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THE D^2* -TRIANGULATION FOR CONTINUOUS DEFORMATION
ALGORITHMS TO COMPUTE SOLUTIONS OF NONLINEAR EQUATIONS

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The $D_2^*$-Triangulation for Continuous Deformation Algorithms to Compute Solutions of Nonlinear Equations

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Abstract

We propose a new triangulation of $(0, 1] \times \mathbb{R}^n$, called the $D_2^*$-triangulation, with continuous refinement of grid sizes for use in continuous deformation algorithms to compute solutions of nonlinear equations. Any positive even integer can be chosen as one of its factors of refinement of grid sizes. We prove that the $D_2^*$-triangulation is superior to the well-known $K_2^*$-triangulation and $J_2^*$-triangulation when we compare the number of simplices. Numerical tests show that the continuous deformation algorithm based on the $D_2^*$-triangulation indeed is more efficient.

Keywords: Triangulations, Measures of Efficiency of Triangulations, Continuous Deformation Algorithms

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1 Introduction

Simplicial methods were originated by Scarf in his seminar paper [18] to compute fixed points of a continuous mapping from the unit simplex to itself. They are also called the fixed point methods in literature. By now, simplicial methods have been developed for over twenty years. As a tool to solve highly nonlinear problems, which are derived from decision-making, economic modelling, and engineering, simplicial methods are very powerful. The so-called continuous deformation algorithm is one of the most successful simplicial methods. It was initiated by Eaves in [9] to compute fixed points on the unit simplex, and generalized to $\mathbb{R}^n$ by Eaves and Saigal in [10] to find solutions of nonlinear equations. This method is also named the simplicial homotopy algorithm. The principles of the continuous deformation algorithm are as follows.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonlinear mapping, $f = (f_1, f_2, \ldots, f_n)^T$. We want to compute a zero point of $f$. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an affinely linear mapping with a zero point $x^0$, i.e., $g(x) = A(x - x^0)$, where $A$ is a $n \times n$ nonsingular matrix. Then the homotopy function $h$ is given by $h(t, x) = (1 - t)f(x) + tg(x)$, for $(t, x) \in [0, 1] \times \mathbb{R}^n$. The underlying space $(0, 1] \times \mathbb{R}^n$ is subdivided into simplices by a triangulation, denoted by $T$, with continuous refinement of grid sizes. The piecewise linear approximation $H$ of $h$ with respect to $T$ is given by, for $(t, x) = \sum_{i=1}^{n} \lambda_i y^i \in \sigma$, a simplex in $T$, with $\lambda_i \geq 0$, for $i = -1, 0, \ldots, n$, and $\sum_{i=-1}^{0} \lambda_i = 1$,

$$H(t, x) = \sum_{i=-1}^{n} \lambda_i h(y^i),$$

where $y^i$ is a vertex of $\sigma$ for $i = -1, 0, \ldots, n$. Then there exist some piecewise linear paths defined by the set of zero points of $H$. In particular, one of the paths starts at $x^0$ and goes to either infinity or converges to a zero point of $f$. One can trace this path with the standard lexicographical pivoting rule. Numerical tests have shown that simplicial algorithms heavily depend on the underlying triangulation. In order to improve the efficiency of the continuous deformation algorithm, a number of triangulations with continuous refinement of grid sizes has been proposed, for example, the $K_3$-triangulation and the $J_3$-triangulation of Todd in [20], the $D_3$-triangulation and the $D_2$-triangulation of Dang in [5] and [6], the arbitrary grid size
refinement triangulation of van der Laan and Talman in [15] and of Shamir in [19], the $K_2$-triangulation, the $J_2$-triangulation, the $K_2^*$-triangulation, and the $J_2^*$-triangulation of Kojima and Yamamoto in [14], the triangulation of Broadie and Eaves in [2], and the triangulation of Doup and Talman in [7]. All these triangulations were derived from the well-known $K_1$-triangulation or $J_1$-triangulation, except the $D_3$-triangulation and the $D_2$-triangulation, which were obtained from the $D_1$-triangulation. The latter triangulation of $R^n$ was proposed in [4] and is superior to the $K_1$-triangulation and the $J_1$-triangulation according to all measures of efficiency of triangulations. Theoretical results and numerical tests have proved that the $D_3$-triangulation is superior to both the $K_3$-triangulation and the $J_3$-triangulation, and that the $D_2$-triangulation is superior to both the $K_2$-triangulation and the $J_2$-triangulation. As mentioned by Kojima and Yamamoto in [14], the $K_3$-triangulation is a special case of the $K_2^*$-triangulation for the factor of refinement equal to two, and the $J_3$-triangulation is a special case of the $J_2^*$-triangulation for the factor of refinement equal to two. Numerical tests have shown in [5] that the continuous deformation algorithm based on the $D_3$-triangulation is very efficient. However, its factors of refinement are also equal to two. Motivated by the results in [14], we construct a new triangulation of $(0, 1] \times R^n$, called the $D_2^*$-triangulation, with continuous refinement of grid sizes for the continuous deformation algorithm, using the $D_1$-triangulation. Any positive even integer can be chosen as one of its factors of refinement. This feature is the same as that of the $K_2^*$-triangulation and of the $J_2^*$-triangulation. Similarly to the $K_3$-triangulation and the $J_3$-triangulation, the $D_3$-triangulation now becomes a special case of the $D_2^*$-triangulation for the factor of refinement equal to two. To compare with the $D_2^*$-triangulation, we also present the $K_2^*$-triangulation and the $J_2^*$-triangulation, which were given by Kojima and Yamamoto in [14] without their algebraic definitions. We prove that the $D_2^*$-triangulation is superior to the $K_2^*$-triangulation and the $J_2^*$-triangulation when we count the number of simplices. Since it is rather complicated to calculate the surface density of these triangulations, we refer for it to [4] and [12]. Numerical tests show that the continuous deformation algorithm based on the $D_2^*$-triangulation indeed is more efficient. We remark that the structure of the $D_2^*$-
triangulation is quite different from that of the $D_2$-triangulation. Numerical tests show that
the $D_2^*$-triangulation is in general faster than the $D_2$-triangulation. Note that there exists a
number of other interesting triangulations of $R^n$, see [17], [16], [22], and [13]. However, it is
not known how these triangulations of $R^n$ can be used to obtain triangulations of $(0,1] \times R^n$
with continuous refinement of grid sizes.

In Section 2, an algebraic definition of the $D_2^*$-triangulation is presented. In Section 3, we
prove that the definition given in Section 2 yields a triangulation. The pivot rules of the $D_2^*$-
triangulation for moving from one simplex to an adjacent simplex are described in Section 4.
Comparison with other triangulations is presented in Section 5.

2 Algebraic Definition of the $D_2^*$-Triangulation

Let $N_0$ denote the index set $\{0,1,\ldots,n\}$ and let $u^i$ be the ith unit vector in $R^{n+1}$ for $i = 0,1,\ldots,n$. Take $\alpha_0 \in (0,1]$ and $\beta_i \in \{1/j \mid j = 1,2,\ldots\}$ for $i = 0,1,\ldots$, and choose $\alpha_j$ such
that $\alpha_{j+1} = \alpha_j \beta_j/2$ for $j = 0,1,\ldots$. Let us set $\beta_{-1} = 1$.

Let $\pi = (\pi(0),\pi(1),\ldots,\pi(n))$ be a permutation of the elements of $N_0$. Let $q$ denote the
integer with $\pi(q) = 0$. Take a vector $y \in (0,1] \times R^n$ such that for an integer $k \geq 0$, $y_0 = 2^{-(k+1)}$, $y_{\pi(i)}/2\alpha_{k+1}$ is an integer for $i = 0,\ldots,q-1$, and $y_{\pi(i)}/\alpha_k$ is odd for $i = q+1,\ldots,n$. Then we
define
\[
\omega_{\pi(i)} = \begin{cases}
[y_{\pi(i)}/\alpha_k] + 1 & \text{if } [y_{\pi(i)}/\alpha_k] \text{ is odd,} \\
[y_{\pi(i)}/\alpha_k] & \text{otherwise,}
\end{cases}
\]
for $i = 0,1,\ldots,q-1$.

Definition 2.1. Let $y$ and $\pi$ be as above. Then the vectors $y^{-1}, y^{0}, \ldots, y^{n}$ are given as follows.

$$y^{-1} = y,$$

$$y^i = y^{i-1} + 2\alpha_{k+1}u_{\pi(i)}, \quad i = 0,1,\ldots,q-1,$$

$$y^q = \alpha_k \sum_{j=0}^{q-1} \omega_{\pi(j)}u_{\pi(j)} + \sum_{j=q+1}^{n}(y_{\pi(j)} - \alpha_k)u_{\pi(j)} + 2y_0u^0,$$

$$y^i = y^{i-1} + 2\alpha_ku_{\pi(i)}, \quad i = q+1,\ldots,n.$$
Let \( y^{-1}, y^0, \ldots, y^n \) be obtained in the above manner. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by \( K_2^*(y, \pi) \). Let \( K_2^* \) denote the collection of all such simplices \( K_2^*(y, \pi) \). It will be shown in the next section that \( K_2^* \) is a triangulation of \((0, 1] \times \mathbb{R}^n\) such that any positive even integer can be chosen as one of its factors of refinement, and when its factor of refinement is always equal to two, the \( K_2^* \)-triangulation is induced as one of its special cases. We call it the \( K_2^* \)-triangulation.

Let \( \pi = (\pi(0), \pi(1), \ldots, \pi(n)) \) be a permutation of the elements of \( N_0 \). Let \( q \) denote the integer with \( \pi(q) = 0 \). Take a vector \( y \in (0, 1] \times \mathbb{R}^n \) such that for an integer \( k \geq 0, y_0 = 2^{-(k+1)} \), if \( 1/\beta_k \) is even, \( y_{\pi(i)}/2\alpha_{k+1} \) is even for \( i = 0, \ldots, q - 1 \), if \( 1/\beta_k \) is odd, \( y_{\pi(i)}/2\alpha_{k+1} \) is odd for \( i = 0, \ldots, q - 1 \), and \( y_{\pi(i)}/\alpha_k \) is odd for \( i = q + 1, \ldots, n \). Then we define

\[
 w_{\pi(i)} = \begin{cases} 
 |y_{\pi(i)}/\alpha_k| - 1 & \text{if } |y_{\pi(i)}/\alpha_k| \text{ is even,} \\
 |y_{\pi(i)}/\alpha_k| + 1 & \text{if } |y_{\pi(i)}/\alpha_k| \text{ is odd and either } y_{\pi(i)}/\alpha_k \neq |y_{\pi(i)}/\alpha_k| \\
 & \text{or both } |y_{\pi(i)}/\alpha_k| = y_{\pi(i)}/\alpha_k \text{ and } s_{\pi(i)} = 1,
\end{cases}
\]

for \( i = 0, 1, \ldots, q - 1 \). If \( 1/\beta_{k-1} \) is odd, let us define

\[
 t_{\pi(i)} = \begin{cases} 
 -1 & \text{if } y_{\pi(i)}/\alpha_k = 1 \pmod{4}, \\
 1 & \text{if } y_{\pi(i)}/\alpha_k = 3 \pmod{4},
\end{cases}
\]

for \( i = q + 1, \ldots, n \), and if \( 1/\beta_{k-1} \) is even, let us define

\[
 t_{\pi(i)} = \begin{cases} 
 1 & \text{if } y_{\pi(i)}/\alpha_k = 1 \pmod{4}, \\
 -1 & \text{if } y_{\pi(i)}/\alpha_k = 3 \pmod{4},
\end{cases}
\]

for \( i = q + 1, \ldots, n \). Take a sign vector \( s = (s_1, s_2, \ldots, s_n)^T \) such that \( s_i \in \{-1, +1\} \) for \( i = 1, 2, \ldots, n \), and \( s_{\pi(i)} = t_{\pi(i)} \) for \( i = q + 1, \ldots, n \).

**Definition 2.2.** Let \( y, \pi \) and \( s \) be as above. Then the vectors \( y^{-1}, y^0, \ldots, y^n \) are given as
follows.

\[ y^{-1} = y, \]
\[ y^i = y^{i-1} + 2\alpha_{k+1}s_{\pi(i)}u^*(i), \quad i = 0, 1, \ldots, q - 1, \]
\[ y^q = \alpha_k \sum_{j=0}^{q-1} w_{\pi(j)}u^*(j) + \sum_{j=q+1}^n (y_{\pi(j)} - \alpha_k s_{\pi(j)})u^*(j) + 2y_0u^0, \]
\[ y^i = y^{i-1} + 2\alpha_k s_{\pi(i)}u^*(i), \quad i = q + 1, \ldots, n. \]

Let \( y^{-1}, y^0, \ldots, y^n \) be obtained in the above manner. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by \( J_2(y, \pi, s) \). Let \( J_2^* \) denote the set of all such simplices \( J_2^*(y, \pi, s) \). It will be shown in the next section that \( J_2^* \) is a triangulation of \( (0, 1] \times \mathbb{R}^n \) such that any positive even integer can be chosen as one of its factors of refinement, and when its factor of refinement is always equal to two, the \( J_3 \)-triangulation is induced as one of its special cases. We call it the \( J_2^* \)-triangulation.

Let \( \pi = (\pi(0), \pi(1), \ldots, \pi(n)) \) be a permutation of the elements of \( N_0 \). Let \( q \) denote the integer with \( \pi(q) = 0 \). Take a vector \( y \in (0, 1] \times \mathbb{R}^n \) such that for an integer \( k \geq 0 \), \( y_0 = 2^{-(k+1)}, \)

if \( 1/\beta_k \) is even, \( y_{\pi(i)}/2\alpha_{k+1} \) is even for \( i = 0, \ldots, q - 1 \), if \( 1/\beta_k \) is odd, \( y_{\pi(i)}/2\alpha_{k+1} \) is odd for \( i = 0, \ldots, q - 1 \), and \( y_{\pi(i)}/\alpha_k \) is odd for \( i = q + 1, \ldots, n \). Then we define

\[ w_{\pi(i)} = \begin{cases} 
[y_{\pi(i)}/\alpha_k] + 1 & \text{if } [y_{\pi(i)}/\alpha_k] \text{ is odd and either } y_{\pi(i)}/\alpha_k \neq [y_{\pi(i)}/\alpha_k] \\
[y_{\pi(i)}/\alpha_k] & \text{if } [y_{\pi(i)}/\alpha_k] \text{ is even,}
\end{cases} \]

\[ s_{\pi(i)} = \begin{cases} 1 & \text{otherwise,}
\end{cases} \]

for \( i = 0, 1, \ldots, q - 1 \). If \( 1/\beta_{k-1} \) is odd, let us define

\[ t_{\pi(i)} = \begin{cases} -1 & \text{if } y_{\pi(i)}/\alpha_k = 1 \pmod{4}, \\
1 & \text{if } y_{\pi(i)}/\alpha_k = 3 \pmod{4},
\end{cases} \]

for \( i = q + 1, \ldots, n \), and if \( 1/\beta_{k-1} \) is even, let us define

\[ t_{\pi(i)} = \begin{cases} 1 & \text{if } y_{\pi(i)}/\alpha_k = 1 \pmod{4}, \\
-1 & \text{if } y_{\pi(i)}/\alpha_k = 3 \pmod{4},
\end{cases} \]
for \( i = q + 1, \ldots, n \). Take a sign vector \( s = (s_1, s_2, \ldots, s_n)^T \) such that \( s_i \in \{-1, +1\} \) for \( i = 1, 2, \ldots, n \), and \( s_{x(i)} = t_{x(i)} \) for \( i = q + 1, \ldots, n \). Then we define

\[
I = \begin{cases} 
\{ \pi(i) \mid w_x(i)/2 \text{ is even and } 0 \leq i \leq q - 1 \} & \text{if } 1/\beta_{k-1} \text{ is odd}, \\
\{ \pi(i) \mid w_x(i)/2 \text{ is odd and } 0 \leq i \leq q - 1 \} & \text{if } 1/\beta_{k-1} \text{ is even}.
\end{cases}
\]

Let \( h \) denote the number of elements in \( I \). Take two integers \( p_1 \) and \( p_2 \) such that \(-1 \leq p_1 \leq q - 2\) if \( q \geq 1 \); \( p_1 = -1 \) if \( q = 0 \); when \( h = 0 \), \( 0 \leq p_2 \leq n - q - 1 \) if \( q < n \), and \( p_2 = 0 \) if \( q = n \); when \( h > 0 \), \( p_2 = n - q \).

**Definition 2.3.** Let \( y, \pi, s, p_1 \) and \( p_2 \) be as above. Then the vectors \( y^{-1}, y^0, \ldots, y^n \) are given as follows.

When \( p_1 = -1 \),

\[
y^{-1} = y,
\]

\[
y^i = y + 2\alpha_{k+1}s_x(i)u_x(i), \quad i = 0, 1, \ldots, q - 1,
\]

and when \( p_1 \geq 0 \),

\[
y^{-1} = y + 2\alpha_{k+1} \sum_{j=0}^{q-1} s_x(j)u_x(j),
\]

\[
y^i = y^{i-1} - 2\alpha_{k+1}s_x(i)u_x(i), \quad i = 0, 1, \ldots, p_1 - 1,
\]

\[
y^i = y + 2\alpha_{k+1}s_x(i)u_x(i), \quad i = p_1, \ldots, q - 1.
\]

When \( h > 0 \),

\[
y^q = \alpha_k \sum_{j=0}^{q-1} w_x(j)u_x(j) + \sum_{j=q+1}^{n} (y_x(j) + \alpha_k s_x(j))u_x(j) + 2y_0u^0,
\]

\[
y^i = y^{i-1} - 2\alpha_k s_x(i)u_x(i), \quad i = q + 1, \ldots, n,
\]

and when \( h = 0 \), if \( p_2 = 0 \), then

\[
y^q = \alpha_k \sum_{j=0}^{q-1} w_x(j)u_x(j) + \sum_{j=q+1}^{n} (y_x(j) - \alpha_k s_x(j))u_x(j) + 2y_0u^0,
\]

\[
y^i = y^i + 2\alpha_k s_x(i)u_x(i), \quad i = q + 1, \ldots, n,
\]

and if \( p_2 \geq 1 \), then

\[
y^q = \alpha_k \sum_{j=0}^{q-1} w_x(j)u_x(j) + \sum_{j=q+1}^{n} (y_x(j) + \alpha_k s_x(j))u_x(j) + 2y_0u^0,
\]

\[
y^i = y^{i-1} - 2\alpha_k s_x(i)u_x(i), \quad i = q + 1, \ldots, q + p_2 - 1,
\]

\[
y^i = y^* + 2\alpha_k s_x(i)u_x(i), \quad i = q + p_2, \ldots, n,
\]
where
\[ y^* = \alpha_k \sum_{j=0}^{q-1} w_{\pi(j)} u^{\pi(j)} + \sum_{j=q+1}^{n} (y_{\pi(j)} - \alpha_k s_{\pi(j)}) u^{\pi(j)} + 2y_0 u^0. \]

Let \( y^{-1}, y^0, \ldots, y^n \) be obtained in the above manner. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by \( D_2^* (y, \pi, s, p_1, p_2) \). Let \( D_2^* \) denote the set of all such simplices \( D_2^* (y, \pi, s, p_1, p_2) \). It will be shown in the next section that \( D_2^* \) is a triangulation of \( (0,1] \times \mathbb{R}^n \) such that any positive even integer can be chosen as one of its factors of refinement, and when its factor of refinement is always equal to two, the \( D_3 \)-triangulation is induced as one of its special cases. We call it the \( D_2^* \)-triangulation.

3 Construction of the \( D_2^* \)-Triangulation

Let \( N \) denote the index set \( \{1, 2, \ldots, n\} \) and let \( Q \) denote the set 
\[ \{w \mid \text{all components of } w \text{ are integers}\}. \]

We take an arbitrary element \( w \in Q \). Then we define
\[ I_o(w) = \{i \in N \mid w_i \text{ is odd}\} \text{ and } I_e(w) = \{j \in N \mid w_j \text{ is even}\}. \]

Furthermore, let \( A(w) \) denote the set
\[ \{x \in \mathbb{R}^n \mid w_i - 1 \leq x_i \leq w_i + 1 \text{ for } i \in I_o(w), \text{ and } x_i = w_i \text{ for } i \in I_e(w)\} \]
and let \( B(w) \) denote the set
\[ \{x \in \mathbb{R}^n \mid x_i = w_i \text{ for } i \in I_o(w), \text{ and } w_i - 1 \leq x_i \leq w_i + 1 \text{ for } i \in I_e(w)\}. \]

Let \( k \) be a nonnegative integer. Then let \( D^k(w) \) denote the convex hull of the set
\[ (\{2^{-k}\} \times A(w)) \cup (\{2^{-(k+1)}\} \times B(w)). \]
The following lemmas can be found in [5] and [14].

**Lemma 3.1.**

\[
D^k(w) = \left\{ d \in [2^{-(k+1)}, 2^{-k}] \times \mathbb{R}^n \left| \begin{array}{l} 
|d_i - w_i| \leq 2^{k+1}d_0 - 1 \text{ for } i \in I_o(w) \\
|d_i - w_i| \leq 2 - 2^{k+1}d_0 \text{ for } i \in I_e(w) \end{array} \right. \right\}.
\]

**Lemma 3.2.** \(\bigcup_{w \in Q} D^k(w) = [2^{-(k+1)}, 2^{-k}] \times \mathbb{R}^n\).

**Lemma 3.3.** For \(w^1, w^2 \in Q\), \(D^k(w^1) \cap D^k(w^2)\) is either empty or a common face of both \(D^k(w^1)\) and \(D^k(w^2)\), and when \(D^k(w^1) \cap D^k(w^2)\) is not empty, it is equal to the convex hull of the set

\[
\left(\left\{2^{-k}\right\} \times (A(w^1) \cap A(w^2))\right) \cup \left(\left\{2^{-(k+1)}\right\} \times (B(w^1) \cap B(w^2))\right).
\]

For convenience of the following discussion, we give the definitions of the \(D_1\)-triangulation, the \(K_1\)-triangulation, and the \(J_1\)-triangulation. For more details, see [4] and [20]. Let \(e^i\) be the \(i\)th unit vector in \(\mathbb{R}^n\) for \(i = 1, 2, \ldots, n\).

Let \(D\) denote either the set

\[
\{x \in \mathbb{R}^n \mid \text{all components of } x \text{ are odd}\}
\]

or the set

\[
\{x \in \mathbb{R}^n \mid \text{all components of } x \text{ are even}\}.
\]

Let \(\pi = (\pi(1), \pi(2), \ldots, \pi(n))\) be a permutation of the elements of \(N\). Take a sign vector \(s = (s_1, s_2, \ldots, s_n)^T\) with \(s_i \in \{-1, 1\}\) for \(i = 1, 2, \ldots, n\). Let \(p\) be an integer with \(0 \leq p \leq n - 1\).

Take a vector \(y\) from the set \(D\).

**Definition 3.1.** Let \(y, \pi, s,\) and \(p\) be as above. Then the vectors \(y^0, y^1, \ldots, y^n\) are as follows.

If \(p = 0\), then \(y^0 = y\), and \(y^j = y + s_{\pi(j)}e^j\), \(j = 1, 2, \ldots, n\).

If \(p \geq 1\), then

\[
y^0 = y + s,
\]

\[
y^j = y^{j-1} - s_{\pi(j)}e^{\pi(j)}, \quad j = 1, 2, \ldots, p - 1,
\]

\[
y^j = y + s_{\pi(j)}e^{\pi(j)}, \quad j = p, p + 1, \ldots, n.
\]
Let $D_1$ denote the collection of all simplices $D_1(y, \pi, s, p)$ that are the convex hull of $y^0, y^1, \ldots, y^n$, as obtained from the above definition. Then $D_1$ is a triangulation of $\mathbb{R}^n$, called the $D_1$-triangulation.

Let $K$ denote the set
\[
\{ x \in \mathbb{R}^n \mid \text{all components of } x \text{ are integers} \}.
\]
Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the elements of $N$. Take a vector $y$ from the set $K$.

**Definition 3.2.** Let $y$ and $\pi$ be as above. Then the vectors $y^0, y^1, \ldots, y^n$ are given as follows
\[
y^0 = y, \text{ and } y^j = y^{j-1} + e^{\pi(j)}, \quad j = 1, 2, \ldots, n.
\]

Let $K_1$ denote the collection of all simplices $K_1(y, \pi)$ that are the convex hull of $y^0, y^1, \ldots, y^n$, as obtained from the above definition. Then $K_1$ is a triangulation of $\mathbb{R}^n$, called the $K_1$-triangulation.

Let $J$ denote either the set
\[
\{ x \in \mathbb{R}^n \mid \text{all components of } x \text{ are odd} \}.
\]
or the set
\[
\{ x \in \mathbb{R}^n \mid \text{all components of } x \text{ are even} \}.
\]
Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the elements of $N$. Take a vector $y$ from the set $J$, and a sign vector $s = (s_1, s_2, \ldots, s_n)^T$ with $s_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, n$.

**Definition 3.3.** Let $y$, $\pi$, and $s$ be as above. Then the vectors $y^0, y^1, \ldots, y^n$ are given as follows.
\[
y^0 = y, \text{ and } y^j = y^{j-1} + s_{\pi(j)} e^{\pi(j)}, \quad j = 1, 2, \ldots, n.
\]

Let $J_1$ denote the collection of all simplices $J_1(y, \pi, s)$ that are the convex hull of $y^0, y^1, \ldots, y^n$, as obtained from the above definition. Then $J_1$ is a triangulation of $\mathbb{R}^n$, called the $J_1$-triangulation.
We take $G$ to be one of these triangulations of $R^n$. Let $\tilde{G}$ denote the set of faces of all simplices in $G$. Then, as before in the second section, we take $\alpha_0 \in (0,1]$ and $\beta_i \in \{1/j \mid j = 1,2,\ldots\}$ for $i = 0,1,\ldots$, and choose $\alpha_j$ such that $\alpha_{j+1} = \alpha_j \beta_j/2$ for $j = 0,1,\ldots$. We set $\beta_{-1} = 1$.

Let $2\alpha_k \tilde{G} \mid \alpha_k A(w)$ be the set given by $$\{\sigma \subseteq \alpha_k A(w) \mid \sigma \in 2\alpha_k \tilde{G} \text{ and } \dim(\sigma) = \dim(A(w))\}$$ and let $2\alpha_{k+1} \tilde{G} \mid \alpha_k B(w)$ be the set given by $$\{\sigma \subseteq \alpha_k B(w) \mid \sigma \in 2\alpha_{k+1} \tilde{G} \text{ and } \dim(\sigma) = \dim(B(w))\}.$$ For the $D_1$-triangulation, the $K_1$-triangulation, and the $J_1$-triangulation, it is obvious that $2\alpha_k \tilde{G} \mid \alpha_k A(w)$ is a triangulation of $\alpha_k A(w)$ and $2\alpha_{k+1} \tilde{G} \mid \alpha_k B(w)$ is a triangulation of $\alpha_k B(w)$.

Let $a$ denote the number of elements in the set $I_0(w)$, and $b$ the number of elements in the set $I_1(w)$. Let $\sigma_A \in 2\alpha_k \tilde{G} \mid \alpha_k A(w)$ be equal to the convex hull of $y_{A_1}, y_{A_2}, \ldots, y_{A_a}$, and let $\sigma_B \in 2\alpha_{k+1} \tilde{G} \mid \alpha_k B(w)$ be equal to the convex hull of $y_{B_1}, y_{B_2}, \ldots, y_{B_b}$. Furthermore, let $\sigma$ denote the convex hull of the set $\left(\{2^{-k}\} \times \sigma_A\right) \cup \left(\{2^{-(k+1)}\} \times \sigma_B\right)$. It can easily be proved that $\sigma$ is a simplex in $[2^{-(k+1)}, 2^{-k}] \times R^n$ and is equal to the convex hull of $(2^{-k}, y_{A_1}^0)^T, (2^{-k}, y_{A_2}^1)^T, \ldots, (2^{-k}, y_{A_a}^a)^T, (2^{-(k+1)}, y_{B_1}^0)^T, (2^{-(k+1)}, y_{B_2}^1)^T, \ldots, (2^{-(k+1)}, y_{B_b}^b)^T$.

Let $T(k,k+1)$ denote the collection of all such simplices $\sigma$. Then, following the conclusions mentioned above, we have that, for $\sigma^1$ and $\sigma^2$ in $T(k,k+1)$, the intersection $\sigma^1 \cap \sigma^2$ is either empty or a common face of both $\sigma^1$ and $\sigma^2$, and that the union of all $\sigma \in T(k,k+1)$ is equal to $[2^{-(k+1)}, 2^{-k}] \times R^n$. Hence, $T(k,k+1)$ is a triangulation of $[2^{-(k+1)}, 2^{-k}] \times R^n$.

Theorem 3.4. The union of $T(k,k+1)$ over all nonnegative integers $k$ is a triangulation of $(0,1] \times R^n$.

Proof. From the choice of $\alpha_j$ and $\beta_j$ for $j = 0,1,\ldots$, the theorem follows immediately.

We call the triangulation obtained in the above manner the $G_2$-triangulation. In this way, we obtain the $K_2$-triangulation, the $J_2$-triangulation, and the $D_2$-triangulation, as described in Section 2. Considering consistency, one can easily prove these results.
4 Pivot Rules of the $D_2^*$-Triangulation

As described in the first section, when a piecewise linear path of zero points is traced, the problem one faces is how to move from a simplex crossed by the path to an adjacent simplex crossed by the path with the standard lexicographic pivoting rule. As follows, the pivot rules of the $K_2^*$-triangulation, the $J_2^*$-triangulation, and the $D_2^*$-triangulation are described. The continuous deformation algorithm based on one of these triangulations can be implemented according to these pivot rules. In these following pivot rules, $y_0 = 2^{-(k+1)}$, $y = (y_1, \ldots, y_n)^T$, $\tilde{y}_0 = 2^{-(k+1)}$, $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)^T$, and $u = (1, 1, \ldots, 1)^T$.

Let a simplex of the $K_2^*$-triangulation, $\sigma = K_2^*(y, \pi)$, be given with vertices $y^{-1}, y^0, \ldots, y^n$. We want to obtain the simplex of the $K_2^*$-triangulation, $\sigma = K_2^*(\tilde{y}, \tilde{\pi})$, such that all vertices of $\sigma$ are also vertices of $\tilde{\sigma}$ except the vertex $y'$. As follows, we show how $\tilde{y}$ and $\tilde{\pi}$ depend on $y$, $\pi$, and $i$.

\[ i = -1: \text{ In case } q = 0, \tilde{y} = y - \alpha_k u, \tilde{\pi} = (\pi(1), \ldots, \pi(n), \pi(0)), \tilde{q} = q = n, \text{ and } \tilde{k} = k - 1. \]

\[ \text{In case } q \geq 1, \text{ if } y_{\pi(0)}^0 = \alpha_k (w_{\pi(0)} + 1), \text{ then } \tilde{y} = y - (y_{w(0)} - \alpha_k (w_{\pi(0)} + 1))u_{\pi(0)}, \tilde{\pi} = (\pi(1), \ldots, \pi(n), \pi(0)), \tilde{q} = q - 1, \text{ and } \tilde{k} = k; \text{ if } y_{\pi(0)}^0 \neq \alpha_k (w_{\pi(0)} + 1), \text{ then } \tilde{y} = y + 2\alpha_{k+1}u_{\pi(0)}, \tilde{\pi} = (\pi(1), \ldots, \pi(q - 1), \pi(0), \pi(q), \ldots, \pi(n)), \tilde{q} = q, \text{ and } \tilde{k} = k. \]

\[ 0 \leq i < q - 1: \tilde{y} = y, \tilde{\pi} = (\pi(0), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n)), \tilde{q} = q, \text{ and } \tilde{k} = k. \]

\[ 0 \leq i < q - 1: \text{ If } y_{\pi(q - 1)} = \alpha_k (w_{\pi(q - 1)} - 1), \text{ then } \tilde{y} = y, \tilde{\pi} = (\pi(0), \ldots, \pi(q), \pi(q - 1), \ldots, \pi(n)), \tilde{q} = q - 1, \text{ and } \tilde{k} = k. \]

\[ \text{If } y_{\pi(q - 1)} \neq \alpha_k (w_{\pi(q - 1)} - 1), \text{ then } \tilde{y} = y - 2\alpha_{k+1}u_{\pi(q - 1)}, \tilde{\pi} = (\pi(q - 1), \pi(0), \ldots, \pi(q - 2), \pi(q), \ldots, \pi(n)), \tilde{q} = q, \text{ and } \tilde{k} = k. \]

\[ q = i < n: \tilde{y} = y, \tilde{\pi} = (\pi(0), \ldots, \pi(q + 1), \pi(q), \ldots, \pi(n)), \tilde{q} = q + 1, \text{ and } \tilde{k} = k. \]

\[ q < i < n: \tilde{y} = y, \tilde{\pi} = (\pi(0), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n)), \tilde{q} = q, \text{ and } \tilde{k} = k. \]

\[ i = n: \text{ In case } q < n, \tilde{y} = y - 2\alpha_{k+1}u_{\pi(n)}, \tilde{\pi} = (\pi(n), \pi(0), \ldots, \pi(n - 1)), \tilde{q} = q + 1, \text{ and } \tilde{k} = k. \]

\[ \text{In case } q = n, \tilde{y} = y + \alpha_{k+1}u, \tilde{\pi} = (\pi(n), \pi(0), \ldots, \pi(n - 1)), \tilde{q} = 0, \text{ and } \tilde{k} = k + 1. \]
Next, let a simplex of the $J_2^*$-triangulation, $\sigma = J_2^*(y, \pi, s)$, be given with vertices $y^{-1}, y^0, \ldots, y^n$. We want to obtain the simplex of the $J_2^*$-triangulation, $\tilde{\sigma} = J_2^*(\tilde{y}, \tilde{\pi}, \tilde{s})$, such that all vertices of $\sigma$ are also vertices of $\tilde{\sigma}$ except the vertex $y'$. As follows, we show how $\tilde{y}$, $\tilde{\pi}$, and $\tilde{s}$ depend on $y$, $\pi$, $s$, and $i$.

$i = -1$: In case $q = 0$, $\tilde{y} = y - \alpha_k s$, $\tilde{s} = s$, $\tilde{\pi} = (\pi(1), \pi(2), \ldots, \pi(n), \pi(0))$, $\tilde{q} = n$, and $\tilde{k} = k - 1$. In case $q > 0$, $\tilde{y} = y + 4\alpha_{k+1} s_{\pi(0)} u^{\pi(0)}$, $\tilde{s} = s - 2s_{\pi(0)} u^{\pi(0)}$, $\tilde{\pi} = \pi$, $\tilde{q} = q$, and $\tilde{k} = k$.

$0 \leq i < q - 1$: $\tilde{y} = y$, $\tilde{s} = s$, $\tilde{\pi} = (\pi(0), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n))$, $\tilde{q} = q + 1$, and $\tilde{k} = k$.

$0 \leq i = q - 1$: In case $y_{\pi(q-1)} = \alpha_k (w_{\pi(q-1)} - s_{\pi(q-1)})$, if $s_{\pi(q-1)} = t_{\pi(q-1)}$, then $\tilde{y} = y$, $\tilde{s} = s$, $\tilde{\pi} = (\pi(0), \ldots, \pi(q), \pi(q - 1), \ldots, \pi(n))$, $\tilde{q} = q - 1$, and $\tilde{k} = k$; if $s_{\pi(q-1)} \neq t_{\pi(q-1)}$, then $\tilde{y} = y$, $\tilde{s} = s - 2s_{\pi(q-1)} u^{\pi(q-1)}$, $\tilde{\pi} = (\pi(0), \ldots, \pi(q - 2), \pi(q), \ldots, \pi(n), \pi(q - 1))$, $\tilde{q} = q - 1$, and $\tilde{k} = k$. In case $y_{\pi(q-1)} \neq \alpha_k (w_{\pi(q-1)} - s_{\pi(q-1)})$, $\tilde{y} = y$, $\tilde{s} = s - 2s_{\pi(q-1)} u^{\pi(q-1)}$, $\tilde{\pi} = \pi$, $\tilde{q} = q$, and $\tilde{k} = k$.

$q = i < n$: $\tilde{y} = y$, $\tilde{s} = s$, $\tilde{\pi} = (\pi(0), \ldots, \pi(q + 1), \pi(q), \ldots, \pi(n))$, $\tilde{q} = q + 1$, and $\tilde{k} = k$.

$q < i < n$: $\tilde{y} = y$, $\tilde{s} = s$, $\tilde{\pi} = (\pi(0), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n))$, $\tilde{q} = q + 1$, and $\tilde{k} = k$.

$i = n$: In case $q < n$, $\tilde{y} = y$, $\tilde{s} = s - 2s_{\pi(n)} u^{\pi(n)}$, $\tilde{\pi} = (\pi(0), \ldots, \pi(q - 1), \pi(n), \pi(q), \ldots, \pi(n - 1))$, $\tilde{q} = q + 1$, and $\tilde{k} = k$. In case $q = n$, $\tilde{y} = y + \alpha_{k+1} s$, $\tilde{s} = s$, $\tilde{\pi} = (\pi(n), \pi(0), \ldots, \pi(n - 1))$, $\tilde{q} = 0$, and $\tilde{k} = k + 1$.

Finally, let a simplex of the $D_2^*$-triangulation, $\sigma = D_2^*(y, \pi, s, p_1, p_2)$, be given with vertices $y^{-1}, y^0, \ldots, y^n$. We want to obtain the simplex of the $D_2^*$-triangulation, $\tilde{\sigma} = D_2^*(\tilde{y}, \tilde{\pi}, \tilde{s}, \tilde{p}_1, \tilde{p}_2)$, such that all vertices of $\sigma$ are also vertices of $\tilde{\sigma}$ except the vertex $y'$. As follows, we show how $\tilde{y}$, $\tilde{\pi}$, $\tilde{s}$, $\tilde{p}_1$, and $\tilde{p}_2$ depend on $y$, $\pi$, $s$, $p_1$, $p_2$, and $i$.

$i = -1$: In case $q = 0$, $\tilde{y} = y - \alpha_k s$, $\tilde{s} = s$, $\tilde{\pi} = (\pi(1), \ldots, \pi(n), \pi(0))$, $\tilde{p}_1 = p_2 - 1$, $\tilde{p}_2 = 0$, $\tilde{q} = n$, and $\tilde{k} = k - 1$. In case $q = 1$, $\tilde{y} = y + 4\alpha_{k+1} s_{\pi(0)} u^{\pi(0)}$, $\tilde{s} = s - 2s_{\pi(0)} u^{\pi(0)}$, $\tilde{\pi} = \pi$, $\tilde{q} = q$, and $\tilde{k} = k$.
\[ p_1 = p_1, \quad p_2 = p_2, \quad q = q, \quad \text{and} \quad k = k. \]

In case \( q > 1 \), when \( p_1 = -1, \quad y = y, \quad \bar{s} = s, \quad \bar{\pi} = \pi, \quad \bar{p}_1 = p_1 + 1, \quad \bar{p}_2 = p_2, \quad \bar{q} = q, \quad \text{and} \quad \bar{k} = k; \]
when \( p_1 = 0, \quad \bar{y} = y, \quad \bar{s} = s, \quad \bar{\pi} = \pi, \quad \bar{p}_1 = p_1 - 1, \quad \bar{p}_2 = p_2, \quad \bar{q} = q, \quad \text{and} \quad \bar{k} = k; \]
when \( p_1 \geq 1 \) and \( y_{\pi(0)} = \alpha_k(w_{\pi(0)} - s_{\pi(0)}), \)
if \( h = 0 \) and \( p_2 = 0, \) then \( \bar{y} = y, \quad \bar{s} = s - 2s_{\pi(0)}u_{\pi(0)}, \quad \bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0), \bar{p}_1 = p_1 - 1, \quad \bar{p}_2 = p_2, \quad \bar{q} = q - 1, \quad \text{and} \quad \bar{k} = k, \) if \( s_{\pi(0)} = t_{\pi(0)} \) and \( h > 1, \) then \( \bar{y} = y, \quad \bar{s} = s, \quad \bar{\pi} = (\pi(1), \ldots, \pi(n), \pi(0)), \quad \bar{p}_1 = p_1 - 1, \quad \bar{p}_2 = p_2 + 1, \quad \bar{q} = q - 1, \quad \text{and} \quad \bar{k} = k, \) if \( s_{\pi(0)} = t_{\pi(0)} \) and \( h > 0, \) then \( \bar{y} = y, \quad \bar{s} = s - 2s_{\pi(0)}u_{\pi(0)}, \quad \bar{\pi} = (\pi(1), \ldots, \pi(q), \pi(q), \pi(q), \pi(q + 1), \ldots, \pi(n)), \quad \bar{p}_1 = p_1 - 1, \quad \bar{p}_2 = p_2 + 1, \quad \bar{q} = q - 1, \quad \text{and} \quad \bar{k} = k; \) when \( p_1 \geq 1 \) and \( y_{\pi(0)} \neq \alpha_k(w_{\pi(0)} - s_{\pi(0)}), \)
\[ y = y, \quad \bar{s} = s - 2s_{\pi(0)}u_{\pi(0)}, \quad \bar{\pi} = \pi, \quad \bar{p}_1 = p_1, \quad \bar{p}_2 = p_2, \quad \bar{q} = q, \quad \text{and} \quad \bar{k} = k. \]

\[ 0 \leq i < q: \]
In case \( p_1 = -1, \) when \( y_{\pi(i)} = \alpha_k(w_{\pi(i)} - s_{\pi(i)}), \)
if \( h = 0 \) and \( p_2 = 0, \) then \( \bar{y} = y, \quad \bar{s} = s - 2s_{\pi(i)}u_{\pi(i)}, \quad \bar{\pi} = (\pi(0), \ldots, \pi(i - 1), \pi(i), \pi(i), \pi(i), \pi(i + 1), \ldots, \pi(n), \pi(i)), \quad \bar{p}_1 = p_1, \quad \bar{p}_2 = p_2, \quad \bar{q} = q - 1, \quad \text{and} \quad \bar{k} = k, \) if \( s_{\pi(i)} = t_{\pi(i)} \) and \( h = 1, \) then \( \bar{y} = y, \quad \bar{s} = s, \quad \bar{\pi} = (\pi(0), \ldots, \pi(i - 1), \pi(i), \pi(i), \pi(i), \pi(i + 1), \ldots, \pi(n), \pi(i)), \quad \bar{p}_1 = p_1, \quad \bar{p}_2 = p_2 + 1, \quad \bar{q} = q - 1, \quad \text{and} \quad \bar{k} = k, \) if \( s_{\pi(i)} \neq t_{\pi(i)} \) and \( h > 0, \) then \( \bar{y} = y, \quad \bar{s} = s - 2s_{\pi(i)}u_{\pi(i)}, \quad \bar{\pi} = (\pi(0), \ldots, \pi(i), \pi(i), \pi(i), \pi(i + 1), \ldots, \pi(n), \pi(i)), \quad \bar{p}_1 = p_1, \quad \bar{p}_2 = p_2 + 1, \quad \bar{q} = q - 1, \quad \text{and} \quad \bar{k} = k; \) when \( y_{\pi(i)} \neq \alpha_k(w_{\pi(i)} - s_{\pi(i)}), \)
\[ y = y, \quad \bar{s} = s - 2s_{\pi(i)}u_{\pi(i)}, \quad \bar{\pi} = \pi, \quad \bar{p}_1 = p_1, \quad \bar{p}_2 = p_2, \quad \bar{q} = q, \quad \text{and} \quad \bar{k} = k. \]
\[ s = s - 2s_{u(i^*)}u_{u(i^*)}, \pi = \pi, \tilde{p}_1 = p_1, \tilde{p}_2 = p_2, \tilde{q} = q, \text{ and } \tilde{k} = k, \text{ where} \]

\[ i^* = \begin{cases} 
q - 1 \text{ if } i = q - 2, \\
q - 2 \text{ if } i = q - 1.
\end{cases} \]

\( i = q: \) In case \( h = 0, \) when \( p_2 = 0, \) if \( q < n - 1, \) then \( \tilde{y} = y, \tilde{s} = s, \tilde{\pi} = \pi, \tilde{p}_1 = p_1, \)
\( \tilde{p}_2 = p_2 + 1, \tilde{q} = q, \) and \( \tilde{k} = k, \) if \( q = n - 1 \) and \( p_1 = -1, \) then \( \tilde{y} = y, \tilde{s} = s, \)
\( \tilde{\pi} = (\pi(0), \ldots, \pi(q + 1), \pi(q), \ldots, \pi(n)), \tilde{p}_1 = p_1, \tilde{p}_2 = p_2, \tilde{q} = q + 1, \) and \( \tilde{k} = k, \) if \( q = n - 1 \) and \( p_1 \geq 0, \) then \( \tilde{y} = y, s, \tilde{\pi} = (\pi(q + 1), \pi(0), \ldots, \pi(q), \pi(q + 2), \ldots, \pi(n)), \tilde{p}_1 = p_1 + 1, \)
\( \tilde{p}_2 = p_2, \tilde{q} = q + 1, \) and \( k; \) when \( p_2 = 1, \) \( \tilde{y} = y, \tilde{s} = s, \tilde{\pi} = \pi, \tilde{p}_1 = p_1, \)
\( \tilde{p}_2 = p_2 - 1, \tilde{q} = q, \) and \( k = k; \) when \( p_2 \geq 2, \) if \( p_1 = -1, \) then \( \tilde{y} = y, \tilde{s} = s - 2s_{u(i^*)}u_{u(i^*)}, \)
\( \tilde{\pi} = (\pi(0), \ldots, \pi(q + 1), \pi(q), \ldots, \pi(n)), \tilde{p}_1 = p_1, \tilde{p}_2 = p_2 - 1, \tilde{q} = q + 1, \) and \( k = k; \) if \( p_1 \geq 0, \) then \( \tilde{y} = y, \tilde{s} = s - 2s_{u(i^*)}u_{u(i^*)}, \)
\( \tilde{\pi} = (\pi(0), \ldots, \pi(q + 1), \pi(q), \ldots, \pi(n)), \tilde{p}_1 = p_1 + 1, \tilde{p}_2 = p_2 - 1, \tilde{q} = q + 1, \) and \( k = k. \) In case \( h \geq 0, \) when \( q < n, \) if \( p_1 = -1, \) then \( \tilde{y} = y, \tilde{s} = s - 2s_{u(i^*)}u_{u(i^*)}, \)
\( \tilde{\pi} = (\pi(0), \ldots, \pi(q - 1), \pi(i), \pi(q), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n)), \tilde{p}_1 = p_1, \)
\( \tilde{p}_2 = p_2, \tilde{q} = q + 1, \) and \( k = k; \) and if \( p_1 \geq 0, \) then \( \tilde{y} = y, \tilde{s} = s - 2s_{u(i^*)}u_{u(i^*)}, \)
\( \tilde{\pi} = (\pi(i), \pi(0), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n)), \tilde{p}_1 = p_1 + 1, \tilde{p}_2 = p_2, \tilde{q} = q + 1, \) and \( k = k. \) In case \( h = 0, \) when \( p_2 = 0, \) if \( q = n - 1, \) then \( \tilde{y} = y, \tilde{s} = s, \tilde{\pi} = \pi, \tilde{p}_1 = p_1, \)
\( \tilde{p}_2 = p_2, \tilde{q} = q, \) and \( \tilde{k} = k, \) if \( q < n - 1 \) and \( p_1 = -1, \) then \( \tilde{y} = y, \tilde{s} = s, \)
\( \tilde{\pi} = (\pi(0), \ldots, \pi(q + 1), \pi(q), \ldots, \pi(n)), \tilde{p}_1 = p_1, \tilde{p}_2 = p_2, \tilde{q} = q + 1, \) and \( \tilde{k} = k, \) if \( q = n - 1 \) and \( p_1 \geq 0, \) then \( \tilde{y} = y, \tilde{s} = s - 2s_{u(i^*)}u_{u(i^*)}, \)
\( \tilde{\pi} = (\pi(q + 1), \pi(0), \ldots, \pi(q), \pi(q + 2), \ldots, \pi(n)), \tilde{p}_1 = p_1, \tilde{p}_2 = p_2 - 1, \tilde{q} = q + 1, \) and \( k = k, \) if \( p_1 \geq 0, \) then \( \tilde{y} = y, \tilde{s} = s - 2s_{u(i^*)}u_{u(i^*)}, \)
\( \tilde{\pi} = (\pi(q + 1), \pi(0), \ldots, \pi(q), \pi(q + 2), \ldots, \pi(n)), \tilde{p}_1 = p_1 + 1, \tilde{p}_2 = p_2 - 1, \tilde{q} = q + 1, \) and \( k = k. \) In case \( q = n, \) \( \tilde{y} = y + \alpha_{k+1}s, \tilde{s} = s, \tilde{\pi} = (\pi(n), \pi(0), \ldots, \pi(n - 1)), \tilde{p}_1 = -1, \)
\( \tilde{p}_2 = p_1 + 1, \tilde{q} = 0 \) and \( k = k + 1. \)

\( q < i \leq n: \) In case \( h = 0, \) when \( p_2 = 0, \) if \( p_1 = -1, \) then \( \tilde{y} = y, \tilde{s} = s - 2s_{u(i^*)}u_{u(i^*)}, \)
\( \tilde{\pi} = (\pi(0), \ldots, \pi(q - 1), \pi(i), \pi(q), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n)), \tilde{p}_1 = p_1, \)
\( \tilde{p}_2 = p_2, \tilde{q} = q + 1, \) and \( k = k; \) if \( i < q + p_2 - 1, \) \( \tilde{y} = y, \)
\( \tilde{s} = s, \tilde{\pi} = (\pi(0), \ldots, \pi(i + 1), \pi(i), \ldots, \pi(n)), \tilde{p}_1 = p_1 + 1, \tilde{p}_2 = p_2, \tilde{q} = q, \) and \( \tilde{k} = k; \) when \( i = q + p_2 - 1, \) \( \tilde{y} = y, \)
\( \tilde{s} = s, \tilde{\pi} = \pi, \tilde{p}_1 = p_1, \tilde{p}_2 = p_2 - 1, \tilde{q} = q, \) and \( \tilde{k} = k; \) when \( i \geq q + p_2 \) and \( 1 \leq p_2 < n - q - 1, \) \( \tilde{y} = y, \tilde{s} = s, \tilde{\pi} = (\pi(0), \ldots, \pi(q + p_2 - 1), \pi(i), \pi(q + p_2), \ldots, \pi(i - 1), \pi(i + 1), \ldots, \pi(n)), \tilde{p}_1 = p_1, \)
\( \tilde{p}_2 = p_2 + 1, \tilde{q} = q, \) and \( \tilde{k} = k; \) when \( i \geq n - 1 \) and \( 1 \leq p_2 < n - q - 1, \) if \( p_1 = -1, \) then \( \tilde{y} = y, \tilde{s} = s, \tilde{\pi} = (\pi(0), \ldots, \pi(q - 1), \pi(i), \pi(q - 1), \ldots, \pi(n)), \tilde{p}_1 = p_1, \)
\( \tilde{p}_2 = p_2, \tilde{q} = q, \) and \( \tilde{k} = k. \)
1), \pi(i^{**}), \pi(q), \ldots, \pi(i^{**}-1), \pi(i^{**}+1), \ldots, \pi(n))

\bar{p}_1 = p_1, \bar{p}_2 = p_2, \bar{q} = q + 1, and \bar{k} = k,

and if \( p_1 \geq 0 \), then \( \bar{y} = y, \bar{s} = s, \bar{\pi} = (\pi(i^{**}), \pi(0), \ldots, \pi(i^{**}-1), \pi(i^{**}+1), \ldots, \pi(n)), \)

\bar{p}_1 = p_1 + 1, \bar{p}_2 = p_2, \bar{q} = q + 1, and \bar{k} = k, where

\[
\begin{align*}
    i^{**} = \begin{cases} 
        n & \text{if } i = n - 1, \\
        n - 1 & \text{if } i = n.
    \end{cases}
\end{align*}
\]

In case \( h > 0 \), when \( i < n, \bar{y} = y, \bar{s} = s, \bar{\pi} = (\pi(0), \ldots, \pi(i - 1), \pi(i), \ldots, \pi(n)), \)

\bar{p}_1 = p_1, \bar{p}_2 = p_2, \bar{q} = q, and \bar{k} = k; when \( i = n, \) if \( p_1 = -1, \) then \( \bar{y} = y, \bar{s} = s, \bar{\pi} = (\pi(0), \ldots, \pi(q - 1), \pi(n), \pi(q), \ldots, \pi(n - 1)), \bar{p}_1 = p_1, \bar{p}_2 = p_2 - 1, \bar{q} = q + 1, \) and \( \bar{k} = k, \) and if \( \bar{p}_1 \geq 0, \) then \( \bar{y} = y, \bar{s} = s, \bar{\pi} = (\pi(n), \pi(0), \ldots, \pi(n - 1)), \bar{p}_1 = p_1 + 1, \)

\bar{p}_2 = p_2 - 1, \bar{q} = q + 1, and \bar{k} = k.

5 Comparison of Triangulations

Since it is very complicated to calculate the surface density of the \( K_2^* \)-triangulation, of the \( J_2^* \)-triangulation, and of the \( D_2^* \)-triangulation, we only compare the number of simplices of these triangulations. For details about the surface density, we refer to [4] and [12]. Let \( H^n \) denote the unit cube \( \{ x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \text{ for } i = 1, 2, \ldots, n \}. \) We set \( \alpha = 1/\beta_k. \)

Theorem 5.1. The number of simplices of the \( K_2^* \)-triangulation and of the \( J_2^* \)-triangulation in the set \( [2^{-(k+1)}, 2^{-k}] \times 2\alpha_k H^n \) is equal to \( p_n(\alpha) \) given by

\[
p_n(\alpha) = (2\alpha n + 1 - 1)n!/(2\alpha - 1).
\]

The number of simplices of the \( D_2^* \)-triangulation in the same set is equal to \( q_n(\alpha) \) given by

\[
q_n(\alpha) = \sum_{m=0}^{n} ((2^m - 1)C_n^m \alpha^m d_m(n - m)! + C_n^m \alpha^m d_m d_{n-m}),
\]

where

\[
d_j = j + j(j - 1) + \cdots + j(j - 1) \cdots 4 \cdot 3 + 2
\]

for \( j \geq 2, d_0 = d_1 = 1, \) and \( C_n^m = n!/m!(n - m)! \).

Proof. Let \( \bar{Q} \) denote the set \( \{ w \in \mathbb{R}^n \mid w_i \in \{0,1,2\} \text{ for } i = 1, 2, \ldots, n \}. \) We take an arbitrary
vector \( w \in \bar{Q} \). Let \( \bar{\mathcal{A}}(w) \) denote the set

\[
\{ x \in \mathbb{R}^n \mid w_i - 1 \leq x_i \leq w_i + 1 \text{ for } i \in I_o(w), \text{ and } x_i = w_i \text{ for } i \in I_e(w) \}
\]

and let \( \bar{\mathcal{B}}(w) \) denote the set

\[
\begin{cases}
  x_i = w_i \text{ for } i \in I_o(w), \\
  w_i \leq x_i \leq w_i + 1 \text{ for } i \in I_e(w) \text{ and } w_i = 0, \\
  w_i - 1 \leq x_i \leq w_i \text{ for } i \in I_e(w) \text{ and } w_i = 2
\end{cases}
\]

Furthermore, let \( \alpha_k \bar{\mathcal{D}}(w) \) denote the convex hull of the set

\[
(\{2^{-k}\} \times \alpha_k \bar{\mathcal{A}}(w)) \cup (\{2^{-(k+1)}\} \times \alpha_k \bar{\mathcal{B}}(w)).
\]

Then it is obvious that

\[
[2^{-(k+1)}, 2^{-k}] \times 2\alpha_k H^n = \bigcup_{w \in \bar{Q}} \alpha_k \bar{\mathcal{D}}(w).
\]

Let \( m \) denote the number of elements in \( I_e(w) \). Then there are \( 2^m C_n^m \) elements in \( \bar{Q} \) such that \( m \) components of each of them are even. Thus the numbers of simplices of the \( K_2^\ast \)-triangulation and of the \( J_2^\ast \)-triangulation in the set

\[
\bigcup_{w \in \bar{Q}, I_e(w) = m} \alpha_k \bar{\mathcal{D}}(w)
\]

is equal to

\[
2^m \alpha^m C_n^m (n - m)! m! = (2\alpha)^m n!
\]

The number of simplices of the \( D_2^\ast \)-triangulation in the same set is equal to

\[
(2^m - 1) C_n^m \alpha^m d_m (n - m)! + C_n^m \alpha^m d_m d_{n-m}.
\]

Since

\[
\bigcup_{m=0}^n \bigcup_{w \in \bar{Q}, I_e(w) = m} \alpha_k \bar{\mathcal{D}}(w) = [2^{-(k+1)}, 2^{-k}] \times 2\alpha_k H^n,
\]

the theorem follows immediately.

**Theorem 5.2.** When \( n \geq 3 \), \( q_n(\alpha) < p_n(\alpha) \). As \( n \) goes to infinity, \( q_n(\alpha)/p_n(\alpha) \) converges to
Proof. The conclusion is obvious, the proof is omitted.

From Theorem 5.2, we have that the number of simplices of the \( D_z^* \)-triangulation is smallest for these three triangulations.

Let us denote the continuous deformation algorithms based on the \( K_z^* \)-triangulation, the \( J_z^* \)-triangulation, and the \( D_z^* \)-triangulation by CDA\( K_z^* \), CDA\( J_z^* \), and CDA\( D_z^* \), respectively. We have made computer codes of these algorithms in PASCAL. As introduced about the principles of the continuous deformation algorithm in the first section, letting \( A \) be the identity matrix and starting at \( x^0 = (0.5, 0.5, \ldots, 0.5)^T \), we have run these computer codes on a few functions for finding a zero point. Numerical tests are given as follows. Let NFE denote the number of function evaluations. The algorithm terminates when the accuracy for \( \max_{1 \leq i \leq n} |f_i(x^*)| \) of less than \( 10^{-5} \) has been reached. In the following tables, if the accuracy has not been satisfied when the number of function evaluations is equal to 40000, a symbol * is marked.

Problem A: The function \( f : R^n \rightarrow R^n \) is given by

\[
  f_i(x) = x_i - \cos(i \sum_{j=1}^{n} x_j), \ i = 1, 2, \ldots, n.
\]

When \( \alpha_0 = 0.25 \) and \( \beta_j = 1 \) for \( j = 0, 1, \ldots \), numerical results are given in the following table.

<table>
<thead>
<tr>
<th>n</th>
<th>NFE(CDA( K_z^* ))</th>
<th>NFE(CDA( J_z^* ))</th>
<th>NFE(CDA( D_z^* ))</th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>376</td>
<td>327</td>
<td>311</td>
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<td>6</td>
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<tr>
<td>10</td>
<td>35797</td>
<td>26006</td>
<td>23735</td>
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</tbody>
</table>

When \( \alpha_0 = 0.25 \) and \( \beta_{2j} = 1 \) and \( \beta_{2j+1} = 0.5 \) for \( j = 0, 1, \ldots \), numerical results are given in the following table.
Problem B: The function $f: \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f_i(x) = x_i - e^{\cos(i \sum_{j=1}^{n} x_j)}, \ i = 1, 2, \ldots, n.$$  

When $\alpha_0 = 0.25$ and $\beta_j = 1$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>NFE(CDAK$\beta$)</th>
<th>NFE(CDAJ$\beta$)</th>
<th>NFE(CDAD$\beta$)</th>
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<td>26418</td>
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</table>

When $\alpha_0 = 0.25$ and $\beta_j = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

<table>
<thead>
<tr>
<th>$n$</th>
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<th>NFE(CDAJ$\beta$)</th>
<th>NFE(CDAD$\beta$)</th>
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</table>

Problem C: The function $f: \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f_i(x) = x_i - e^{\sin(i \sum_{j=1}^{n} x_j)}, \ i = 1, 2, \ldots, n.$$  

When $\alpha_0 = 0.25$ and $\beta_j = 1$ for $j = 0, 1, \ldots$, numerical results are given in the following table.
When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

<table>
<thead>
<tr>
<th>n</th>
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<th>NFE($CDAJ_z^i$)</th>
<th>NFE($CDAD_z^i$)</th>
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<td>22559</td>
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<td>21398</td>
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</tr>
<tr>
<td>10</td>
<td>*</td>
<td>36616</td>
<td>15170</td>
</tr>
</tbody>
</table>

Problem D: The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$f_i(x) = x_i - \sin(t \sum_{j=1}^{n} x_j), \quad i = 1, 2, \ldots, n.$$  

When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

<table>
<thead>
<tr>
<th>n</th>
<th>NFE($CDAK_z^i$)</th>
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<td>18954</td>
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<tr>
<td>10</td>
<td>*</td>
<td>*</td>
<td>16392</td>
</tr>
</tbody>
</table>

When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

<table>
<thead>
<tr>
<th>n</th>
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<th>NFE($CDAJ_z^i$)</th>
<th>NFE($CDAD_z^i$)</th>
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<td>*</td>
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</tr>
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</table>

When $\alpha_0 = 0.25$ and $\beta_{2j} = 1$ and $\beta_{2j+1} = 0.5$ for $j = 0, 1, \ldots$, numerical results are given in the following table.

<table>
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<tr>
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<td>32636</td>
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</table>
From these numerical examples, it appears that the continuous deformation algorithm based on the $D_2^*$-triangulation indeed is more efficient.

References


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