 - \varphi_{nj}(u)| \rightarrow 0
\]

From

\[
|\varphi_{nj}(u) - 1 - iu \sigma^2_{nj}| \leq \frac{1}{2} |u|^2 \epsilon \sigma^2_{nj} + 2|u|E\{x^2_{nj} I(\left| x_{nj} \right| \geq \epsilon)\}
\]
we obtain

$$\sum_{j=1}^{k_n} |\phi_{n_j}(u)-1-\nu \sigma_{n_j}^2| \rightarrow 0.$$ 
Together this gives

$$\sum_{j=1}^{k_n} (\log \phi_{n_j}(u) - iu \nu \sigma_{n_j}^2) \rightarrow 0,$$
proving the result for $\lambda_{n_j} = 1$.

For non-negative $\lambda_{n_j}$, the result follows from this. Replace only $x_{n_j}$ by $\lambda_{n_j} x_{n_j}$.

For arbitrary $\lambda_{n_j}$ the result follows by splitting up the sum for positive and negative $\lambda_{n_j}$.

**Corollary.** By taking $x_{n_j} = \epsilon_j / \sqrt{s_n}$ we see that the LF-condition for $(x_{n_j})$ follows from (3.6). Hence,

(3.7) \[ s_n^{-1} \max_{1 \leq t \leq n} |\epsilon_t| \rightarrow 0 \]

(3.8) \[ m_n / s_n \rightarrow 0 \]

and for bounded nonstochastic $(\lambda_{nt})_{1 \leq t \leq n}$, $n \in \mathbb{N}$:

(3.9) \[ s_n^{-1} \sum_{t=1}^{n} \lambda_{nt} (\epsilon_t^2 - \sigma_t^2) \rightarrow 0 \]

**Lemma 3.2.** For nonstochastic $\phi_h$ and $\psi_h$ ($h = 0, 1, \ldots$) let

$$y_t = \sum_{h=0}^{\infty} \phi_h \epsilon_{t-h}, z_t = \sum_{h=0}^{\infty} \psi_h \epsilon_{t-h}, t \in \mathbb{Z},$$

where $\epsilon_t = 0$ if $t < -p$, $\sigma_t^2 = E[\epsilon_t^2] < \infty$ if $1 - p \leq t \leq 0$.

If $E \phi_h^2, E \psi_h^2$ have convergence radii larger than 1, then for any bounded nonstochastic array $(\lambda_{nt})_{1 \leq t \leq n}$, $n \in \mathbb{N}$ we have

(3.10) \[ s_n^{-1} \sum_{t=1}^{n} \lambda_{nt} (y_{t-i} z_{t-j} - E[\sum_{t=1}^{n} \lambda_{nt} y_{t-i} z_{t-j}]) \rightarrow 0, \quad i, j \in \mathbb{Z} \]

**Proof.** Note that $\phi_h = o(\rho^h)$, $\psi_h = o(\rho^h)$ for some $\rho$ with $0 < \rho < 1$.

Set $\phi_r = \psi_r = 0$ if $r < 0$. Then
\begin{equation}
\sum_{t=1}^{n} \lambda t^{t-i} t^{-j} = \sum_{t=1}^{n} \lambda t^{t-i} \sum_{s} \psi_{t-s}^{t-j} = \sum_{r,s} \sum_{nrs} \epsilon_{r,s, t=i-j-s}
\end{equation}

where

\[ a_{nrs} = \lambda \sum_{t} t^{-i-r-s} \]

Note that \( a_{nrs} = 0 \) if \( r > n - i \) or \( s > n - j \). For any \( a < b \) we have

\[ \left| \sum_{t=1}^{n} \lambda t^{a} \psi_{t-b} \right| \leq c'. \sum_{t=a}^{b} \psi_{t-b} \leq c''. \sum_{t=a}^{b} \rho_{a-b} \leq c'''. \]

for some constants \( c', c'', c''' \). Therefore there exists a constant \( c \) not depending on \( n, r, s \) such that

\[ |a_{nrs}| \leq c \rho |r-s|. \]

In particular, \( a_{nrs} \) is bounded in \( n, r, s \). We split up

\[ \sum_{r,s} \sum_{nrs} \epsilon_{r,s, t=i-j-s} = \sum_{k=1}^{5} A_k(n) \]

according to the ranges \((r,s \leq 0), (r \geq 1, s \leq 0), (r \leq 0, s \geq 1), (r,s \geq 1 \text{ and } r \neq s), (r, s \geq 1 \text{ and } r = s)\), respectively.

The number of terms in \( A_1(n) \) is finite and the terms themselves are bounded in \( n \). Furthermore,

\[ E|A_2(n)| \leq \sum_{s=0}^{n-1} \sum_{r=1}^{n-1} |a_{nrs}| E|\epsilon_{r,s,t}| \leq c'. \sum_{s=0}^{n-1} \sum_{r=1}^{n-1} \rho_{r-s} \sigma_{r,s} \leq c'' \sqrt{m_{n-1}}, \]

for some constants \( c', c'' \). A similar relation holds for \( A_3(n) \). Hence, with (3.8) we get

\begin{equation}
(3.12) \quad s^{-1} E|A_k(n)| \to 0 \quad , k = 1, 2, 3.
\end{equation}

Since \( E[A_4(n)] = 0 \) and

\[ V[A_4(n)] = \sum_{r \neq s} a_{nrs} (a_{nrs} + a_{nsr}) \sigma_{r,s}^{2} \sigma_{r,s}^{2} \leq 2c^{2} \sum_{r \neq s} \rho_{r-s} \sigma_{r,s}^{2} \sigma_{r,s}^{2} = \]

\begin{align*}
\sum_{r \neq s} a_{nrs} (a_{nrs} + a_{nsr}) & \leq 2c^{2} \sum_{r \neq s} \rho_{r-s} \sigma_{r,s}^{2} \sigma_{r,s}^{2} \\
& = 2c^{2} \sum_{r \neq s} \rho_{r-s} \sigma_{r,s}^{2} \sigma_{r,s}^{2} \\
& = 2c^{2} \sum_{r \neq s} \rho_{r-s} \sigma_{r,s}^{2} \sigma_{r,s}^{2}
\end{align*}
for some $c'$, we get with (3.8) that

\[(3.13) \quad s_n^{-1}(A_4(n) - \mathbb{E}[A_4(n)]) \to 0.\]

Finally, with (3.9) we get

\[(3.14) \quad s_n^{-1}(A_5(n) - \mathbb{E}[A_5(n)]) = s_n^{-1} \max_{i,j} \sum_{t=1}^{n} \mathbf{a}_{ntt} (\epsilon_t^2 - \sigma_t^2) \to 0.\]

Combining (3.11) - (3.14) gives (3.10).

Lemma 3.3. Let $(x_t)_{t \in \mathbb{N}}$ be a sequence of $k \times 1$ vectors and $(\phi_h)_{h \in \mathbb{Z}}$ a sequence of scalars. Let

\[y_{nj} = \sum_{t=1}^{n} x_t \phi_{t-j}, \quad n \in \mathbb{N}, \quad j \in \mathbb{Z},\]

If $\sum_{h} |\phi_h| < \infty$ then

\[\sum_{j} y_{nj} y_{nj}' \leq c \sum_{t=1}^{n} x_t x_t', \quad n \in \mathbb{N},\]

for some constant $c$ not depending on $n$.

Proof. Set

\[\varphi(\lambda) = \sum_{j} \phi_{j} e^{i\lambda j}, \quad X_n(\lambda) = \sum_{j=1}^{n} x_t e^{i\lambda t}, \quad Y_n(\lambda) = \sum_{j} y_{nj} e^{i\lambda j},\]

then $Y_n(\lambda) = X_n(\lambda) \varphi(-\lambda)$. With

\[c = \sup_{|\lambda| \leq \pi} |\varphi(-\lambda)|^2\]

this gives

\[\sum_{j} y_{nj} y_{nj}' = \frac{1}{2\pi} \int_{-\pi}^{\pi} y_{nj} y_{nj}'(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(-\lambda)|^2 X_n(\lambda) X_n(\lambda) d\lambda \leq c \frac{1}{2\pi} \int_{-\pi}^{\pi} X_n(\lambda) X_n(\lambda) d\lambda = c \sum_{t=1}^{n} x_t x_t'.\]
Lemma 3.4. Let \((x_t)_{t \in \mathbb{Z}}\) be a sequence of \(k \times 1\) vectors with \(x_t = 0\) if \(t \leq -p\) for some \(p \geq 0\), and let \((\varphi_h)_{h \geq 0}\) be a sequence of scalars. Let
\[
y_t = \sum_{h=0}^{\infty} \varphi_h x_{t-h}, \quad t \in \mathbb{Z}.
\]
If \(\varphi(z) = \sum \varphi_h z^h\) has convergence radius larger than 1 and \(\varphi(z) \neq 0\) if \(|z| < 1\), then there exists constants \(c_1, c_2 > 0\), only depending on \((\varphi_h)\), such that
\[
\sum_{t=1-p}^{n} \sum_{t'=1-p}^{n} c_1 x_t x_{t'} t' t \leq \sum_{t=1-p}^{n} \sum_{t'=1-p}^{n} c_2 y_t y_{t'} t' t, \quad n \in \mathbb{N}.
\]

Proof. Let \(A(z) = 1/\varphi(z) = \sum \alpha_n z^h\), then \(\sum \alpha_m \varphi_{h-m} = \delta_{0h}\) for \(h = 0, 1, \ldots\).

Note that
\[
\sum_{h=0}^{\infty} |\varphi_h| < \infty, \quad \sum_{h=0}^{\infty} |\alpha_h| < \infty.
\]

Let \(L_n\) be the \((n+p) \times (n+p)\) lag matrix with unit elements just below the diagonal and zero elements elsewhere. Then \(L_n^{m} [x_{1-p}, \ldots, x_n]^\prime = [x_{1-p-m}, \ldots, x_{n-m}]^\prime\) with the convention \(L_n^0 = I\). Note that \(L_n^m = 0\) for \(m > n + p\). Introduce
\[
\phi_n = \sum_{m=0}^{n+p-1} \varphi_m L_n^m = \sum_{m=0}^{\infty} \varphi_m L_n^m
\]
then
\[
\phi_n^{-1} = \sum_{h=0}^{\infty} \alpha_h L_n^h = \sum_{h=0}^{\infty} \alpha_h L_n^h.
\]

Set \(x_n' = [x_{1-p}, \ldots, x_n]\), \(y_n' = [y_{1-p}, \ldots, y_n]\) then \(y_n = \phi_n x_n\).

With \(\|L_n\| < 1\) this gives
\[
\sum_{t=1-p}^{n} y_t y_t' = \sum_{t=1-p}^{n} y_n y_n' = \sum_{t=1-p}^{n} x_t x_t'\|L_n\|^2 x_n x_n' < \sum_{t=1-p}^{n} y_t y_t' \sum_{t=1-p}^{n} x_t x_t'\|\phi_n^{-1}\|^2. \quad \sum_{t=1-p}^{n} x_t x_t'\|L_n\|^2 x_n x_n' < \sum_{t=1-p}^{n} y_t y_t' \sum_{t=1-p}^{n} x_t x_t'\|\phi_n^{-1}\|^2.
\]

and, in the same way,
\[
\sum_{t=1-p}^{n} x_t x_t' \leq \sum_{t=1-p}^{n} y_t y_t' \sum_{t=1-p}^{n} x_t x_t'\|\phi_n^{-1}\|^2. \quad \sum_{t=1-p}^{n} x_t x_t' \leq \sum_{t=1-p}^{n} y_t y_t' \sum_{t=1-p}^{n} x_t x_t'\|\phi_n^{-1}\|^2.
\]

This completes the proof.
Lemma 3.5. Let $(x_t)_{t \in \mathbb{N}}$ be a nonstochastic sequence of $k \times 1$-vectors.

a) Let $S_n := \sum_{t=1}^{n} x_t x'_t$ with $S_n > 0$ for some $n$. We have

$$\max_{1 \leq t \leq n} x'_t S_n^{-1} x_t = S_n^{-1} x_n x'_n = 0,$$

b) Let $Z_n$ be a sequence of nonstochastic $k \times k$-matrices with $0 \leq Z_1 \leq Z_2 \leq \ldots$ and $Z_n > 0$ for some $n$. We have

$$Z_n^{-1} x_n x'_n = 0 \Rightarrow \max_{1 \leq t \leq n} x'_t Z_n^{-1} x_t = 0.$$

Proof. a) Take some fixed $c \in \mathbb{R}^k$. Let $N$ be such that $S_N > 0$. Then $c = \sum_{t=1}^{N} x_t x'_t$ for some $a_1, \ldots, a_N$ not depending on $n$. Hence, for $n \geq N$:

$$c'S^{-1}c \leq \sum_{t=1}^{N} \sum_{s=1}^{N} a_t a_s \|x'_t S^{-1} x_s\| \leq \left(\sum_{t=1}^{N} |a_t|\right)^2 \max_{1 \leq t \leq n} x'_t S^{-1} x_t \to 0.$$

This holds for any $c$, implying $S_n^{-1} \to 0$.

b) For given $n$ let $T$ be the largest index for which the maximum is attained. If $(T_n)$ is bounded then $(x_{T_n})$ is bounded. So,

$$x_{T_n} Z_n^{-1} x_{T_n} \leq \|x_{T_n}\|^2 \|Z_n^{-1}\| \to 0.$$

If $(T_n)$ is unbounded then $T_n \to \infty$. Since $T_n < n$ we have $Z_n^{-1} \geq Z_n^{-1}$. So,

$$x_{T_n} Z_n^{-1} x_{T_n} \leq x_{T_n} Z_n^{-1} x_{T_n} \to 0.$$

Proof of theorem 2.2.

The condition (2.2) is fulfilled. So we have only to prove that (2.5) - (2.7) imply (2.3) - (2.4).

From (2.6) and $s_n \leq n_l$ we get

$$\|Z_n^{-1}\| = O(s_n^{-1}).$$

In particular $Z_n^{-1} \to 0$. With (2.7) and lemma 3.5,b this leads to

$$\max_{1 \leq t \leq n} |z_n^{-1} v_t | \to 0.$$

We derive a suitable expression for $x_t$. Set $y_t = 0$ if $t \leq -p$,

$$w_t = 0 \text{ if } t \leq 0.$$

Define $\epsilon_t$ for $t \leq 0$ by the relation (1.2). Then this relation holds for all $t \in \mathbb{Z}$. In particular $\epsilon_t = 0$ if $t \leq -p$. Let $\varphi(z) = 1/A(z) = \sum_{m=0}^{\infty} z_m$. 
Then \( \psi(z) \) has convergence radius larger than 1 and \( \psi(z) \neq 0 \) if \( |z| < 1 \). In particular \( \sum \psi_m | < \infty \). The solution of (1.2) can be written as

\[
y_t = \sum_{h=0}^{\infty} \psi_h (y_{t-h} + e_{t-h}), \quad t \in \mathbb{Z}.
\]

Furthermore, since \( \sum \psi \alpha_{k-j} = \delta_{0k} \), we have

\[
\begin{align*}
W_t &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_j \alpha_{k-j} w_{t-j} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_k \sum_{j=0}^{\infty} \alpha_{k-j} w_{t-j} = \sum_{k=0}^{\infty} \psi_k \sum_{j=0}^{\infty} \alpha_{k-j} w_{t-j-k}
\end{align*}
\]

Therefore,

\[
x_t' = (w'_t, y'_{t-1}, \ldots, y'_{t-p}) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \psi_k \alpha_{k-j} w_{t-j} + (0, \varepsilon_{t-k-1}, \ldots, \varepsilon_{t-k-p}).
\]

This gives

\[
(3.17) \quad x_t = \mu_t + \xi_t = \sum_{k=0}^{\infty} \psi_k (v_{t-k} + \eta_{t-k}), \quad t \in \mathbb{Z},
\]

where \( \eta_t' = (0, \varepsilon_{t-1}, \ldots, \varepsilon_{t-p}) \) and

\[
\begin{align*}
\mu_t &= \sum_{k=0}^{\infty} \psi_k \alpha_{k-j} w_{t-j}, \quad \xi_t = \sum_{k=0}^{\infty} \psi_k \eta_{t-k}, \quad t \in \mathbb{Z}.
\end{align*}
\]

With (3.17) we can derive a lower bound for \( \bar{S}_n \). From lemma 3.4 it follows that

\[
\sum_{t=1}^{n} x_t x'_t - \sum_{t=1}^{n} (v_t + \eta_t) (v_t + \eta_t)' > c_2 \sum_{t=1}^{n} (v_t + \eta_t) (v_t + \eta_t)',
\]

Since

\[
\sum_{t=1}^{n} E(\eta_t \eta_t') = \text{diag} (0, \sum_{t=1}^{n} \sigma_{t-1}^2, \ldots, \sum_{t=1}^{n} \sigma_{t-p}^2) \geq (n-p)1.1_0
\]

we get, by taking expectations,

\[
0 \leq \sum_{t=1}^{n} E(x_t x'_t) + \bar{S}_n - c'.Z_n
\]

for some \( c' > 0 \). Since \( Z_n^{-1} \to 0 \) this implies

\[
(3.18) \quad \bar{S}_n \geq cZ_n
\]
for some $c > 0$.

With these preparations we can prove (2.3), (2.4). First we prove (2.4). From (3.17) and (3.18) we get

$$
(c \max_{x_t} x_t^{-1} S_{n-1}^{-1} x_t) - (\max_{x_t} x_t^{-1} S_{n-1}^{-1} x_t) \leq \max_{z_t} z_t^{-1} = \max_{z_t} z_t^{-1} x_t
$$

$$
\leq \max_{z_t} z_t^{-1} + \max_{z_t} z_t^{-1} \varepsilon_t
$$

$$
\leq \sum_{k=0}^{\infty} \left( \max_{z_t} z_t^{-1} + \max_{z_t} z_t^{-1} \right)
$$

According to (3.16) the first term on the right-hand side tends to 0, and with (3.15) and (3.7) we see that the second term on the right-hand side tends to 0 in probability. Together this proves (2.4).

Finally we prove (2.3). From (3.15) and (3.17) we get

$$
\| S_{n-1}^{-1} (\Sigma_{\lambda t} x_{t,i} x_{t,j} - E(\Sigma_{\lambda t} x_{t,i} x_{t,j})) S_{n-1}^{-1} \| \leq \| S_{n-1}^{-1} \| Z_t^{-1} \| Z_{n-1}^{-1} \| (\Sigma_{\lambda t} x_{t,i} x_{t,j} - E(\Sigma_{\lambda t} x_{t,i} x_{t,j})) Z_t^{-1} \| + \| 2Z_t^{-1} \| E(\Sigma_{\lambda t} x_{t,i} x_{t,j}) Z_t^{-1} \| + \| \| S_{n-1}^{-1} \| \| S_{n-1}^{-1} \| (\Sigma_{\lambda t} x_{t,i} x_{t,j} - E(\Sigma_{\lambda t} x_{t,i} x_{t,j})) Z_t^{-1} \|
$$

for some constants $c', c''$. So it suffices to prove that

$$
\| S_{n-1}^{-1} (\Sigma_{\lambda t} x_{t,i} x_{t,j} - E(\Sigma_{\lambda t} x_{t,i} x_{t,j})) \| \leq 0, \text{ } i, j \in \mathbb{Z}
$$

$$
\| S_{n-1}^{-1} \| Z_{n-1}^{-1} (\Sigma_{\lambda t} x_{t,i} x_{t,j} - E(\Sigma_{\lambda t} x_{t,i} x_{t,j})) \| \leq 0, \text{ } i \in \mathbb{Z},
$$

where

$$
\xi_t = \sum_{h=0}^{\infty} \varphi_t e_t \xi_t
$$

The relation (3.19) follows from lemma 3.2 and the fact that $(\lambda_t)$ is bounded. For the proof of (3.20) we write
(3.21) \[ \sum_{i=1}^{\infty} \lambda_t t^i t^{-i} = \sum_{i=1}^{\infty} \lambda_t t^i (\sum_{h=1}^{i} \phi_t t^{-h}) = \sum_{j \geq 0} a_{nj} \epsilon_{j-i} \]

where

\[ a_{nj} = \sum_{i=1}^{\infty} \lambda_t t^i t^{-j} \]

From (3.16) we get for some \( c > 0 \) that

\[ |Z_n^{-\frac{1}{2}} a_{nj}^k| \leq c |\phi_k| \max_{h=0}^{\infty} |Z_n t^{-h} u_t| \leq c \max_{h=0}^{\infty} |Z_n^{-\frac{1}{2}} v_t| \]

and so

\[ (3.22) \quad s_n^{-\frac{1}{2}} \sum_{j=1}^{\infty} a_{nj} \epsilon_{j-i} \leq s_n^{-\frac{1}{2}} \sum_{j} \sup_{j} |Z_n a_{nj}| \cdot \mathbb{E}\{ \sum_{j<i} |\epsilon_j| \} \to 0 \]

Since \( (\lambda_t) \) is bounded and \( Z_n^{-1} \to 0 \) we get from lemma 3.3 and lemma 3.4 for some \( c, c' > 0 \) that

\[ \|Z_n^{-\frac{1}{2}} (\sum_{j=1}^{\infty} a_{nj}^j) Z_n^{-\frac{1}{2}} \| \leq c \|Z_n^{-\frac{1}{2}} (\sum_{j=1}^{\infty} \lambda_t^j t^j t^{-j}) Z_n^{-\frac{1}{2}} \| \]

\[ \leq c \|Z_n^{-\frac{1}{2}} Z_n (\sum_{j=1}^{\infty} v_t v^j t^{-j}) Z_n^{-1} \| \leq c, \]

where the last inequality follows from

\[ \sum_{j=1}^{\infty} v_t v^j t^{-j} = Z_n. \]

Since

\[ \text{Cov}\{ \sum_{j>i} a_{nj} \epsilon_{j-i} \} = \sum_{j>i} \sigma_{j-i}^2 a_{nj} a_{nj'} < \sum_{j>i} a_{nj} a_{nj'} \]

this leads with (3.8) to

\[ \text{Cov}\{ s_n^{-\frac{1}{2}} \sum_{j=1}^{\infty} a_{nj} \epsilon_{j-i} \} \leq cm / s \to 0. \]

This gives

\[ (3.23) \quad s_n^{-\frac{1}{2}} \sum_{j>i} a_{nj} \epsilon_{j-i} \to 0. \]

Combining (3.21) - (3.23) yields (3.20), which completes the proof.

Proof of theorem 2.3

We restrict ourselves to the general case \( r < \min (s, p+q) \) (The proof for other special cases follows along similar lines). We write
At first we derive some properties of $P_n$ and $Q_n$. We have

\[ P_n = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \]

and so, with (2.8), (2.9) it follows that

\[ (3.24) \quad \|P_n^{-1}\| = O(1/n). \]

Furthermore,

\[ Q_n = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \]

and with (2.8) this gives

\[ \|P_n^{-1}Q_n\| < \| (\phi_{00} \phi_{01}^{-1})^{-1} \| = \| (\phi_{00}^{-1} (\phi_{10}^{-1} + D_{n0}^{-1} D_{n1}^{-1}) \| \]

Using (2.10) this gives

\[ (3.25) \quad \|P_n^{-1}Q_n\| = O(1). \]

Next we derive a lower bound for $Z_n^{-1}$. We have for some constants $c, c' > 0$:

\[ Z_n = \sum_{t=0}^{n} v_t v_t' + nI_0 \geq c' \phi D_n \phi' + nI_0 \]

or

\[ (3.26) \quad Z_n \geq c' A_n \]

where
\[ A_n = \begin{bmatrix} p_n & q_n \\ q'_n & R_n + nI \end{bmatrix} \]

Note that
\[ A^{-1}_n = \begin{bmatrix} p_n^{-1} + T_n^{-1}T' & -T_n^{-1} \\ -A_n^{-1} & A_n^{-1} \end{bmatrix} \]

where \( T_n = p_n^{-1}q_n \), \( \Delta_n = nI + R_n - Q'(p_n^{-1}q_n) \). Since \( \phi D \phi' \geq 0 \) we have \( R_n - Q'(p_n^{-1}q_n) \geq 0 \) and so \( \| \Delta_n^{-1} \| = O(1/n) \). Then from (3.24), (3.25) it follows that \( \| A_n^{-1} \| = O(1/n) \) and with (3.26) this proves (2.6).

Finally, with (3.26) we get
\[
\begin{align*}
\langle v', z_n \rangle &= \langle q'(n), D \phi' z_n \rangle - \langle \phi D q'(n) \rangle \\
&\leq \frac{1}{c} |q'(n)|^2 \| D \phi' \|_n \| \phi D q' \|_n \\
&\leq \frac{1}{c} |q'(n)|^2 \text{ tr}(A_n^{-1} \phi D \phi') .
\end{align*}
\]

Therefore (2.7) follows from (2.12) if we can prove that \( \text{tr}(A_n^{-1} \phi D \phi') = O(1) \).

However, this follows from
\[
\begin{align*}
\text{tr}(A_n^{-1} \phi D \phi') &= \text{tr}( (p_n^{-1} + T_n^{-1}T')p_n - T_n^{-1}Q' ) + \\
&\quad + \text{tr}( -A_n^{-1}T'Q_n + A_n^{-1}R_n ) = \text{tr}( I + A_n^{-1}(R_n - Q'(p_n^{-1}q_n)) ) = \\
&= p + \text{tr}( A_n^{-1}(R_n - Q'(p_n^{-1}q_n)) A_n^{-1} ) \leq 2p + q - r ,
\end{align*}
\]

completing the proof.
4. **Weak consistency.**

We make some final remarks about weak consistency of the LS-estimator in the autoregressive model (1.2).

The conditions of theorem 2.2 imply $b_n \xrightarrow{P} \beta$. This follows from a short inspection of the proofs of theorems 2.1 and 2.2. The condition (2.6) implies $Z_n^{-1} \rightarrow 0$ and therefore also $C_n \rightarrow 0$. Then the convergence in distribution of $C_n^{-1}(b_n - \beta)$ gives $b_n \xrightarrow{P} \beta$.

In fact, for weak consistency the conditions of theorem 2.2 can be weakened considerably because we need only to prove that $S_n^{-1}S_n \xrightarrow{P} I$ and $C_n \rightarrow 0$.

This can be done under the stability assumption (2.5), the assumption that the $\varepsilon_t$ satisfy the LF condition instead of the Eicker conditions (see (3.6)), and the assumption $\|Z_n^{-1}\| = o(s_n^{-1})$, where in the definition of $Z$ the term $nI_0$ is replaced by $s_n I_0$. The behaviour of $Z_n$ can be investigated as in theorem 2.3. For other conditions about weak consistency see e.g. Drygas [2], remark 4.5, and Eicker [3], Satz 1.
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15. H. Peer
Economische groei en uitputtelijke grondstoffen
Building and analyzing an econometric model with the use of a hybrid computer; part I.

System properties of the interplay model

Decentralisatie en regionaal sociaal-economisch beleid

Een monetaristisch model voor de Nederlandse economie

Morfologie van de "Wolstad". Over het ontstaan en de ontwikkeling van de ruimtelijke geleding en structuur van Tilburg.

Over de (on)mogelijkheden van het model van Knoester.

De betekenis van het monetaire beleid voor de Nederlandse economie, presentatie van een analyse aan de hand van een eenvoudig model

The use of non-linear transformation in ARIMA-Models when the data are non-Gaussian distributed