Global games and equilibrium selection
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Abstract

A global game is an incomplete information game where the actual payoff structure is determined by a random draw from a given class of games and where each player makes a noisy observation of the selected game. For $2 \times 2$ games, it is shown that equilibrium play in a global game with vanishing noise forces the players to conform to Harsanyi and Selten's risk dominance criterion. When the uncertainty is one-dimensional, the result may be obtained by repeated elimination of dominated strategies in the global game.
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1 Introduction

A fundamental assumption in most studies of game theory is that the rules of the game, including its payoff structure and the rationality of the players are common knowledge. This assumption, or the weaker assumption that each player is completely informed about all payoffs in the matrix, seems to imply that players have unbounded capabilities of discrimination. Real life players certainly do not have such powers (see e.g. Millar (1957)) and, hence, one would like to know what influence a slight weakening of the common knowledge assumption has on the game theoretic predictions.

In this paper we analyse the consequences of relaxing the assumption that the payoffs in a normal form game are common knowledge. The context in which we picture our players is one in which it is common knowledge that a game with a certain structure has to be played, however, the exact payoffs of the game to be played are not known. Players have a prior on the class of possible games and we assume that, when the actual game is presented to them, each of them makes a slight observation error and, hence, may think he is playing a slightly different game. The observation errors force players to analyse all games in the class simultaneously and we show that not every equilibrium of a given game from the class can be part of an equilibrium rule of how to play the entire class of games.

Specifically, we will show that this approach makes it possible to derive unique solutions for almost all $2 \times 2$ games having 2 strict Nash equilibria. In particular we show that, if (in a certain sense) the initial uncertainty about which game has to be played is sufficiently large, players will, when the observation noise vanishes, coordinate on the risk dominant equilibrium (Harsanyi and Selten (1988)) for every game that they

\[1\]The only games for which we do not get uniqueness are those that are best reply equivalent to completely symmetric games. Hence, we do not say anything about how to play a pure coordination game. With Schelling (1960) we agree that, in this case, the usual game theoretic description is not rich enough to say anything meaningful.
actually play.\textsuperscript{2} To our knowledge, the theory presented in this paper is the first purely noncooperative theory that is capable of eliminating strict equilibria.\textsuperscript{3}

As an illustration of our general approach and to provide some basic intuition for the main results, let us consider two examples. First consider the coordination game from Fig. 1a.

[insert Figure 1 about here]

The authors are confident that, when asked to play the game $G_1$ from Table 1 in a purely noncooperative fashion, and without receiving any outside guidance about how to play, each reader would choose $\beta$. Common sense dictates that one should play the game in this way, but up to now there is no formal, purely noncooperative\textsuperscript{4} theory that singles out $(\beta_1, \beta_2)$ as the unique rational solution of $G_1$. Indeed, also the strategy pair $(\alpha_1, \alpha_2)$

\textsuperscript{2}Technically, we perturb the game, investigate limits of sequences of equilibria of perturbed games, and show that only the risk dominant equilibrium of the base game can be approximated.

\textsuperscript{3}The point of view that strict equilibria possess all nice properties that one can hope for is, among others taken in Fudenberg et al. (1988) and Kohlberg and Mertens (1986). In Footnote 3 of the latter paper the intuition for this point of view is expressed as follows: Whatever “incentive to deviate” may mean, once the players expect $\alpha$ to be played (in the game $G_1$ from Figure 1) neither player will have an incentive to unilaterally deviate. There is experimental evidence that refutes this point of view. The results Van Huyck et al. (1988) obtained for a game with a similar structure strongly suggests that if it is recommended to the players that they play $\alpha$ in the game $G_1$, then (even without communication) they will succeed to jointly deviate to $\beta$. Our explanation for the subjects' behavior in these experiments is that players are firmly convinced that only $\beta$ makes sense, hence, that the experimenter is simply trying to confuse them and that they can safely ignore his recommendation to play $\alpha$. Our theory justifies why only $\beta$ makes sense in game $G_1$.

\textsuperscript{4}We emphasize the term 'purely noncooperative' by which we mean 'based solely on considerations of individual utility maximization'. There exist theories that single out $\beta$ as the unique solution, the most prominent example being the selection theory from Harsanyi and Selten (1988), but all these adopt as a principle that one should not play a Pareto inferior equilibrium, hence, they assume away the difficulty in $G_1$. Also the literature on cheap-talk (that starts with Farrell (1983)) singles out $\beta$ as the unique solution of $G_1$, however, in our view the basic assumption from this literature, i.e. that any suggestion to play an equilibrium will necessarily be followed is not tenable. After the first version of this paper was completed we became aware of Anderlini (1990) in which it is roughly shown that when players are turing machines, perfect equilibria of the preplay communication game necessarily lead to
is a Nash equilibrium. In fact, this is a strict equilibrium (each player strictly looses by deviating unilaterally) so that it satisfies the conditions imposed by the most refined noncooperative equilibrium notions proposed today, such as, for example, stability à la Kohlberg and Mertens (1986). Our approach is based on the idea that players will not analyse the game from Fig. 1a in isolation but that they rather will view the game as a special case of a $2 \times 2$ unanimity game with common interests, and that they will implement the action that their behavioral rule for playing these games prescribes in this instance. The class of all $2 \times 2$ unanimity games with common interests may be parametrized as in Fig. 1b, where $\theta \in \mathbb{R}$. (Note that $G_1 = 3G_1(7/3)$.) If players can observe the actual value of $\theta$ only with noise and if the observations of the two players are not perfectly correlated, then it is clear that a player cannot analyse $G_1(7/3)$ without analysing games in the neighborhood of $G_1(7/3)$. After all his opponent may think that he is playing $G_1(7/3 \pm \varepsilon)$ and a player cannot know exactly what game the opponent thinks he is playing.

Let us assume that the players’ prior beliefs are that $\theta$ is uniformly distributed on some interval $\Theta$ and that $[0, 1] \subset \text{int } \Theta$. Also assume that players’ observation errors are independent and are identically distributed with support contained in $[-\varepsilon, \varepsilon]$. Note that, if the observed value of $\theta$ is sufficiently small, each player knows that the actual game is dominance solvable and each player $i$ will play $\alpha_i$. Similarly, players will coordinate on $\beta_i$ if the actual value (and, hence, the observed value) of $\theta$ is sufficiently large. Intuitively one expects that the noise forces the players, in equilibrium, to play simple strategies, i.e. player $i$ doesn’t switch irregularly from $\alpha_i$ to $\beta_i$. Let us consider the simplest possible type of strategy: There exists some $\theta^*_i$ such that player $i$ chooses $\alpha_i$ if he observes $\theta_i < \theta^*_i$, while he chooses $\beta_i$ if $\theta_i > \theta^*_i$. By continuity each player $i$ should be indifferent between his two actions at the switching point $\theta^*_i$. By the symmetry of the payoffs and the observation errors, it should be obvious that the optimal $\beta$ being played in $G_1$. Finally, let us mention that Aumann and Sorin (1989) and Matsui (1989) have constructed models that force players to choose $\beta$ when $G_1$ is repeated sufficiently often. Throughout this paper we restrict our attention to one-shot games.
switching points must be identical, i.e. $\theta_1^* = \theta_2^*$. Suppose, for instance, that $\theta_1^* > \theta_2^*$ and that player 1 is different between $\alpha_1$ and $\beta_1$ when observing $\theta_1^*$. Then player 2 must strictly prefer $\alpha_2$ to $\beta_2$ when he observes $\theta_2^*$ since both the conditional probability of player 1 choosing $\alpha_1$ and the expected payoff when players coordinate on $\alpha$ (resp. $\beta$) would be larger (resp. smaller) than the corresponding quantities for player 1 at $\theta_1^*$. Exploiting the symmetry once more we see that, if player i observes $\theta_i^*$, then it is just as likely that the opponent observed $\theta_j < \theta_i^*$ as that he observed $\theta_j > \theta_i^*$, hence, at the observation $\theta_i^*$, player i is playing against the mixed strategy that chooses both $\alpha_j$ and $\beta_j$ with probability $\frac{1}{2}$. Since observation errors are small we therefore must have $\theta_i^* \approx \frac{1}{2}$ and $\theta_i^* \to \frac{1}{2}$ as $\epsilon \to 0$. In the limit as observations become perfect players coordinate on the Pareto best equilibrium for every game from the class described in Fig. 1b.

Let us hasten to add that our approach does not justify playing the Pareto dominant equilibrium in general. The class of games from Fig. 1b is special in the sense that, for these games, the payoff dominance relation coincides with the risk dominance relation. The latter has been introduced in Harsanyi and Selten (1988) and will be formally defined in Sect. 3. The main result from this paper is that, given certain assumptions, the imprecise observations will force players to coordinate on the risk dominant equilibrium in $2 \times 2$ games. The game of Figure 2 may illustrate the intuition for this result. (The game is a slight modification of one discussed in Rubinstein (1989), we compare our work to Rubinstein’s in Sect. 9.)

Assume that, in Fig. 2, that players make slightly imprecise observations on $\theta$. Also assume that, a priori, players do not exclude that $\theta$ might be negative. If $0 < \theta < 100$ there are 2 strict equilibria, viz. $\alpha$ and $\beta$, and the latter Pareto dominates the former.

---

5There is a unique risk dominant equilibrium as long as the game does not have the same best reply structure as a completely symmetric game, cf. Fn 1.
However, for small values of $\theta$ playing $\beta_i$ is quite a risky strategy, player $i$ looses 100 if the opponent does not match that choice. Hence, player $i$ will choose $\beta_i$ only if he is confident enough that player $j$ will play $\beta_j$. Now it is clear that $i$ will not be confident enough if his observation of $\theta_i$ is close to zero. Namely, in that case there is a reasonably large chance that player $j$ knows that $\theta$ is actually negative and in this case, player $j$ will certainly play $\alpha_j$ since this is a dominant strategy. Hence, we conclude that for small positive $\theta$ the players will not play the Pareto dominant equilibrium $\beta$. Now, of course, one can carry the argument further by induction: Once it is known that players choose $\alpha$ if $\theta < \theta^*$ then it becomes risky to play $\beta$ if the observed value of $\theta$ is just marginally above $\theta^*$. On the other hand, the switch has to occur somewhere since for $\theta > 100$ the strategy $\alpha_i$ is strictly dominated by $\beta_i$. Using the same argument as in the discussion of Fig. 1b the reader may verify that, under the same assumptions as the ones discussed above, the switch occurs approximately at $\theta^* = 50$. In the limit, as observations become perfect, players coordinate automatically on the risk dominant equilibrium of $G_2(\theta)$ for any value of $\theta$: They play $\alpha$ if $\theta < 50$ and $\beta$ if $\theta > 50$.

The remainder of the paper is devoted to formalization and extension of the above intuitive argument. We show that, for the special case of $2 \times 2$ games, the argument is completely general, i.e. it does not depend on the underlying class of games being one-dimensional, nor on the symmetry assumed in the above examples, nor on the assumption that prior is uniform. As the reader can already see from the above argument, however, it is essential that players conceive of games that are dominance solvable. The proof is by showing that such games exert a kind of remote influence on games with multiple equilibria. The reason that such action from a distance occurs is that, although when the noise is small the players may know almost exactly which game they are playing, it can never be common knowledge that the game is not dominance solvable.

The paper is organized as follows: Section 2 provides a general definition and existence proof for global games. In Sections 3 - 6, we present and prove our main result. We then show that, in a one-dimensional setting, the result does not require a Nash equilibrium
condition but may be obtained by repeated deletion of dominated strategies in the global game (Section 7). The remainder of the paper (Sections 8 - 11) is devoted to discussing the results and their relation to the literature.

2 Global Games

The basic idea behind the notion of a global game is that the information available to the players can be partitioned into two sets: (i) the structure of the game and (ii) the game's parameters, and that, although the structure of the game is common knowledge, each player is somewhat uncertain about the actual parameters. This uncertainty forces players to analyse entire families of games with the same structure simultaneously. For the purpose of this paper attention will be restricted to situations in which the common knowledge includes the strategies that the players have at their disposal. Hence, the structure consists of the normal form, (some of) the payoffs of the game are the parameters.

Formally, for $i = 1, \ldots, n$, let $A_i$ be a finite set, let the parameter space $\Theta$ be an open subset of $\mathbb{R}^m$ for some $m \in \mathbb{N}$ and for $\theta \in \Theta$ let $g(\theta)$ be an $n$-person normal form game in which player $i$'s set of pure actions is $A_i$. For a pure action combination $a \in A := A_1 \times \cdots \times A_n$, let $g_i(\theta, a)$ be player $i$'s payoff in $g(\theta)$ if $a$ is played, and assume that $g_i(\theta, a)$ is continuous in $\theta$ and bounded, for each $i$ and $a$. Consider the family of games $\mathcal{G} = \{g(\theta); \theta \in \Theta\}$. We picture players in the situation where it is common knowledge that a game from $\mathcal{G}$ has to be played, but players do not know which one. Each player then receives a signal about which game will be played but observations are noisy. We denote player $i$'s signal by $\theta^i$ and write $\Theta^i$ for the support of the random variable $\theta^i$. Letting $F$ denote the joint probability distribution of $(\theta, \theta^1, \ldots, \theta^n)$ we thus consider as our model of the situation the incomplete information game $I^\prime$ described by the following rules:

To avoid confusion between strategies in ordinary normal form games and strategies in global games, the former are referred to as actions.
(\theta, \theta^1, \ldots, \theta^n) \text{ is drawn according to } F \quad (2.1)

Player i is informed about \theta^i (i = 1, \ldots, n). \quad (2.2)

Player i chooses, possibly by using a random device, an action \( a_i \in A_i \)
(players choose simultaneously) \quad (2.3)

If \theta was selected in (2.1) and \( a \) was chosen in (2.3), then player i receives the payoff \( g_i(\theta, a) \). \quad (2.4)

This game \( \Gamma \) will be called a \textit{global game}. Note that a global game is an incomplete information game in which player's payoff does not depend on his type, a type (i.e. an observation) just describes a player's beliefs. A pure strategy of player i in \( \Gamma \) is a (Borel) measurable function from \( \Theta^i \) into \( A_i \). Letting \( \Delta(A_i) \) denote the set of probability distributions on \( A_i \), a behavioral strategy of player i is a measurable function \( s_i \) from \( \Theta^i \) into \( \Delta(A_i) \). Denote by \( \pi_i(s \backslash a_i; \theta^i) \) player i's expected payoff when he has observed \( \theta^i \), he chooses \( a_i \) and the opponents play according to the behavior strategy profile \( s_i \). The profile \( s_i \) is said to be a \textit{(Bayesian Nash) equilibrium} of \( \Gamma \) if

\[
\pi_i(s; \theta^i) \geq \pi_i(s \backslash a_i; \theta^i) \quad \text{for all } i, a_i, \theta^i \quad (2.5)
\]

We have

Proposition 1 If the joint distribution \( F \) of \((\theta, \theta^1, \ldots, \theta^n)\) as well as the marginal distribution \( F^i \) of each \( \theta^i \) has a density, then the global game \( \Gamma \) has an equilibrium.

Proof. Proposition 1 in Milgrom and Weber (1985) shows that \( \Gamma \) has a Nash equilibrium (i.e. a strategy profile in which the conditions from (2.5) are satisfied for almost all \( \theta^i \)) if
(i) payoffs are equicontinuous, i.e. the family of functions \( \{ g_i(\theta, \cdot); \theta \in \Theta \} \) is equicontinuous, and

(ii) information is absolutely continuous, i.e. the measure \( dF \) is absolutely continuous with respect to the product measure \( dF_0 \times dF_1 \times \ldots \times dF^n \).

The first condition is satisfied since \( g_i(\theta, a) \) is multilinear in \( a \) and bounded in \( \theta \). The second condition directly follows from our assumption about nonatomic distributions. Since, given our assumptions, a strategy of each player \( i \) can be changed at sets of measure zero without changing any player \( j \)'s payoff, it follows immediately that also a Bayesian Nash equilibrium exists. \( \Box \)

Obviously, when the signals are very noisy, the equilibria of the global game \( \Gamma \) need not bear any relationship to the equilibria of the underlying class of games \( \mathcal{G} \). We will investigate the opposite extreme case when observation errors vanish and we will study the associated limiting behavior of the equilibria of the global game. We have not been able to carry through the mathematical analysis in the general case. However, we managed to carry out the program for the class of \( 2 \times 2 \) normal form games and we obtained very interesting results for this special case. Therefore, we turn to that class next.

3 Best Reply Structure and Risk Dominance in \( 2 \times 2 \) Games

From now on we restrict ourselves to the case where \( \mathcal{G} = \{ g(\theta); \theta \in \Theta \} \) is a subclass of the set of 2 player \( 2 \times 2 \) normal form games. In this section we introduce some notation for this family, analyse the best reply structure, and introduce and briefly discuss Harsanyi and Selten's (1988) concept of risk dominance.

We write \( A_i = \{ \alpha_i, \beta_i \} \) for the set of pure strategies of player \( i \). A mixed strategy of
this player is identified with the probability \( a_i \) that it assigns to \( \alpha_i \). We write \( \alpha = (\alpha_1, \alpha_2) \), 
\( \beta = (\beta_1, \beta_2) \) and \( \alpha \setminus \beta = (\alpha_i, \beta_j) \). Let \( v_i^\alpha(\theta) \) be the loss that player \( i \) incurs if he deviates unilaterally from \( \alpha \) in game \( g(\theta) \) and let \( v_i^\beta(\theta) \) be the corresponding loss when deviating from \( \beta \). Hence

\[
v_i^\alpha(\theta) = g_i(\theta, \alpha) - g_i(\theta, \alpha \setminus \beta_i)
\]

\[
v_i^\beta(\theta) = g_i(\theta, \beta) - g_i(\theta, \beta \setminus \alpha_i)
\]

For \( \gamma \in \{\alpha, \beta\} \), let \( \Theta_\gamma \) denote the region of \( \Theta \) where \( \gamma \) is a strict Nash equilibrium of \( g(\theta) \):

\[
\Theta_\gamma := \{\theta \in \Theta; v_i^\gamma(\theta) > 0 \text{ for } i = 1, 2\}, \gamma = \alpha, \beta
\]

We let \( D_{i\gamma} \) denote the region where \( \gamma \) is a strict equilibrium and \( \gamma_i \) is a strictly dominant strategy for player \( i \):

\[
D_{i\alpha} := \{\theta \in \Theta_\alpha; v_i^\alpha(\theta) < 0\}
\]

\[
D_{i\beta} := \{\theta \in \Theta_\beta; v_i^\beta(\theta) < 0\}
\]

We let \( D_\alpha \) (resp. \( D_\beta \)) denote the parameter region where \( g(\theta) \) is dominance solvable with solution \( \alpha \) (resp. \( \beta \)), hence, \( D_\alpha := D_{1\alpha} \cup D_{2\alpha} \) and \( D_\beta := D_{1\beta} \cup D_{2\beta} \).

\( \Theta_{\alpha\beta} \) will denote the region where both \( \alpha \) and \( \beta \) are strict equilibria, hence \( \Theta_{\alpha\beta} := \Theta_\alpha \cap \Theta_\beta \). If \( \theta \in \Theta_{\alpha\beta} \), then \( g(\theta) \) has, besides \( \alpha \) and \( \beta \), an equilibrium in mixed strategies. In the latter, player \( i \) chooses \( \alpha_i \) with probability \( \bar{a}_i = \bar{a}_i(\theta) \) given by

\[
\bar{a}_i = \frac{v_i^\beta}{v_i^\alpha + v_i^\beta}
\]

When player \( i \) chooses \( a_i \) as in (3.1), then player \( j \) is indifferent between \( \alpha_j \) and \( \beta_j \), if player \( i \) chooses \( \alpha_i \) with a probability larger than \( \bar{a}_i \), then player \( j \) strictly prefers \( \alpha_j \).
Hence, if $\bar{a}_i$ is small, then player $j$ only needs to attach a relatively small probability to $i$ playing $\alpha_i$ in order to make it optimal for him to play $\alpha_j$. In other words, if $\bar{a}_i$ is small, then $\alpha_j$ is a relatively safe strategy for player $j$ and a lower value of $\bar{a}_i$ makes $\alpha_j$ more attractive for this player. This is the idea that underlies Harsanyi and Selten's (1988) notion of risk dominance\(^8\). Formally $\alpha$ is said to \textit{risk dominate} $\beta$ if the Nash product of the deviation losses associated with $\alpha$ is larger than the Nash product associated with $\beta$, i.e.

$$v'_1 v'_2 > v''_1 v''_2,$$

(3.2)

and that $\beta$ risk dominates $\alpha$ if the reverse inequality is satisfied. From (3.1) one sees that $\alpha$ risk dominates $\beta$ if and only if

$$(1 - \bar{a}_1)(1 - \bar{a}_2) > \bar{a}_1 \bar{a}_2$$

(3.3)

i.e. if the area of the stability region\(^9\) of $\alpha$ is larger than the area of the stability region

\(^8\)For general games, the risk dominance relation is defined by means of the tracing procedure and need not be transitive. For $2 \times 2$ games, the relation is trivially transitive and Harsanyi and Selten characterize it axiomatically. For these games the relation also corresponds quite well with intuition. Furthermore, Harsanyi and Selten give a heuristic justification for $2 \times 2$ games. It seems worthwhile to briefly describe the latter since it illustrates that our theory is not trivial: On the face of it, our approach has nothing to do with risk dominance, still it generates risk dominant equilibria as its solutions.

Harsanyi and Selten argue as follows. Consider a game $g(\theta)$ with $\theta \in \Theta_{\alpha\beta}$ and assume $\theta$ is common knowledge. There are three equilibria, so that there is initial uncertainty about which equilibrium to play. Player $j$'s beliefs about which strategy is played by player $i$ may be represented by a mixed strategy $z \alpha_i + (1 - z) \beta_j$. Player $i$ doesn't know $j$'s beliefs and, applying the principle of insufficient reason, he considers all values of $z$ to be equally likely (i.e. $i$ considers $z$ to be uniformly distributed on $[0, 1]$). Player $i$ reasons that, whatever the value of $z$, player $j$ will play a best response against his beliefs. Noting that $j$'s best response is $\alpha_j$ if and only if $z > \bar{a}_i$ and averaging over $z$, player $i$ concludes that player $j$ will play $\alpha_j$ with the probability $1 - \bar{a}_i$ where $\bar{a}_i$ is given in (3.1). Player $i$ will play his best response against this mixed strategy, i.e. he will play $\alpha_i$ if $1 - \bar{a}_i > \bar{a}_j$ and he will play $\beta_j$ if $1 - \bar{a}_i < \bar{a}_j$. Since player $j$ reasons in exactly the same way, the players coordinate on $\alpha$ if $\bar{a}_1 + \bar{a}_2 < 1$ while they coordinate on $\beta$ if $\bar{a}_1 + \bar{a}_2 > 1$.

\(^9\)This is the set of mixed strategy pairs $\alpha$ such that $\alpha_i$ is a best reply against $\alpha_j$ ($i, j \in \{1, 2\}, i \neq j$).
of $\beta$. Obviously it is therefore true that $\alpha$ risk dominates $\beta$ if and only if

$$a_1 + a_2 < 1$$

(3.4)

and that $\beta$ risk dominates $\alpha$ if and only if the reverse strict inequality is satisfied. For our purposes this simple characterization is most helpful.

For analytical purposes it is useful to extend the definition of $\bar{a}_i$ continuously from $\Theta_{\alpha\beta}$ to $\Theta_\alpha \cup \Theta_\beta$. Hence we define

$$\bar{a}_i(0) = \begin{cases} 0 & \text{if } \theta \in \Theta_\alpha \text{ and } v_{ij}^\theta(0) \leq 0 \\ \text{as in (3.1)} & \text{if } \theta \in \Theta_{\alpha\beta} \\ 1 & \text{if } \theta \in \Theta_\beta \text{ and } v_{ij}^\theta(0) \leq 0 \end{cases}$$

(3.5)

Finally we write $R_\alpha$ (resp. $R_\beta$) for the subset of $\Theta_\alpha \cup \Theta_\beta$ where either $\alpha$ is the unique equilibrium or $\alpha$ risk dominates $\beta$ (resp. $\beta$ is the unique equilibrium or $\beta$ risk dominates $\alpha$). Hence, using (3.4) and (3.5)

$$R_\alpha = \{ \theta \in \Theta_\alpha; a_1(\theta) + a_2(\theta) < 1 \}$$

$$R_\beta = \{ \theta \in \Theta_\beta; a_1(\theta) + a_2(\theta) > 1 \}$$

Note that $D_\gamma \subset R_\gamma$ and that the sets $D_i, D_{\gamma}$ and $R_\gamma$ are open, $\gamma = \alpha, \beta$.

To conclude this section we list some assumptions that will be assumed to be fulfilled throughout the remainder. Assumption 1a) repeats an assumption made above and 1b) is a mild regularity condition. 1c) guarantees that $\mathcal{G}$ includes games that are dominance solvable and that have solution $\alpha$ (resp. $\beta$), in combination with 1d) this assumption guarantees that each game in $R_\alpha$ (resp. $R_\beta$) can be connected with a game in $D_\alpha$ (resp. $D_\beta$).

Assumption 1. The class $\mathcal{G} = \{ g(\theta); \theta \in \Theta \}$ satisfies
a) $\Theta$ is an open subset of $\mathbb{R}^m$, each game $g(\theta)$ is a 2-person $2 \times 2$ normal form game,  

b) $v_i^\gamma$ is continuously differentiable on $\Theta$ (for $i = 1, 2$ and $\gamma = \alpha, \beta$) and the partial derivatives $\partial v_i^\gamma / \partial \theta_j$ are all bounded, 

c) $D_\alpha$ and $D_\beta$ are non-empty, and 

d) $R_\alpha$ and $R_\beta$ are connected.

The reader can easily verify that this assumption is satisfied in the examples from Section 1 if $\Theta \supset [0, 1]$ (resp. $\Theta \supset [0, L]$). Another important example in which these assumptions hold is the case with $\Theta = \mathbb{R}^s$ and $g(\theta) = \theta$, i.e. players a priori just know that they have to play a $2 \times 2$ bimatrix game.

4 Statement of the Theorem

We now picture players in the situation where it is common knowledge that a game from $\mathcal{G}$ has to be played but players do not yet know which one. Players make observations on which game is played, but observations are noisy. Note that some games in $\mathcal{G}$ (viz. those with $\theta \in D_\alpha \cup D_\beta$) are easy but that for those corresponding to $\theta \in \Theta_{\alpha\beta}$ there is an equilibrium selection problem. We will show that the uncertainty about which game is played may help players to resolve their coordination problem in this area. We assume that the joint distribution of the actual game and the measurement errors is common knowledge so that the overall situation can be modelled as a global game. We are particularly interested in the situation where measurements are almost correct, i.e. we will investigate sequences of global games in which the noise vanishes.

Formally, for $\varepsilon > 0$ and random variables $\theta, e^1$ and $e^2$ taking values in $\mathbb{R}^m$ we consider the global game $\Gamma^\varepsilon$ played according to the rules (2.1) – (2.4) with the observation of player $i$ being given by

$$\theta^i = \theta + \varepsilon e^i$$  \hspace{1cm} (4.1)
In addition we will assume that the following Assumption holds.

Assumption 2

a) \( \theta \) has distribution function \( H \), and admits a continuously differentiable density \( h \) that is positive on \( \Theta \) and bounded.

b) \( e^1 \) and \( e^2 \) are independent of \( \theta \) and have a joint distribution \( \Phi \) with a density \( \varphi \).

c) The support of \( \Phi \) is contained in a ball around zero with radius \( \frac{1}{2} \) and the density \( \varphi \) is bounded.

The set of equilibria of \( \Gamma^e \) is denoted by \( E(\Gamma^e) \). Proposition 1 in Sect. 2 guarantees that \( E(\Gamma^e) \neq \emptyset \). If \( s \in E(\Gamma^e) \), then we write \( s_i^\gamma(\theta^i) \) for the probability with which player \( i \) chooses \( \gamma_i \) if he observes \( \theta^i \) (\( \gamma \in \{\alpha, \beta\} \)). Our main result may be stated as follows

**Theorem 1.** Let \( \theta \in R_\alpha \) (resp. \( \theta \in R_\beta \)) and let \( s \in E(\Gamma^e) \). If the assumptions 1 and 2 are satisfied and \( \epsilon \) is small, then \( s_i^\gamma(\theta) = 1 \) (resp. \( s_i^\theta(\theta) = 1 \)). In other words, in the limit as the noise vanishes, the players coordinate on the risk dominant equilibrium of the actual game.

The following section introduces notation and results that will be used in the proof of Theorem 1. The proof itself is in Section 6. Section 7 considers the special case where \( \Theta \) is one-dimensional. In this case we are able to show that one doesn’t have to assume equilibrium behavior in the global game to justify the coordination on the risk dominant equilibrium in the limit: The global game is dominance solvable in this case, hence, rationalizability suffices to obtain a unique equilibrium solution.
5 Posterior Beliefs and Equilibrium Properties

In this section we consider a fixed global game $\Gamma'$. We first derive a symmetry property of the players' posterior beliefs (Lemma 2) that plays an essential role in the proof of Theorem 1. Thereafter we derive some properties that equilibria of global games satisfy.

5.1 Posterior beliefs

Let $F_1^*(B|B')$ and $F_2^*(B|B')$ denote the distribution of player 1's posterior beliefs about, respectively, $B'$ and $(B, B')$ when he has observed $B'$, the corresponding densities being $f_1^*(B|B')$ and $f_2^*(B, B'|B')$. We will derive a fundamental property, Lemma 2, which links the players' posterior beliefs about each other's observations. This property takes its simplest form when $\theta$ is one-dimensional and has a uniform prior. Then, the posteriors $F_1^*(\theta^2|\theta^1)$ and $F_2^*(\theta^1|\theta^2)$ add up exactly to 1:

$$F_1^*(\theta^2|\theta^1) + F_2^*(\theta^1|\theta^2) = 1$$

(5.1)

The reason is that, when the prior is uniform, a player's observation does not give him any (additional) information about the distribution of the observation errors. Hence, e.g., $F_2^*(\theta^1|\theta^2)$ is simply equal to the (prior) probability that the difference between the observation errors (i.e. $e_1^1 - e_1^2$) is no greater than $\theta^1 - \theta^2$.

When the prior is non-uniform (5.1) holds approximately for small $\varepsilon$ since the prior will still be locally constant. Lemma 2 generalizes this result to the case of an $m$-dimensional $\Theta$. For the sequel, it is useful to note the similarity between (5.1) and the definition of risk dominance in (3.4), a similarity which will be exploited in the proof of Theorem 1.

Next we formalize the above intuition. Some notation first. For $x \in \mathbb{R}^m$ and $r \in \mathbb{R}$, let $B(x, r)$ denote the (Euclidian) closed ball with radius $r$ centered at $x$: 
\[ B(x, r) := \{ y \in \mathbb{R}^m; \| y - x \| \leq r \}. \]

If \( S \) is a subset of \( \mathbb{R}^m \) then \( S(r) \) denotes the points \( x \) in \( S \) that are at least \( r \) inside of \( S \)

\[ S(r) := \{ x \in S; B(x, r) \subset S \}. \]

To derive the posterior beliefs of player \( i \) in \( \Gamma^e \) after having observed \( \theta^i \) explicitly, first note that the joint density of the triple \((\theta, \theta^1, \theta^2)\) is given by

\[ f^\varepsilon(\theta, \theta^1, \theta^2) = \varphi^\varepsilon(\theta^1 - \theta, \theta^2 - \theta)h(\theta) \quad \text{for } \theta \in \Theta, \]

where \( \varphi^\varepsilon \) is the density of \((\varepsilon \epsilon^1, \varepsilon \epsilon^2)\), i.e.

\[ \varphi^\varepsilon(x) = \varepsilon^{-2m} \varphi(\varepsilon^{-1}x) \quad (x \in \mathbb{R}^{2m}). \]

Write \( \Psi^\varepsilon \) for the distribution of \( \varepsilon \epsilon^1 - \varepsilon \epsilon^2 \) and let \( \psi^\varepsilon \) be the associated density. Then

\[ \psi^\varepsilon(\ell^1 - \ell^2) = \int \varphi^\varepsilon(t, t - \ell^1 + \ell^2)dt = \int \varphi^\varepsilon(\ell^1 - t, \ell^2 - t)dt \]

(5.2)

Player \( i \)’s posterior density of \( \theta^i \) conditional on having observed \( \theta^i \) is given by

\[ f^\varepsilon_i(\theta^i|\theta^i) = \frac{\int \varphi^\varepsilon(\theta^1 - \theta, \theta^2 - \theta)h(\theta)d\theta}{\int \varphi^\varepsilon(\theta^1 - \theta, \theta^2 - \theta)h(\theta)d\theta d\theta^i} \]

Assume \( \theta^i \in \Theta(\varepsilon) \) and let \( h^+_\varepsilon(\theta^i) \) (resp. \( h^-_\varepsilon(\theta^i) \)) denote the maximum (resp. minimum) of \( h \) in \( B(\theta^i, \varepsilon) \). Then in view of (5.2)

\[ \psi^\varepsilon(\theta^1 - \theta^2) \frac{h^-_\varepsilon(\theta^i)}{h^+_\varepsilon(\theta^i)} \leq f^\varepsilon_i(\theta^i|\theta^i) \leq \psi^\varepsilon(\theta^1 - \theta^2) \frac{h^+_\varepsilon(\theta^i)}{h^-_\varepsilon(\theta^i)} \]

(5.3)

Let \( K \) be a compact subset of \( \Theta \). Assumption 2a implies that there exists a constant \( k \) such that for all \( \theta^i \in K(\varepsilon) \) and all sufficiently small \( \varepsilon \)
\[ \frac{h_+^e(\theta^i)}{h_-^e(\theta^i)} - 1 \leq k\varepsilon \]

Hence, we have

**Lemma 1.** For each compact set \( K \) with \( K \subset \Theta(\varepsilon) \) there exists a constant \( k \) such that for all \( \theta^i \in K \)

\[ |f_1^e(\theta^i|\theta^j) - \psi^e(\theta^1 - \theta^2)| \leq k\varepsilon\psi^e(\theta^1 - \theta^2) \]  

(5.4)

Note that, if the prior \( h \) would be uniform, then a player's posterior would be exactly equal to \( \psi^e(\theta^1 - \theta^2) \), hence, in this case \( f_1^e(\theta^2|\theta^1) = f_2^e(\theta^2|\theta^1) \). The content of Lemma 1 is that this symmetry property remains approximately valid, i.e.

\[ f_1^e(\theta^2|\theta^1) \approx f_2^e(\theta^1|\theta^2) \]

if the prior is locally almost constant. This symmetry property allows us to derive Lemma 2.

**Lemma 2.** For \( x \in \mathbb{R}^m \) and \( p \in \mathbb{R}^m \setminus \{0\} \), let \( H_p(x) \) be the hyperplane in \( \mathbb{R}^m \) with normal vector \( p \) going through \( x \), i.e. \( H_p(x) = \{ y \in \mathbb{R}^m; py = px \} \). Furthermore, let \( H_p^-(x) \) (resp. \( H_p^+(x) \)) be the closed halfspace below (resp. above) \( H_p(x) \). Let \( K \subset \Theta(\varepsilon) \) be compact and let \( k \) be as in Lemma 1. Then for each \( x^1, x^2 \in K \)

\[ 1 - 2k\varepsilon \leq \text{Prob} \{ \theta^2 \in H_p^-(x^2)|\theta^1 = x^1 \} + \text{Prob} \{ \theta^1 \in H_p^-(x^1)|\theta^2 = x^2 \} \leq 1 + 2k\varepsilon. \]  

(5.5)

**Proof.** By Lemma 1 we have

\[ \text{Prob} \{ \theta^2 \in H_p^-(x^2)|\theta^1 = x^1 \} = \int_{p\theta^2 \leq px^2} f_1^e(\theta^2|x^1)d\theta^2 \leq \int_{p\theta^2 \leq px^2} \psi^e(x^1 - \theta^2)d\theta^2 + k\varepsilon \]
Making the linear transformation \( \theta^1 = x^1 + x^2 - \theta^2 \), we see that the latter integral is equal to

\[
\int_{p\theta^1 \geq x^1} \psi^*(\theta^1 - x^2) d\theta^1;
\]

and, applying Lemma 1 once more we have

\[
\int_{p\theta^1 \geq x^1} \psi^*(\theta^1 - x^2) d\theta^1 \leq \int_{p\theta^1 \geq x^1} f^*_2(\theta^1|\theta^2) d\theta^1 + k\varepsilon
\]

\[
= 1 - \text{Prob} \{ \theta^1 \in H^-_p(x^1) | \theta^2 = x^2 \} + k\varepsilon
\]

Hence we have shown that

\[
\text{Prob} \{ \theta^2 \in H^-_p(x^2) | \theta^1 = x^1 \} + \text{Prob} \{ \theta^1 \in H^-_p(x^1) | \theta^2 = x^2 \} \leq 1 + 2k\varepsilon.
\]

The second inequality needed to establish the lemma is proved in exactly the same way.

\[\square\]

5.2 Equilibrium properties

In this subsection we derive some basic properties of equilibria of global games. Again, some notation first. Given a strategy pair \( s \) in \( \Gamma^e \), we write \( \alpha^*_j(s|\theta^i) \) (resp. \( \beta^*_j(s|\theta^i) \)) for the conditional probability upon observing \( \theta^i \) that \( j \) will choose \( \alpha_j \) (resp. \( \beta_j \)) when playing \( s_j \), hence

\[
\alpha^*_j(s|\theta^i) = \int s^*_j(\theta^i) dF^*_i(\theta^i|\theta^i)
\]

(5.6)

where \( s^*_j(\theta^i) \) is the probability that player \( j \) chooses \( \alpha_j \) when he observes \( \theta^j \). Also write \( V^*_i(s|\theta^i) \) for the difference in expected payoffs between \( \alpha_i \) and \( \beta \), when \( i \) has observed \( \theta^i \) and when \( j \) plays \( s_j \), hence
\[ V_i^\varepsilon(s|\theta^i) = \int \left[ s_j^\alpha(\theta^j) v_i^\varepsilon(\theta) - s_j^\beta(\theta^j) v_i^\varepsilon(\theta) \right] dF_i^\varepsilon(\theta, \theta^j|\theta^i). \]  

(5.7)

The Assumptions 1 and 2 together with Lebesgue's bounded convergence theorem immediately imply

**Lemma 3.** For every strategy combination \( s \), the functions \( \alpha^\varepsilon_i(s|\theta^i) \) and \( V_i^\varepsilon(s|\theta^i) \) are continuous in \( \theta^i \).

Clearly, a strategy pair \( s \) is an equilibrium of \( \Gamma^\varepsilon \) if and only if \( s \) satisfies the following two conditions for \( i = 1, 2 \) and all \( \theta^i \)

\[
\begin{align*}
\text{if } V_i^\varepsilon(s|\theta^i) < 0, \text{ then } s_i^\alpha(\theta^i) = 0 & \quad (5.8) \\
\text{if } V_i^\varepsilon(s|\theta^i) > 0, \text{ then } s_i^\alpha(\theta^i) = 1 & \quad (5.9)
\end{align*}
\]

A point \( \theta^i \) with \( V_i^\varepsilon(s|\theta^i) = 0 \) will be called a switching point for player \( i \) given \( s \). (Switching from a given pure action to some other action at \( \theta^i \) can be optimal only if \( V_i^\varepsilon(\theta^i) = 0 \).) Intuitively, it is clear that if \( \varepsilon \) is small, \( s \) is an equilibrium of \( \Gamma^\varepsilon \), and \( \theta^i \) is a switching point of player \( i \), then \( \alpha^\varepsilon_j(s|\theta^i) \) must be close to \( \bar{a}_j(\theta^i) \), that is, it must be close to the probability that \( j \) has to attach to \( s_j(\theta^i) \) in order to make \( i \) indifferent in the game \( g(\theta^i) \).

The following Lemma formally states this result.

**Lemma 4.** Let \( \bar{M} \) be an upper bound on \( |\partial v_i^\gamma/\partial \theta^j| \) and let \( M = m\bar{M} \). If \( s \in E(\Gamma^\varepsilon) \) and \( \theta^i \in \Theta_\alpha \cup \Theta_\beta \) is a switching point for \( i \) given \( s \), then

\[
|\alpha^\varepsilon_j(s|\theta^i) - \bar{a}_j(\theta^i)| \leq \frac{M \varepsilon}{\max \{v_i^\alpha(\theta^i), v_i^\beta(\theta^i)\}}. \]

(5.10)

Proof. If \( f_i^\varepsilon(\theta, \theta^j|\theta^i) > 0 \), then \( ||\theta - \theta^i|| \leq \varepsilon \), hence \( |v_i^\gamma(\theta) - v_i^\gamma(\theta^i)| \leq M \varepsilon \) for \( \gamma \in \{\alpha, \beta\} \). Since \( \theta^i \) is a switching point \( V_i^\varepsilon(s|\theta^i) = 0 \), hence,
\[-M\varepsilon \leq \int \left[ v_i^\omega(\theta^i) s_j^\omega(\theta^j) - v_i^\beta(\theta^i) s_j^\beta(\theta^j) \right] dF_i(\theta, \theta^i | \theta^i) \leq M\varepsilon,\]

or, equivalently

\[-M\varepsilon \leq \left[ v_i^\omega(\theta^i) + v_i^\beta(\theta^i) \right] \alpha_j^\omega(s | \theta^i) - v_i^\beta(\theta^i) \leq M\varepsilon. \tag{5.11}\]

If both \(v_i^\omega(\theta^i)\) and \(v_i^\beta(\theta^i)\) are strictly positive, then (5.10) follows immediately from (3.1) and (5.11). Assume \(v_i^\omega(\theta^i) \leq 0\). Hence \(v_i^\beta(\theta^i) > 0\) and \(a_j(\theta^i) = 1\). In this case the first inequality from (5.11) yields

\[v_i^\beta(\theta^i)(1 - \alpha_j^\omega(s | \theta^i)) \leq v_i^\omega(\theta^i) \alpha_j^\omega(s | \theta^i) + M\varepsilon \leq M\varepsilon\]

which implies (5.10). Finally, if \(v_i^\beta(\theta^i) \leq 0\), then \(v_i^\omega(\theta^i) > 0\) and \(a_i(\theta^i) = 0\), and we have

\[v_i^\omega(\theta^i) \alpha_j^\omega(s | \theta^i) \leq v_i^\omega(\theta^i) \alpha_j^\omega(s | \theta^i) - v_i^\beta(\theta^i)(1 - \alpha_j^\omega(s | \theta^i)) \leq M\varepsilon\]

where the last inequality follows from (5.11). This again implies (5.10). \(\square\)

The next lemma says that the switching points of the two players are close together

**Lemma 5.** Let \(s \in E(\Gamma^e)\). a) If \(\theta^i \in \Theta_\alpha(\varepsilon)\) and \(V_j^\omega(s | \theta^j) \leq 0\), then there exists \(\theta^j \in B(\theta^i, \varepsilon)\) with \(V_j^\omega(s | \theta^j) \leq 0\). b) If \(\theta^i \in \Theta_\beta(\varepsilon)\) and \(V_j^\beta(s | \theta^i) \geq 0\), then \(V_j^\beta(s | \theta^j) \geq 0\) for some \(\theta^j \in B(\theta^i, \varepsilon)\).

**Proof.** We show a). Let \(\theta^i \in \Theta_\alpha(\varepsilon)\). If \(V_j^\omega(s | \theta^j) > 0\) for all \(\theta^j \in B(\theta^i, \varepsilon)\), then \(s_j^\omega(\theta^j) = 1\) for all \(\theta^j\) with \(f_j^\omega(\theta^j | \theta^i) > 0\). Furthermore \(V_i^\omega(\theta) > 0\) for all \(\theta\) with \(f_j^\omega(\theta, \theta^j | \theta^i) > 0\). Hence \(V_i^\omega(s | \theta^j) > 0\). \(\square\)

The final lemma in this section states that if player \(i\)'s observation is well inside \(D_\gamma\), then player \(i\) chooses \(\gamma_i\) in any equilibrium of \(\Gamma^e (\gamma \in \{\alpha, \beta\})\).
Lemma 6. If \( s \in E(\Gamma^e) \), then \( s^\gamma_i(\theta^i) = 1 \) for each \( \theta^i \in D_1(\gamma) \cup D_2(\gamma) \) (\( \gamma = \alpha, \beta \)).

Proof. Take \( i = 1 \) and \( \gamma = \alpha \). First consider the case where \( \theta^1 \in D_{1\alpha}(\epsilon/2) \). Then for all \( \theta \) with \( f_1^\gamma(\theta, \theta^2|\theta^1) > 0 \) we have \( \theta \in D_{1\alpha} \), hence \( v_1^\gamma(\theta) > 0 \) and \( v_1^\alpha(\theta) < 0 \). Hence player 1 knows that \( \alpha_1 \) is a strictly dominant strategy for each game that he might play. Therefore \( s^\gamma_1(\theta^1) = 1 \).

Next consider the case where \( \theta^1 \in D_{2\alpha}(\epsilon) \). Then \( \theta^2 \in D_{2\alpha}(\epsilon/2) \) for all \( \theta^2 \) with \( f_1^\gamma(\theta, \theta^2|\theta^1) > 0 \). Hence, by the first half of the proof (with the roles of the players reversed), \( s^\gamma_2(\theta^2) = 1 \) for all such \( \theta^2 \). Hence, player 1 knows that player 2 chooses \( \alpha_2 \) for sure. Since \( v_1^\gamma(\theta) > 0 \) for all \( \theta \) with \( f_1^\gamma(\theta, \theta^2|\theta^1) > 0 \), player 1's unique best response is \( \alpha_1 \) for any game that he might play, therefore, \( s^\gamma_1(\theta^1) = 1 \). \( \square \)

6 Proof of Theorem 1

Assume that there exists \( \theta' \in R_\alpha \) for which the statement from Theorem 1 does not hold, i.e. we can find sequences \( \epsilon_n \to 0 \) and \( s^n \in (\Gamma^{\epsilon_n}) \) such that \( w_i^n(\theta') := V_i^{\epsilon_n}(s^n|\theta') \leq 0 \) for some \( i \in \{1, 2\} \) and all \( n \). The proof, which is by contradiction, is divided into four steps: First we construct a compact subset \( K \) of \( R_\alpha \), which has a connected interior and in which both \( \theta' \) and some \( \theta'' \in D_\alpha \) are interior points. Using Lemmas 5 and 6, it is easily seen that \( K \) must contain switching points for each player when \( \epsilon \) is sufficiently small. The central part of the proof (Steps 2 and 3) consists in constructing a particular pair of switching points \( x^1 \) and \( x^2 \) and corresponding halfspaces which allow us to apply Lemma 2. These switching points are close together and they have the property that, at least locally, each player \( i \) chooses \( \alpha_i \) for each observation \( \theta^i \) that is to the left of \( x^i \). Lemma 2 then allows us to conclude that the sum of the posterior probabilities that \( i \) assigns to \( j \) playing \( \alpha_j \) (\( i, j \in \{1, 2\}, i \neq j \)) must be approximately equal to 1. Using Lemma 4 it is then straightforward to establish a contradiction in Step 4.

Step 1. Construction of a compact set relevant for the remainder of the proof.
Since $R_\alpha$ is connected and since $D_\alpha \neq \phi$ we can find $\theta'' \in D_\alpha$ and a curve $C$ connecting $\theta''$ to $\theta'$ with $C \subset R_\alpha$. W.l.o.g. assume $\theta'' \in D_{1\alpha}$. By Lemma 5 we may also assume that $w_\alpha^n(\theta') \leq 0$ for all $n$. Since $D_{1\alpha}$ and $R_\alpha$ are open and $C$ is compact, we can find $\eta > 0$ such that $B(\theta'', 2\eta) \subset D_{1\alpha}$ and such that $B(c, 2\eta) \in R_\alpha$ for each $c \in C$. Let $K$ be the compact set

$$K := \bigcup_{c \in C} B(c, 2\eta)$$

Then $K \subset R_\alpha$ and since $\bar{a}_i(\theta)$ and $w_\alpha^i(\theta)$ depend continuously on $\theta$

$$\max_{\theta \in K} \bar{a}_1(\theta) + \bar{a}_2(\theta) < 1 \tag{6.1}$$

$$\min_{\theta \in K} v_1^\alpha(\theta) > 0 \text{ and } \min_{\theta \in K} v_2^\alpha(\theta) > 0 \tag{6.2}$$

**Step 2.** Construction of switching points. (See Figure 3.)

Take $n$ large enough so that $\varepsilon_n$ is small relative to $\eta$, specifically $2\varepsilon_n < \eta$. To simplify notation, write $\varepsilon = \varepsilon_n$, $s = s_n$ and $w_\alpha = w_\alpha^n$. Let $c$ be a continuous parametrization of $C$ with $c(0) = \theta''$ and $c(1) = \theta'$. Define the continuous function $f$ by

$$f(\lambda) = \min\{w_\alpha(\theta); \theta \in B(c(\lambda), \eta)\}$$

Then, by Lemma 6, $f(0) > 0$ since $B(\theta'', \eta) \subset D_{1\alpha}$. On the other hand $f(1) \leq 0$ since $w_\alpha(\theta') \leq 0$. Let $\lambda^*$ be the smallest zero of $f$ and write $x^* = c(\lambda^*), B^* = B(x^*, \eta)$. Then $w_\alpha(\theta) > 0$ for $\theta$ in the interior of $B^*$ (hence, $s_\alpha^i(\theta) = 1$ for such $\theta$) and there exists $x^1 \in \partial B^*$ with $w_\alpha(x^1) = 0$. Write $p = x^1 - x^*$ and let $H_p(\theta), H^-_p(\theta)$ be defined as in Lemma 2. There exists an interval $[\lambda^-, \lambda^+]$ with $\lambda^- < 1 < \lambda^+$ such that

$$H_p((1 - \lambda)x^* + \lambda x^1) \cap B(x^1, \varepsilon) \neq \phi \quad \text{for } \lambda \in [\lambda^-, \lambda^+]$$

Note that for $\lambda = \lambda^-$ the intersection consists of a single point, say $\theta^2$, and that
B(\(\theta^2, \epsilon\)) \subset B^*$. Hence, when player 2 observes \(\theta^2\), he is sure that player 1 will play \(\alpha\). Therefore, \(w_2(\theta^2) > 0\). Define the continuous map \(g\) on \([\lambda^-, \lambda^+]\) by

\[
g(\lambda) = \min\{w_2(\theta); \theta \in H_p^-((1 - \lambda)x^* + \lambda x^1) \cap B(x^1, \epsilon)\}
\]

Hence, \(g(\lambda^-) > 0\). On the other hand \(g(\lambda^+) \leq 0\) since for \(\lambda^+\) the ball \(B(x^1, \epsilon)\) is completely contained in the halfspace \(H_p^-((1 - \lambda^+)x^* + \lambda^+ x^1)\) and \(w_1(x^1) = 0\), so that there must exist \(\theta \in B(x^1, \epsilon)\) with \(w_2(\theta) < 0\) (cf. Lemma 5). Let \(\bar{\lambda}\) be the smallest zero of \(g\). Then there exists a point \(x^2 \in H_p((1 - \bar{\lambda})x^* + \bar{\lambda} x^1)\) with \(w_2(x^2) = 0\) while \(w_2(\theta) > 0\) for all \(\theta\) that are in \(B(x^1, \epsilon)\) and that are strictly to the left of this hyperplane.

[insert Figure 3 about here]

**Step 3.** Preparation to apply Lemma 2.

In view of the construction from Step 2 we have

\[
\alpha_2(s|x^1) \geq \text{Prob}\left(\theta^2 \in H_p^-(x^2)|\theta^1 = x^1\right) \tag{6.3}
\]

and

\[
\alpha_1(s|x^2) \geq \text{Prob}\left(\theta^1 \in B^*|\theta^2 = x^2\right) \tag{6.4}
\]

We want to show that the RHS of (6.4) is approximately equal to \(\text{Prob}\left(\theta^1 \in H_p^- (x^1)|\theta^2 = x^2\right)\). Specifically, we will show that there exists a constant \(\tilde{k}\), independent of \(\epsilon\), such that for all \(x \in B(x^1, \epsilon)\)

\[
\text{Prob}\left(\theta^1 \in H_p^- (x^1) \backslash B^*|\theta^2 = x\right) \leq \tilde{k}\epsilon \tag{6.5}
\]
To prove (6.5) note first that for all such $x$, the intersection of the set from (6.5) with $B(x, \varepsilon)$ is contained in $B(x^1, 2\varepsilon)$. Write $D := (H_p^-(x^1)) \setminus B \cap B(x^1, 2\varepsilon)$ and let $\delta$ be such that the ball with radius $\eta + \delta$ centered at $x^*$ just contains $D$ (see Figure 4). By the Pythagorean theorem $(\eta + \delta)^2 = \eta^2 + 4\varepsilon^2$, hence $\delta \leq 2\varepsilon^2/\eta$. Furthermore, one sees that, $\mu(D)$, the Lebesgue measure of $D$ is bounded above by $(4\varepsilon)^{m-1}\delta$, hence, $\mu(D) \leq 4^m\varepsilon^{m+1}/\eta$. Let $M$ be an upper bound for the density $\psi$ of the difference $e^1 - e^2$. Noting that $\varepsilon^{-m}M$ is an upper bound for $\psi^e$, we see from (5.3) that for all $x \in B(x^1, \varepsilon)$

$$
\text{Prob} (\theta^1 \in H_p^-(x^1) \setminus B^* | \theta^2 = x) \leq \text{Prob} (\theta^1 \in D | \theta^2 = x) \leq \mu(D)\varepsilon^{-m}M \leq 4^mM\varepsilon/\eta
$$

which proves (6.5).

**Step 4. Wrapping up.**

From (6.3) - (6.5) and Lemma 2 we may conclude that there exists a constant $\tilde{k}$, independent of $\varepsilon$, such that

$$
\alpha_2(s|x^1) + \alpha_1(s|x^2) \geq 1 - \tilde{k}\varepsilon
$$

Lemma 4 and (6.2) therefore show that there exists another constant $k$, again independent of $\varepsilon$ such that

$$
\tilde{a}_1(x^1) + \tilde{a}_2(x^2) \geq 1 - k\varepsilon
$$

Let $\bar{x} \in K$ be a limit point of $x^1$ as $\varepsilon$ tends to zero. Then $\bar{x}$ is also a limit point of $x^2$ (Lemma 5) and since $\tilde{a}_1$ and $\tilde{a}_2$ are continuous on $R_\alpha$ we have
\[ \bar{a}_1(\bar{x}) + \bar{a}_2(\bar{x}) \geq 1 \]

but this contradicts (6.1). \[\Box\]

7 The One-Dimensional Case: Dominance Solvability

It is worthwhile to consider the special case where \( \Theta \) is one-dimensional in more detail since in this case, by making a slightly more restrictive regularity assumption, we can prove a considerably stronger result, viz. that the global game \( \Gamma^* \) is almost strictly dominance solvable. Hence, in order to justify that players should coordinate on the equilibrium \( \alpha \) if \( \theta \in R_\alpha \), we do not have to rely on the assumption that players should play an equilibrium of \( \Gamma^* \), we arrive at this result by iterative elimination of strictly dominated strategies in \( \Gamma^* \). Given that the notion of rationalizability (Bernheim (1984), Pearce (1984)) is generally considered to be less objectionable than the Nash concept, this result is of considerable independent interest.

The intuition for this result is actually quite simple and can be illustrated by means of the family of games from Fig. 1b. Obviously \( \beta_i \) (resp. \( \alpha_i \)) is a strictly dominant strategy for each player if \( \theta \) is sufficiently large, say \( \theta > \bar{x}_1 \), (resp. if \( \theta \) is sufficiently small, say \( \theta < \underline{x}_1 \)). Consider an observation \( \theta^i \) of player \( i \) slightly below \( \bar{x}_1 \). Player \( i \) knows that his opponent will play \( \beta_j \) if \( \theta^j > \bar{x}_1 \), hence \( i \)'s payoff if he chooses \( \beta_i \) at \( \theta^i \) is at least (approximately) \( \theta^i \) times the probability that \( \theta^j \) is above \( \bar{x}_1 \). For \( \theta^i \) close to \( \bar{x}_1 \) this payoff is approximately \( \bar{x}_1/2 \). A similar reasoning shows that the expected payoff to \( \alpha_i \) is at most approximately \( (1 - \bar{x}_1)/2 \). Hence, if \( \bar{x}_1 > 1/2 \), then there exists \( \bar{x}_2 < \bar{x}_1 \) such that \( \beta_i \) is strictly dominant for \( \theta^i > \bar{x}_2 \) in the reduced game where player \( j \) is constrained to play \( \beta_j \) if \( \theta^j > \bar{x}_2 \). In a similar way we can construct \( \bar{x}_2 > \bar{x}_1 \) and continuing inductively we find sequences \( \underline{x}_n \) and \( \bar{x}_n \) such that \( \alpha_i \) is iteratively dominant if \( \theta^i < \underline{x}_n \) while \( \beta_i \) is iteratively dominant for \( \theta^i > \bar{x}_n \). From the above argument it is also clear that when \( \varepsilon \) is small then \( \bar{x}_n \) and \( \underline{x}_n \) must be approximately \( 1/2 \) for \( n \) large.
Next, we formalize the above intuition. Suppose the Assumptions 1 and 2 hold and assume, without loss of generality, that there exist $\theta' \in D_\alpha$, $\theta'' \in D_\beta$ with $\theta' < \theta''$. Assumption 2 then implies that there exist $\theta_\alpha, \theta_\beta, \theta^*, \bar{\theta}^*$ with $\theta < \theta_\alpha < \theta^* < \bar{\theta}^* < \theta_\beta < \bar{\theta}$ such that

$$\Theta_\alpha = (\theta, \theta_\beta) \quad D_\alpha = (\theta, \theta_\alpha) \quad R_\alpha = (\theta, \theta^*),$$
$$\Theta_\beta = (\theta_\alpha, \bar{\theta}) \quad D_\beta = (\theta_\beta, \bar{\theta}) \quad R_\beta = (\bar{\theta}^*, \bar{\theta}).$$

$$\bar{a}_1(\theta) + \bar{a}_2(\theta) = 1 \quad \text{for} \quad \theta \in [\theta^*, \bar{\theta}^*]$$

Note that, in the examples from Section 1, $\theta^* = \bar{\theta}^*$ and that all games in $[\theta^*, \bar{\theta}^*]$ have the same best reply structure as a completely symmetric game (cf. Fn 1), in particular, all the games in this interval are best reply equivalent to each other. Hence, the condition for dominance solvability in the second part of Theorem 2 is only a mild one.

Note that $v_i^\alpha(\theta_\alpha) > 0$ and $v_i^\beta(\theta_\beta) \geq 0$ for all $i$, and that similar inequalities hold at $\bar{\theta}$, so that $v_i^\alpha(\theta) + v_i^\beta(\theta)$ is strictly positive on a neighborhood of $[\theta_\alpha, \theta_\beta]$. Let $K^{-1}$ be a lower bound for this function on $[\theta_\alpha - 2\varepsilon, \theta_\beta + 2\varepsilon]$. For this interval let $k$ be as in (5.5) and let $L$ be an upper bound on the derivatives of $\bar{a}_i(\theta)$, $i = 1, 2$. Finally, the continuity of the $\bar{a}_i$ functions implies that for every $\eta > 0$ there exists $\delta(\eta) > 0$ with $\delta(\eta) \to 0$ as $\eta \to 0$ such that for all $\theta$

$$\text{if the distance between } \theta \text{ and } [\theta^*, \bar{\theta}^*] \text{ is at least } \delta(\eta),$$
$$\text{then } |\bar{a}_1(\theta) + \bar{a}_2(\theta) - 1| > \eta. \quad (7.1)$$

Let $S$ be a set of strategies in $\Gamma^\varepsilon$. We say that $\alpha_i$ (resp. $\beta_i$) is dominant at $\theta^i$ given $S$ if $V_i^\varepsilon(s|\theta^i) > 0$ (resp. $V_i^\varepsilon(s|\theta^i) < 0$) for all $s \in S$. Let $S^0$ be the set of all strategies and inductively define
\[ D_{r_i}^\varepsilon(S^k) := \{ \theta^i; \gamma_i \text{ is dominant at } \theta^i \text{ given } S^k \} \]

where \( i \in \{1, 2\}, \gamma \in \{\alpha, \beta\}, \) and

\[ S^{k+1} := \{ s \in S^k; s_i^\gamma(\theta^i) = 1 \text{ if } \theta^i \in D_{r_i}^\varepsilon(S^k), \text{ for all } i, \gamma \}. \]

We write \( \bar{D}_{r_i}^\varepsilon \) for the set of all observations at which \( r_i \) is iterated dominant

\[ \bar{D}_{r_i}^\varepsilon := \bigcup_k D_{r_i}^\varepsilon(S^k) \]

(Note that the sets \( D_{r_i}^\varepsilon(S^k) \) are ordered by set inclusion.) Finally, \( \bar{D}_{r_i}^\varepsilon \) is the set where \( r_i \) is iterated dominant

\[ \bar{D}_{r_i}^\varepsilon := \bar{D}_{r_1}^\varepsilon \cap \bar{D}_{r_2}^\varepsilon. \]

The main result of this section is

**Theorem 2.** For every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) with \( \lim_{\delta \to 0} \delta(\varepsilon) = 0 \) such that \( \bar{D}_\alpha^\varepsilon \supseteq (\theta, \theta^* - \delta) \) and \( \bar{D}_\beta^\varepsilon \supseteq (\bar{\theta}, \delta, \bar{\theta}). \) Hence, if \( \theta^* = \bar{\theta}, \) then the game \( \Gamma^\varepsilon \) is almost dominance solvable (it is dominance solvable in the limit). If \( \alpha \) (resp. \( \beta \)) risk dominates \( \beta \) (resp. \( \alpha \)) at \( \theta \), then only \( \alpha \) (resp. \( \beta \)) is rationalizable at \( \theta \) if \( \varepsilon \) is sufficiently small.

**Proof.** For \( \varepsilon > 0 \) let \( \eta = 2\varepsilon(MK + 2L + k) \) (where \( K, L \) and \( k \) are given above while \( M \) is defined as in Lemma 4) and let \( \delta = \delta(\eta) \) be as in (7.1). We will show that \( \bar{D}_\alpha^\varepsilon \supseteq (\theta, \theta^* - \delta) \).

Recall that in Lemma 6 we have shown that \( \bar{D}_\alpha^\varepsilon \supseteq D_\alpha^\varepsilon(S^0) \supseteq (\theta, \theta_\alpha - \varepsilon). \) We will show that

\[ \text{if } x \in [\theta_\alpha - \varepsilon, \theta^* - \delta) \text{ and } \bar{D}_\alpha^\varepsilon \supseteq (\theta, x), \]

then there exists \( x' > x \) such that \( \bar{D}_\alpha^\varepsilon \supseteq (\theta, x') \quad (7.2) \)
Clearly, (7.2) together with the initial step provided by Lemma 6 establish the proposition. Hence, let \( x \) satisfy the condition in (7.2). Write \( s_{ix} \) for the strategy defined by

\[
    s_{ix}(\theta^i) = \begin{cases} 
        1 & \text{if } \theta^i < x \\
        0 & \text{otherwise}
    \end{cases}
\]

and define \( s_x = (s_{1x}, s_{2x}) \). Since \( v_i^\theta(\theta) + v_i^\theta(\theta) \) is nonnegative for \( \theta \geq \theta_\alpha - 2\varepsilon \) we have for any strategy \( s_j \) with \( s_j(\theta^i) = 1 \) if \( \theta^i < x \)

\[
    V_i^x(s|\theta^i) \geq V_i^x(s_x|\theta^i) \quad \text{if } \theta^i \geq \theta_\alpha - \varepsilon \tag{7.3}
\]

Furthermore, we have

\[
    V_i^x(s_x|\theta^i) > F_i^x(x|\theta^i)(v_i^\theta(\theta^i) + v_i^\theta(\theta^i)) - v_i^\theta(\theta^i) - M\varepsilon \\
    > (v_i^\theta(\theta^i) + v_i^\theta(\theta^i))(F_i^x(x|\theta^i) - \bar{a}_j(\theta^i) - MK\varepsilon) \tag{7.4}
\]

Consider, for \( i \in \{1, 2\} \) and \( j \neq i \), the continuous function

\[
    \theta^i \to F_i^x(x|\theta^i) - \bar{a}_j(\theta^i) - MK\varepsilon \tag{7.5}
\]

defined on \((\theta, \bar{\theta})\). If \( \theta^i \) is small, the function value is positive while the value is negative if \( \theta^i \) is close to \( \theta_\beta \). Hence, there exists a zero. Note that, in view of (5.5), and because of the definition of \( \delta \)

\[
    (F_i^x(x|x) - \bar{a}_2(x) - MK\varepsilon) + (F_2^x(x|x) - \bar{a}_1(x) - MK\varepsilon) \\
    \geq 1 - 2k\varepsilon - \bar{a}_1(x) - \bar{a}_2(x) - 2MK\varepsilon > 0,
\]

so that at least one of the functions defined in (7.5) is strictly positive at \( x \) and hence must have a zero to the right of \( x \). Assume, w.l.o.g., that the function with a zero to the right of \( x \) has the index \( i = 1 \) and let \( x_1^* \) be the smallest such zero. Hence
and \( x_1 > x \) with \( \tilde{D}_{x_{1 \alpha}} \supset (\emptyset, x_1) \). Let \( x_2 \) be the smallest value such that

\[
F_2 (x_1 | x_2) = \tilde{a}_1 (x_2) + MK \epsilon.
\]  

(7.7)

Then \( \tilde{D}_{x_{2 \alpha}} \supset (\emptyset, x_2) \) so that the proof is complete if we can show that \( x_2 > x \). Assume \( x_2 \leq x \). Then from (7.6) we obtain

\[
F_1 (x_1 | x_2) \leq \bar{a}_1 (x_1) + MK \epsilon,
\]

and combining this latter inequality with (7.7) and Lemma 2 yields

\[
1 - 2k \epsilon \leq \bar{a}_1 (x_1) + \bar{a}_2 (x_1) + 2MK \epsilon.
\]

(7.8)

Now it follows from (7.7) that \( x_1 \) and \( x_2 \) cannot be more than \( 2 \epsilon \) apart (because in this case the LHS is either 0 or 1), hence \( x_1 \) and \( x_2 \) are at most \( 2 \epsilon \) from \( x \). Therefore, (7.8) yields

\[
1 \leq \bar{a}_1 (x) + \bar{a}_2 (x) + 2 \epsilon (MK + 2L + k)
\]

but this contradicts (7.2) given the definition of \( \delta \). Hence we must have \( x_2 > x \) and \( x' > x \) can be constructed. \( \Box \)

8 Discussion

As is clear from the proofs, our main result is driven by the fact that the regions \( D_{\alpha} \), resp. \( D_{\beta} \) of dominance solvable games with solution \( \alpha \), resp. \( \beta \), exert a remote influence
on the region $\Theta_{\alpha\beta}$ where both $\alpha$ and $\beta$ are strict equilibria and in this way they determine the equilibrium selected in that region. To illustrate that the nonemptiness of $D_\alpha$ and $D_\beta$ is necessary for our result to hold, consider a global game $\Gamma^*$ based on the class of games $g(\theta)$ from Fig. 5 with $\theta$ being uniform on $(\bar{\theta}, \bar{\theta})$. (Note that $\alpha$ risk dominates $\beta$ if and only if $\theta < 4$ but that $\alpha$ payoff dominates $\beta$ whenever $\alpha$ is an equilibrium. If $\theta > 8$, $\alpha$ is not an equilibrium and for $\theta \in (8, 9)$ the game is a prisoner's dilemma.)

If $\bar{\theta} \leq 8$, then $\alpha$ is a strict equilibrium of $g(\theta)$ for every value of $\theta$, hence, in this case the global game $\Gamma^*$ trivially has a Bayesian Nash equilibrium where each player chooses $\alpha_i$ for each observation $\theta^i$. Similarly, $\Gamma^*$ has an equilibrium in which players always choose $\beta$ if $\theta \geq 0$. As long as $4 \in (\bar{\theta}, \bar{\theta})$ and $\varepsilon$ is small, the game $\Gamma^*$ also has an equilibrium in which the players coordinate (approximately) on the risk dominant equilibrium of $g(\theta)$ for each $\theta$, i.e. they choose $\beta$ iff they observe $\theta^i > 4$ ($i = 1, 2$). (The latter result follows easily from the Propositions 1 and 2, since, if $4 \in (\bar{\theta}, \bar{\theta})$, then the posterior beliefs in the neighborhood of 4 do not depend on $\theta$ and $\bar{\theta}$.) We see that the deletion of a strong dominance region may result in non-uniqueness of the limit equilibria. On the other hand our result may be paraphrased by saying that if the initial uncertainty is sufficiently large (in the sense made precise by the Assumptions 1 and 2), then the limit equilibrium necessarily coincides with the risk dominant equilibrium for each observation.

In our view, the results of this paper add strongly to the intuitive appeal of the risk dominance criterion.\textsuperscript{10} An objection that is frequently raised against risk dominance is

\textsuperscript{10}Note especially that our justification for selecting in accordance with risk dominance does not depend on the prior being uniform. Harsanyi and Selten, in their justification, (cf. Fn 8) truncate the hierarchy of beliefs by the ad hoc assumption of uniformity at the second level. In this case, different truncations produce different outcomes. For example, in the game of Fig. 5 with (it being common knowledge that) $\theta = 7$, if each player $i$ assigns a sufficiently high probability to player $j$ believing that $i$ will play $\alpha_i$, then players will finally coordinate on the Pareto dominant equilibrium $\alpha_i$. (Formally,
that it may lead to the selection of an equilibrium that is Pareto dominated by another equilibrium. This conflict between payoff dominance and risk dominance also occurs in the class of games from Fig. 5. Several authors, including Harsanyi and Selten, argue that precedence should be given to payoff dominance in case of such a conflict. On p. 356 of Harsanyi and Selten (1988) this choice is motivated as follows:

"(...) risk dominance is important only in those situations where the players would be initially uncertain whether the other players would choose one equilibrium or the other. Yet, if one equilibrium would give all players higher payoffs than the other would (...) every player can be quite certain that the other players will opt for this equilibrium which will make risk dominance considerations irrelevant." (Harsanyi and Selten (1988, p. 356), emphasis added)

The global games approach is not incompatible with this point of view for, if $\Theta$ is sufficiently small, the global game admits an equilibrium that selects the Pareto dominant equilibrium for each game $g(\theta)$. (See the above discussion of Fig. 5.) However, we do show that if the initial uncertainty is sufficiently large, players can never be “quite certain” that the others will opt for the Pareto dominant equilibrium: Coordination on the Pareto dominant equilibrium for each $g(\theta)$ cannot be an equilibrium of the global game. If, in Fig. 5, player $i$ would switch at $\theta^i = 8$, then player $j$’s best response is to switch at a slightly lower value and then in turn $i$ wants to switch at yet a lower value, etc. (See the proof of Theorem 2.)

In this context it is also worthwhile to quote Luce and Raiffa (1957, p. 110) in which the notion of risk dominance is discussed informally under the name of “psychological dominance”. Luce and Raiffa discuss the game from Fig. 6, first with $x = 8$ and then with $x = 12$. (For our convenience, we have relabeled the strategies.)

using the notation from Fn. 8, if $z$ is distributed according to $F(z) = z^2$ (which represents an initial bias in favor of Pareto dominance), then the argument from that footnote leads to the equilibrium $\alpha$.

We do not truncate the hierarchy of beliefs: An equilibrium of the global game determines beliefs at every level.
Luce and Raiffa write that in the game from Fig. 6 with \( x = 8 \), the equilibrium \( \alpha \) "psychologically dominates" \( \beta \), for "if 2 has any reason to fear that 1 will take \( \alpha_1 \), then he dare not take \( \beta_2 \) in fear of getting \(-300\), but 1 knowing this has every reason to take \( \alpha_1 \), which gives him his best return. But now the argument is cyclic, for 2, having some rationalization for 1’s adoption of \( \alpha_1 \), has all the more reason to avoid \( \beta_2 \)." They then go on to discuss the game with \( x = 12 \) of which they note that \( \beta \) payoff dominates \( \alpha \). They write (using our labeling of strategies)

"Yet if we were player 1 we would hesitate to use \( \beta_1 \) on the grounds that player 2 would argue that \( \alpha_2 \) psychologically dominates \( \beta_2 \), and as long as 2 can give any rationale for 1’s choosing \( \alpha_1 \), 2 does not dare choose \( \beta_2 \). The argument is cyclic and it reinforces \( \alpha \) even though it is jointly dominated by \( \beta \)." (Luce and Raiffa (1957, p. 110), emphasis added.)

The Harsanyi/Selten point of view is that, if (joint) rationality is common knowledge, 2 can give no rationale for 1’s choosing \( \alpha_1 \) and then \( \beta \) is a perfectly acceptable solution. In a global game based on the class from Fig. 6, however, it is easy to rationalize 1’s choice of \( \alpha_1 \): Player 1 may believe that \( x \) is low and that \( \alpha_1 \) is a dominant strategy.

Nevertheless at first it appears somewhat paradoxical that, for example in the game\(^{11}\) from Fig. 5 with \( \theta = 7 \), players are forced to select \( \beta \) even for small \( \epsilon \), although they know that a uniformly better equilibrium is available. Upon closer investigation this is not paradoxical at all, it is a most natural consequence of the informational conditions under

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\(^{11}\)The game from Fig. 5 with \( \theta = 7 \) is due to Aumann who argues that, if players are initially convinced that they should play \( \beta \), no amount of preplay communication can convince them that they should switch to \( \alpha \). (The point is that player i always gains if he can get his opponent to play \( \alpha_j \).) In the global game, communication (after players have received their private information) might make a difference. We have not yet performed a formal analysis of this interesting issue.
which the global game is played. The key to the understanding of the phenomenon is the realization that there is a sharp separation between knowledge and common knowledge in global games. Even if in the global game \( \Gamma^\varepsilon \) with \( \varepsilon \) small, players know upon observing \( \theta^i \approx 7 \) that \( \alpha \) payoff dominates \( \beta \), this fact is not common knowledge\(^{12} \). In fact, it cannot even be common knowledge that \( \alpha \) is an equilibrium of \( g(\theta) \), or more precisely, that \( \alpha \) is a best response against \( \alpha_j \) \( (i, j \in \{1, 2\}, i \neq j) \). Namely, let \( E \) be the event that \( \alpha \) is a best response against \( \alpha_i \), i.e. \( E = \{ \omega \in \Omega; \theta \leq 8 \} \). Player \( i \) knows that \( E \) occurs if and only if \( \theta^i \leq 8 - \varepsilon /2 \), player \( i \) knows that \( j \) knows that \( E \) occurs if and only if \( \theta^i \leq 8 - 3\varepsilon /2 \). Continuing inductively one sees that, if \( \varepsilon > 0 \), the event \( E \) cannot be common knowledge. The phenomenon that lack of common knowledge enables remote areas to exert an influence also supplies a deeper motivation for using the term global games: When determining rational behavior in a given game situation it is not enough to look at the equilibrium structure that is known to prevail, it must also be ensured that the chosen action is part of a consistent plan for all situations that could have occurred in the underlying class of games.

Returning to the role played by the assumptions, let us note the importance of the prior having a density. Specifically it is important that the class of games is connected and that each game in \( \Theta_{\alpha, \beta} \) can be connected continuously to some games in \( D_\alpha \cup D_\beta \). (Hence, also the connectedness assumption 1d is important, but there is no need to be concerned about this one: All the examples that we studied satisfy 1d, in particular, this holds for the ‘natural’ parametrization \( \Theta = \mathbb{R}^3 \) and \( g(\theta) = \Theta \).) To see this most clearly, consider a global game based on Fig. 5 where \( \Theta \) is a discrete subset of \( \mathbb{R} \). A natural way of letting the observation errors vanish is by requiring that

\[
\text{Prob}_\varepsilon^i(\theta = x|\theta^i = x) \to 1 \quad \text{as} \quad \varepsilon \to 0 \quad \text{(all) x}
\]  

\(^{12}\)The definition of common knowledge is given in Aumann (1976). For the extension to continuous state spaces, see Brandenburger and Dekel (1987). In our special case the definition is as follows. Let \( \Omega = \{ \omega; \omega = (\theta, \theta^1, \theta^2) \} \) be the state space and let \( E \) be an event. Denote by \( KE \) the event that both players know \( E \), i.e. \( KE = \{ \omega; F^1_1(E|\theta^1) = 1 \text{ and } F^2_2(E|\theta^2) = 1 \} \), and write \( K^{n+1}E = K(K^nE) \). The event \( E \) is said to be common knowledge at all states \( \omega \) belonging to \( \cap_n K^nE \).
Hence, in $\Gamma^*$, each player’s observation coincides with the actual game with probability close to 1. Let $\theta \in \Theta$. If player $i$ observes $\theta$ and $\varepsilon$ is small, then he knows that the actual game is very likely to be $\theta$ and he also knows that his opponent very likely made the same observation as he did. This enables players to analyse the game $g(\theta)$ as if it would occur in isolation, the games $\{g(\theta') : \theta' \neq \theta\}$ just add some minor noise which does not influence the decision too much if both $\alpha$ and $\beta$ are strict equilibria (i.e. if $\theta \in \Theta_{\alpha\beta}$). Hence, if one expects the opponent to play $\alpha$ (resp. $\beta$) at $\theta \in \Theta_{\alpha\beta}$ then in the global game $\Gamma^*$ with $\varepsilon$ small, the best response will be to also choose $\alpha$ (resp. $\beta$): The global game admits many equilibria and for $\theta \in \Theta_{\alpha\beta}$ one can obtain any selection in the limit. (In the limit one can even obtain the mixed equilibrium of $g(\theta)$ if $\theta \in \Theta_{\alpha\beta}$.)

Of course, the above is not really surprising. What is (perhaps) more surprising is that, in the case of discrete $\Theta$, the results no longer hold even if there is a real link between neighboring games. To see this, assume $\Theta$ is a finite subset of $[-1, 9]$, write $\theta^+$ (resp. $\theta^-$) for the smallest (resp. largest) value in $\Theta$ larger (resp. smaller) than $\theta$, assume $\theta^+ - \theta^- < \varepsilon$ for all $\theta$ and let $g(\theta)$ be as in Fig. 5. Consider the global game $\Gamma$ in which $\theta$ is drawn from the uniform prior on $\Theta$ and where player 1 always observes the value of $\theta$ exactly, but where (if $\theta$ is not an endpoint of $\Theta$) player 2 either observes $\theta^+$ (with probability $\lambda$) or $\theta^-$ (with probability $1 - \lambda$). Assume $\lambda \geq \frac{1}{2}$ and assume $x \in \Theta$ is such that $8(1 - \lambda) < x < 8\lambda$. Then one easily verifies that, for small enough $\varepsilon$, the strategy combination $s$ given by

$$s(\theta) = \begin{cases} (\alpha_1, \alpha_2) & \text{if } \theta < x \\ (\beta_1, \alpha_2) & \text{if } \theta = x \\ (\beta_1, \beta_2) & \text{if } \theta > x \end{cases} \quad (8.2)$$

is an equilibrium of the global game. Hence, if $\lambda > \frac{1}{2}$, then the “limit equilibrium” as the grid size $\varepsilon$ of $\Theta$ tends to zero is not unique, (although one certainly can only approximate the risk dominant equilibrium if the risk dominance margin is large enough) and one gets results that necessarily are in agreement with risk dominance only if the initial situation
is symmetric, i.e. if $\lambda = \frac{1}{2}$. The reason for the discrepancy between the discrete and the continuous model is that in the former model there need not exist a switching point $\theta^i$ with $V_i^c(s|\theta^i) = 0$; in the continuum case such a switching point always exists since $V_i^c$ is continuous in $\theta^i$.

The above discussion also makes clear that the assumptions that the errors are independent of the true game, that they have densities and that the noise enters additively, are to some extent relevant to establish our main result. Certainly, if the errors were perfectly correlated with $\theta$, or if a player's observation would be completely uninformative, the results would be completely different. On the other hand, it seems that Assumption 2b and (4.1) can be weakened without destroying the main result. We did not yet pursue this avenue of research. It should be stressed, however, that Assumption 2c, viz. that the error has a bounded support, has been made only to simplify some of the arguments somewhat; this assumption is not essential.

Of course, the main restriction of the paper is that the results only cover $2 \times 2$ games. In the final section we return to the issue of whether this assumption can be relaxed.

## 9 Relation to the Literature

The approach taken in this paper is based on two related but distinct ideas.

(i) The standard assumption in game theory that not only the game structure but also the game's parameters are common knowledge is too strong.

(ii) Players do not analyse each game separately, rather they analyse classes of games with similar characteristics simultaneously, and they search for equilibrium rules rather than equilibrium actions.

It should be clear that the second idea and the recognition that extra mileage can be obtained by exploiting it, i.e. by requiring that a solution should be part of a plan that
is consistent across a larger domain is motivated by the seminal work of Nash (1953) on bargaining and that of Schelling (1960) on focal points. Schelling argued forcefully (and convincingly) that the context in which the game is played (in our setup the class of games) may provide invaluable clues of how to solve each game played in this context. Nash realized that by imposing consistency requirements on a bargaining solution across a large enough domain, one can obtain a unique solution for each bargaining game. A recent paper that also tries to exploit the idea that players will analyze similar games in a similar way and in which (as in this paper) games are said to be similar if their payoffs are close together is Fudenberg and Kreps (1990). (Also see Kreps (1990, Chapter 6) in particular the discussion of Fig. 6.4.) Their (preliminary) results are not incompatible with ours.\textsuperscript{13} Also Harsanyi and Selten (1988, Sect. 3.8) in their axiomatic characterization of risk dominance for $2 \times 2$ games follow the approach of linking solutions of different games to each other.

However, in contrast with Harsanyi and Selten, we stay entirely within a noncooperative framework. Moreover, our results are obtained by relaxing the rather restrictive informational assumptions of traditional game theory. Admittedly, it is still an open question whether global games provide an appropriate model of uncertainty in game situations. In particular, the assumption that the rules of a global game are common knowledge may appear unduly rationalistic. An important task for future research will be to investigate whether this assumption can also be relaxed. We feel that the result of Theorem 2 in conjunction with recent progress with learning models indicates that this may indeed be possible. In particular, Milgrom and Roberts (1989) show that, for a great variety of learning processes, the sequence of strategy choices will eventually be confined to the set of strategies which survives iterated removal of strongly dominated strategies.

Nash (1953) also deserves credit for contributing the idea that by adding (continuous) noise to the game one can strongly reduce the number of equilibria. Consider the simple

\textsuperscript{13}We became aware of the work of Fudenberg and Kreps only after the first version of this paper was completed.
bargaining game in which two risk-neutral players have to divide \( \theta \). Simultaneously, each player states a demand \( d_i \) and player \( i \) receives \( d_i \) if \( d_1 + d_2 \leq \theta \). (If demands are incompatible each player gets zero.) There are infinitely many equilibria to the demand game, but Nash shows that if there is slight (continuous) common uncertainty about \( \theta \), then each player will demand approximately \( \theta/2 \). Although Nash's approach was not completely successful (any particular perturbed game still may have many equilibria, but only \( \theta/2, \theta/2 \) is a necessary limit of equilibria of the perturbed game no matter what the perturbation is), his basic intuition has been confirmed in recent studies by Binmore (1987) and Carlsson (1987). These authors assume that \( \theta \) is common knowledge but they introduce noise by assuming that players may tremble. Each player's demand gets slightly perturbed: If player \( i \) intends to demand \( \tilde{d}_i \), then his actual demand is \( d_i = \tilde{d}_i + \epsilon e^i \) where \( e^i \) has an atomless density. Binmore and Carlsson show that if \( \epsilon \) is small, each player demands approximately \( \theta/2 \) — or more generally, the Nash-solution when utilities are non-linear — in any equilibrium of the perturbed game. It should be noted that the mathematical analysis for this type of games is very similar to that of simple one-dimensional global games. (Specifically, consider \( 2 \times 2 \) unanimity games where the diagonal payoffs are \( (\theta, 1) \) and \( (1, 1 - \theta) \), respectively). Of course there is also an obvious analogy between equilibrium selection according to risk dominance in \( 2 \times 2 \) games and the Nash solution in the demand game: Both selection criteria are based on the maximization of Nash products.

Games, like all models, are idealizations that abstract away from many of the imperfections of real-life situations. Sometimes the idealization is carried on too far, too many relevant aspects are abstracted away from. Multiplicity of equilibria, as well as nonexistence of equilibria in pure strategies may be viewed as manifestations that the model under consideration is overidealized. Since the publication of the seminal Selten (1975) paper on trembling hand perfection it has been realized that including additional elements into the model (i.e. perturbing the game slightly) may enable to cut down on the number of solutions. In this paper, while retaining the assumption that the game structure is common knowledge, we slightly relax the standard assumption that all parameters
of the game are common knowledge. In this way, the paper fits into the refinements pro-
gram that was initiated by Selten's paper. This program has recently been criticized by
Fudenberg et.al. (1988). They argue that an equilibrium that is unreasonable (i.e. that
is eliminated by refinements) in a given game may not be unreasonable in nearby games
so the analyst may be wise to have second thoughts about rejecting this outcome unless
he has absolute faith in the model (i.e. in the game that he analyses). These authors
take the point of view that every strict equilibrium is reasonable and they roughly show
that every normal form perfect equilibrium can be approximated by strict equilibria of
"nearby" games, hence, that any such equilibrium is reasonable as well. Technically,
their paper differs from ours in the definition of nearness of games and (in their Section
3) in the assumption that the analyst knows much less about the game than the players
do. (In their Section 3, Fudenberg et.al. assume that only the analyst does not know
the payoffs, the payoffs are, however, common knowledge among the players themselves.)

In the previous section we already discussed the question of whether a global game
\( \Gamma^e \) in which the players make observation errors of order \( \varepsilon \) is actually only a 'slight'
perturbation of the game \( \Gamma^0 \) in which the players can observe the parameter vector \( \theta \)
extactly, hence, in which the game \( g(\theta) \) that has to be played is common knowledge for
each observation \( \theta \). (In \( \Gamma^0 \) each game \( g(\theta) \) occurs as a subgame.) We have seen that this
is a delicate issue because of the intricacies associated with the concept of knowledge
and since situations of common knowledge are difficult to visualize. On this issue our
paper is related to Rubinstein (1989).

Consider again the game from Fig. 2 discussed in the Introduction and write
\( A = G_2(-10) \) and \( B = G_2(10) \). Then \( A \) is dominance solvable with solution \( \alpha \), while
in \( B \) \( \beta \) payoff dominates \( \alpha \) but \( \alpha \) risk dominates \( \beta \). Rubinstein considers the following
situation.\(^{14}\) First one of the games, \( A \) or \( B \), is selected, each with probability \( 1/2 \). Player
1 always gets to hear which game is played. If the game is \( A \), player 2 does not get to

\(^{14}\)Our games are slightly different from Rubinstein's but the arguments and the 'paradox' are exactly
the same.
hear it. If the game is $B$, player 1 automatically sends a message to 2 (saying the game is $B$) and 2, upon receiving the message, automatically acknowledges the receipt, an acknowledgement which when received by 1 is again automatically acknowledged, etc. The communication technology, however, is slightly imperfect: each message gets lost with probability $\varepsilon$. Rubinstein shows that there is only one Nash equilibrium in which player 1 chooses $\alpha_1$ in state $A$ and that, in this equilibrium, players choose $\alpha$ irrespective of how many messages they receive. Hence, also in this example the game $A$ (which roughly corresponds to zero messages) exerts a remote influence on game $B$ (roughly infinitely many messages). The proof is simple (induction on the number of messages) and relies on the fact that in game $B$ it is optimal to choose $\alpha_i$ if player $i$ expects his opponent to randomize equally. Hence, the crucial aspect is that, in game $B$, $\bar{a}_1 = \bar{a}_2 > \frac{1}{2}$, i.e. $\beta$ risk dominates $\alpha$. If we substitute higher payoffs $-L(0 < L < 100)$ for the two $-100$ entries in Fig. 2 and, specifically, if we choose $L < 20$, then $\beta$ risk dominates $\alpha$ and there exists an equilibrium where player $i$ chooses $\alpha_i$ if he doesn't receive a message while he chooses $\beta_i$ if he receives at least one message.\(^{15}\)

Rubinstein considers this example, and in particular the phenomenon that the perturbation excludes the equilibrium that is (in Rubinstein’s opinion) most reasonable, paradoxical since it shows

"(...) that the game theoretic “prediction” for the “almost common knowledge” situation is very different from the situation with common knowledge."

(Rubinstein (1989, p.385.)

We are not convinced. We do not know what the game theoretic ‘prediction’ is for the common knowledge situation and we consider it premature to identify this ‘prediction’ with the Pareto dominant equilibrium. In our view, if there is a conflict between Pareto dominance and risk dominance, the prediction should depend on the context in which the game is played. It is certainly easy to visualize situations where the context

\(^{15}\)This last result depends on the fact that the prior probability of $A$ is $\frac{1}{2}$. If this prior probability $\lambda$ would be higher, then one can switch from $\alpha$ to $\beta$ only if $L < 10/\lambda$. Hence, this model is very much like the discrete model discussed at the end of the previous section.
forces players to choose the risk dominant equilibrium. We have described one set of circumstances in this paper. An experimental setting where players choose the risk dominant equilibrium rather than the Pareto dominant one is described in Van Huyck et al. (1990). Rubinstein's setting is just another example of such a situation.

Another paper dealing with issues of knowledge and common knowledge and with the discontinuity in Rubinstein's example is Monderer and Samet (1989). These authors study (using our terminology) $\varepsilon$-equilibria of global games based on a finite state space $\Theta$. They generalize the notion of knowledge to that of belief and of common knowledge to common $p$-belief ($p \in [0, 1]$). (Common 1-belief is almost the same as common knowledge.) Roughly speaking they show that, if the set of states of the world for which there is a game that is common $p$-believed is large enough, then for any selection $s(\theta)$ with $s(\theta) \in E(g(\theta))$ there exists an $\varepsilon$-equilibrium of the global game for which the payoffs are close to the payoffs that would result if each game $g(\theta)$ were common knowledge with $s(\theta)$ being played in $g(\theta)$. (Hence, the payoffs are close to $\int g(\theta, s(\theta)) h(\theta) d\theta$.) The difference with our paper is that we work with exact equilibria and with a continuous state space. The latter is important as we have already seen in the previous section. Indeed, in our global game, only the set of all games is common $p$-believed for any $p > \frac{1}{2}$.

10 Comparison with Harsanyi's Games with Randomly Disturbed Payoffs

The paper that is most closely related to ours is Harsanyi's (1973) paper on games with randomly disturbed payoffs. In this paper Harsanyi notes that there is no need for players to actively randomize if there is slight uncertainty about the payoffs of the game: A mixed strategy equilibrium of a normal form game can be interpreted as a somewhat imprecise description of a pure strategy equilibrium of a larger (perturbed) game that takes this uncertainty explicitly into account. We now describe the similarities and differences between Harsanyi's paper and ours. (Harsanyi (1973) covers general $n$-person normal form games; we will restrict attention to 2-player $2 \times 2$ games to avoid additional
Note that the class of all $2 \times 2$ games can be viewed as an 8-dimensional space. Writing $\Theta = \mathbb{R}^8$ we can identify $\theta$ with $g(\theta)$ and we may write $\theta = (\theta_1, \theta_2) = (g_1(\theta), g_2(\theta))$ with $\theta_i \in \mathbb{R}^4$. Let $\theta^* = (\theta_1^*, \theta_2^*) \in \Theta$ and let $e_i$ be a random variable taking values in $\mathbb{R}^4$ with a distribution function $H_i$ that admits a continuous density $h_i$ that is positive only in an $\varepsilon$-neighborhood of 0. Consider the random game $\theta^*$, defined by $\theta^*_i = \theta_i^* + \varepsilon e_i$ in which players are almost sure that they play the game $\theta^*$. Harsanyi considers the sequence of global games $\{G^*\}_{\varepsilon > 0}$ where $G^*$ is defined by the following rules

\begin{align}
(e_1, e_2) & \text{ is drawn from } H_1 \times H_2 \\
\text{player } i \text{ is informed about } \theta^*_i &= \theta^* + \varepsilon e_i \\
\text{players simultaneously choose actions} \\
\text{payoffs are determined according to } \theta^* 
\end{align}

(10.1) (10.2) (10.3) (10.4)

Comparing Harsanyi's sequence $\{G^*\}_{\varepsilon > 0}$ with our sequence $\{\Gamma^*\}_{\varepsilon > 0}$ one sees that in both the uncertainty vanishes about which game is played, but that the uncertainty vanishes in different ways. The models also differ in the signals that the players receive and Harsanyi's model in addition assumes independence of the players' payoffs. Specifically the differences are

1. Harsanyi lets the prior uncertainty vanish. In our approximating sequence the prior remains constant but, ex post, after the players have made their observations the residual uncertainty vanishes for each observation.

2. In Harsanyi's setup players' payoffs are independently distributed. In our model payoffs may be correlated, but we do not exclude the independent case.

Harsanyi makes some additional assumptions, e.g. that different components of $e_i$ are independent. These assumptions are not essential.
3. In Harsanyi's model players learn their own payoffs exactly but they don't receive information about the opponent's payoff. Since payoffs are independent, a player's observation doesn't tell him anything about the opponent's payoffs. In our model, players make (imperfect) observations on the entire game, i.e. they learn something about both players' payoffs. In addition, players' observations are correlated.

A final important difference is that we are interested in pointwise convergence of the equilibrium strategies $s^*$ of the global game $\Gamma^*$, whereas Harsanyi studies convergence of averages. To put it differently, in Harsanyi's model equilibria should be interpreted as beliefs. Formally, let player $i$'s beliefs induced by the strategy profile $s$ from the global game, $\alpha_i^*(s|\theta^i)$, be defined as in (5.6) and note that in Harsanyi's case these beliefs are independent of the observation $\theta^i$ so that we may write $\alpha_i^*(s|\theta^i) = \alpha_i^*(s)$. The final difference between the models is

4. Harsanyi investigates, for sequences $\{s^i\}_{i=1}^{\infty}$ with $s^i \in E(G^i)$, the limit behavior of the equilibrium beliefs $\alpha_i^*(s^i)$. We investigate for sequences of equilibria of global games the limit behavior of the equilibrium actions $s_i^*(\theta)$ for each observation $\theta$.

We think it is fair to say that, although the models look similar at first, these differences are actually considerable. Hence, it should not be surprising that the results are completely different as well. Harsanyi shows that for a $2 \times 2$ game with three equilibria (i.e. $\theta \in \Theta_{\alpha\beta}$) all three equilibrium beliefs can be approximated (including the beliefs associated with the mixed equilibrium), whereas with our model we can approximate only the risk dominant equilibrium. We conclude this section with an explicit example of a model that is a hybrid of ours and Harsanyi's, and that contains our model as well as Harsanyi's as extreme special cases. The mathematical analysis clearly brings out the differences between the various models. Consider the game with the payoff matrix from Figure 7

[insert Figure 7 about here]

and associated with it let us investigate the Bayesian game $\Gamma(\eta, \epsilon)$ described by the following rules
(i) Given are three independent random variables \( \theta, e_1 \) and \( e_2 \) each having a normal distribution with mean zero and variance 1,

(ii) \( \theta_i = \eta \theta + e \varepsilon_i \), the realization of \( \theta_i \) is revealed to player \( i \),

(iii) player \( i \) chooses between \( \alpha_i \) and \( \beta_i \) (players choose simultaneously),

(iv) players receive payoffs as in the game from Fig. 7.

This model is a hybrid of ours and Harsanyi's: each player knows his own payoffs, but knowing the own payoffs gives information about the payoffs of the opponent. The case with fixed \( \eta \) and \( \varepsilon \) tending to zero corresponds more or less to our model, the case with \( \varepsilon \) large relative \( \eta \) corresponds more closely to Harsanyi's model. We will investigate the Bayesian equilibria of \( \Gamma(\eta, \varepsilon) \) and show that in the first case one indeed finds results as in this paper, whereas in the second case, one obtains results as in Harsanyi [1973].

Note that the game \( \Gamma(\eta, \varepsilon) \) is symmetric (if player \( i \) receives the information "\( \theta_i = x \)" he is in exactly the same situation as when player \( j \) receives the information "\( \theta_j = x \)"), so that it is natural to look for symmetric equilibria. Let us restrict attention to these\(^{17}\). Note that player \( i \) will play \( \alpha_i \) if \( \theta_i > 1 \) and that he will play \( \beta_i \) if \( \theta_i < 1 \). We will look for simple equilibria of the form

\[
\begin{align*}
s^2_i(\theta_i) &= \begin{cases} 
1 & \text{if } \theta_i > x \\
0 & \text{if } \theta_i < x 
\end{cases}
\end{align*}
\]

where \( s^2_i(\theta_i) \) is defined as in Sect. 4. The condition that player \( i \) is indifferent if \( \theta_i = x \) may be written as

\[
\frac{1 + x}{2} = \text{Prob}(\theta_j < x|\theta_i = x)
\]

Now, conditional on \( \theta_i \), taking the value \( x \), \( \theta_j \) is normally distributed with mean

\(^{17}\)One can show that this restriction is without loss of generality.
\[ \mu = \frac{\eta^2 x}{\eta^2 + \varepsilon^2} \]  
\[ (10.6) \]

and standard deviation
\[ \sigma = \sqrt{\frac{2\varepsilon^2 \eta^2 + \varepsilon^4}{\varepsilon^2 + \eta^2}} \]  
\[ (10.7) \]

so that (8.5) is equivalent to
\[ \frac{1 + x}{2} = \Phi \left( \frac{x - \mu}{\sigma} \right) \]

with \( \Phi \) being the standard normal distribution function. This last equation in turn is equivalent to
\[ \frac{1 + x}{2} = \Phi \left( x \sqrt{\varepsilon^2/(2\eta^2 + \varepsilon^2)(\eta^2 + \varepsilon^2)} \right) \]  
\[ (10.8) \]

One obvious solution is given by \( x = 0 \). One also sees that \( x \) is a solution if and only if \(-x\) is a solution. Finally, since \( \Phi \) is concave on \([0, \infty)\), we have that there is a unique solution if the derivative of the RHS of (8.8) evaluated at \( x = 0 \) is less than or equal to \( \frac{1}{2} \), and that there are three solutions if this derivative is larger than \( \frac{1}{2} \). Hence, the condition for a unique symmetric equilibrium is
\[ \frac{\varepsilon^2}{2\pi(2\eta^2 + \varepsilon^2)(\eta^2 + \varepsilon^2)} \leq \frac{1}{2}, \]

a condition that is satisfied if and only if \( \eta \) is relatively large compared to \( \varepsilon \).

Let us now analyse what happens when \( \eta \) and \( \varepsilon \) tend to zero. Note from (8.6) that \( \theta_1 \) and \( \theta_2 \) are independent in the limit if \( \eta^2/(\eta^2 + \varepsilon^2) \to 0 \), hence, in this case we approximate the situation considered by Ilarsanyi. From (8.8) we see that in this case the second (resp. third) solution converges to +1 (resp. −1). In the limit the switching
point \( x = 1 \) gives rise to beliefs that players play their second pure strategies, while \( x = -1 \) corresponds to playing the first strategies with probability 1. Of course, the switching point \( x = 0 \) yields beliefs corresponding to the mixed equilibrium of Fig. 7 (with \( \theta_1 = \theta_2 = 0 \)), viz. each player believes the opponent's pure strategies both occur with probability \( \frac{1}{2} \). Hence, when \( \eta^2 / (\eta^2 + \varepsilon^2) \to 0 \) we replicate the results obtained by Harsanyi.

The other extreme case is the one where \( \varepsilon^2 / (\eta^2 + \varepsilon^2) \to 0 \). Then \( \theta_1 \) and \( \theta_2 \) become perfectly correlated in the limit. A player chooses his first strategy if \( \theta_1 > 0 \) and his second strategy otherwise. This choice is in agreement with equilibrium selection in the game of Fig. 7 according to the risk dominance criterion, hence, this case replicates the main results obtained in this paper. Note that the beliefs generated by the Bayesian equilibrium in this case do not converge to a Nash equilibrium of the limit game (this is the game of Fig. 7 with \( \theta_1 = \theta_2 = 0 \)), the beliefs converge to the correlated equilibrium in which both pure equilibria are played with probability \( \frac{1}{2} \).

11 Conclusion

Two central aims in recent game-theoretic research have been to arrive at unique solutions\(^{18}\), on the one hand, and to incorporate more realistic informational assumptions on the other. There is often thought to exist a conflict, and thus a necessary trade-off between these two goals. The concept of global games that has been presented here indicates that this need not be the case. On the contrary, the use of an informational setup implying a considerable weakening of the common knowledge assumption has been shown to generate a model with interesting equilibrium selection properties.

\(^{18}\) The importance of uniqueness is stressed by Robert Aumann, who writes in the foreword to Harsanyi and Selten (1988): “Nash equilibrium makes sense only if each player knows which strategies the others are playing: if the equilibrium recommended by the theory is not unique, the players will not have this knowledge. Thus it is essential that for each game, the theory selects one unique equilibrium from the set of all Nash equilibria.”
The paper's main message is that something can be gained by moving from the conventional local analysis of individual games to a global analysis of classes of games, i.e. not every equilibrium of a given game need be consistent with an equilibrium rule for the entire class of games. The global game approach provides a natural way to force players to link games together and to analyse them simultaneously. Unfortunately, this particular approach turned out not to be easily tractable mathematically and we were able to pursue its implications only for the restricted class of 2-person $2 \times 2$ normal form games. Extensions to other classes of games are therefore urgently called for. In this connection two distinct questions become relevant. The first concerns the possibility to extend the uniqueness result while the second concerns the nature of the solution whenever it is unique.

The authors are confident that the uniqueness result can be extended to other interesting classes of games. As an example we recall the class of bargaining games discussed in Sect. 7: Players are slightly uncertain about the amount $\theta$ that is to be divided and each player receives a signal $\theta^i$ correlated to $\theta$. A second example is the Stag Hunt Game discussed in Van Huyck et. al. (1990): Players simultaneously choose numbers from a finite set $A$ and player $i$'s payoff is $\min_j a_j - \theta a_i$, where $\theta \in \mathbb{R}$. (The interpretation is that $a_i$ is player $i$'s effort, $\theta$ is the cost of effort and there is extreme complementarity in the production process so that the output is only $\min_j a_j$.) If $\theta$ can only be observed imperfectly then uniqueness results. A third example is the class of pure coordination games, i.e. the class of common interest unanimity games. The common feature in these examples is that we have a meaningful one-dimensional global game satisfying monotonicity properties similar to that from Sect. 6, and we conjecture that in such a case uniqueness obtains.

On the basis of a result contained in Carlsson (1989) we also conjecture that, in case of uniqueness of the limit equilibrium, the selection criterion will be a rather straightforward generalization of the risk dominance rule that we found for $2 \times 2$ games.\footnote{Carlsson (1989) analyses $m \times m$ unanimity games defined on a one-dimensional parameter space.} Unfortunately,
selection rules based on Nash products of deviation losses may become intransitive in
general (see Harsanyi and Selten (1988, p. 112 and pp. 216-217)) and, hence, may be
incapable of selecting a solution. Therefore there is a reason to be pessimistic about the
chances of the present approach generating uniqueness in general. Nevertheless it would
be an important achievement if uniqueness could be established for games where risk
dominance based on the Nash product criterion is transitive.

Naturally we hope that it will be possible to apply our approach to even broader classes
of games. In particular applications to extensive form games could yield interesting
comparisons with the refinements literature. (See Carlsson and Dasgupta (1990) for a
related application to signalling games.) At the same time we are aware of the technical
difficulties which may arise. Although global games are based on a very simple idea, the
analysis tends to become rather involved.

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Under some additional assumptions he shows that, in the region where α risk dominates any alternative
equilibrium β, there can be no switching away from α in the global game.


Fig. 1a: Game $G_1$  
Fig. 1b: Game $G_1(\theta)$  

Fig. 2: Game $G_2(\theta)$
Figure 3
Figure 4 (The shaded area is $D$)
\begin{figure}
\begin{center}
\begin{tabular}{|c|c|}
\hline
$\alpha_1$ & $\beta_2$ \\
\hline
9 & 0 \\
9 & $1 + \theta$ \\
\hline
$\beta_1$ & $\theta$ \\
$1 + \theta$ & $\theta$ \\
\hline
0 & $\theta$ \\
\hline
\end{tabular}
\end{center}
\caption{Fig. 5}
\end{figure}

\begin{figure}
\begin{center}
\begin{tabular}{|c|c|}
\hline
$\alpha_1$ & $\beta_2$ \\
\hline
10 & 4 \\
6 & -3000 \\
\hline
$\beta_1$ & x \\
5 & 4 \\
\hline
4 & 8 \\
\hline
\end{tabular}
\end{center}
\caption{Fig. 6}
\end{figure}
Fig. 7
Discussion Paper Series, CentER, Tilburg University, The Netherlands:

(For previous papers please consult previous discussion papers.)

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