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THE GENERALIZED EXTREME VALUE RANDOM UTILITY MODEL FOR CONTINUOUS CHOICE

by John K. Dagsvik

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by

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Abstract

The Generalized Extreme Value Model was developed by McFadden for the case with discrete choice sets. The present paper extends this model to the case with continuous choice sets. The development relies on a representation result for stochastic processes with finite-dimensional distributions of the multivariate extreme value type. The continuous Luce model emerges as a special case.

The paper also discusses a testable property of the continuous GEV model which may be considered an extension of Luce Axiom.

Finally, an attempt is given to justify the stochastic structure of the utility function from theoretical arguments.
1. Introduction

The Generalized Extreme Value random utility model (GEV) for discrete choice was developed by McFadden (see McFadden, 1981). The motivation was to extend Luce choice model so as to relax the IIA property and still retain this property as a special case. So far there have been few attempts to extend this model to the case where the choice set is a continuum. Ben-Akiva et al (1985) introduce a continuous logit model obtained by postulating a discrete random utility model from which the continuous version is established by a limiting argument. They do, however, not demonstrate the existence of a random utility structure consistent with this limiting case nor do they consider more general continuous GEV.

Cosslett (1988) considers a class (although quite general) of GEV models where he uses a particular approach to discuss the existence of a continuous GEV and the implied functional forms for the choice probabilities. He also considers statistical inference in such models.

The present approach take a different point of departure. By applying a representation result of stochastic processes generated from multidimensional extreme value finite-dimensional distributions we demonstrate that the choice probabilities of the continuous GEV as well as of the continuous Luce model can be obtained quite readily. Moreover, this representation result enables us to link this type of random utility models to the more traditional textbook models of consumer demand.

The motivation for studying random utility models is to obtain structural choice models where parameters related to preferences can be separated from parameters characterizing the choice environment. A particularly appealing property of the Luce model is that it admits a behavioral characterization of the revealed preference type. Specifically, this model is equivalent to the IIA axiom. While a similar type of characterization of the discrete GEV model has not been obtained we demonstrate here that the continuous GEV implies, under quite general assumptions, a relaxed version of IIA.

The organisation of the paper is as follows. In Section 2 the random utility choice setting is discussed and in Section 3 the corresponding choice probabilities are derived. In Section 4 the continuous Luce model is discussed and Section 5 is devoted to a theoretical justification of the GEV model. That is, we provide theoretical assumptions that imply the type of utility functions postulated in Section 2. The final Section
discusses briefly the application of the GEV setting to a two-good consumer demand example.

2. Max-stable random utility processes

Let us start by recalling some fundamental properties of the multivariate extreme value distribution which is the basis for GEV. Let \((U_1, U_2, \ldots, U_m)\) be a random vector with probability distribution function \(F(u_1, u_2, \ldots, u_m)\). If for all \(y \in \mathbb{R}\)

\[
\log F(u_1, u_2, \ldots, u_m) = e^{-ay} \log F(u_1 - y, u_2 - y, \ldots, u_m - y)
\]

for some constant \(a > 0\), then \(F\) is a type III multivariate extreme value distribution (MEV). From this characterization it follows immediately that the univariate marginals have distribution \(\exp(-be^{-au})\), \(b > 0\), which is the type III extreme value distribution (see Galambos, 1978).

Now consider stochastic processes \(\{U(x), x \geq 0\}\) for which the finite-dimensional distributions are MEV. This class of processes is called max-stable processes (cf. de Haan, 1984). It is denoted max-stable since it follows from (2.1) that the maximum of independent copies of max-stable processes is max-stable. We shall now state a very useful representation result for max-stable processes given by de Haan (1984).

**Theorem 1 (de Haan):** Assume that \(U = \{U(x), x \geq 0\}\) is max-stable and continuous in probability. Then there exists a (measurable and finite) function \(v(\cdot)\) and a finite measure \(\lambda\) such that \(U\) has the same finite-dimensional distributions as

\[
\max \{v(x, T(z)) + c(z)\}
\]

where \(\{T(z), c(z)\}\) is an enumeration of a Poisson process on \([0,1] \times \mathbb{R}\) with intensity measure

\[
\lambda(dt) \cdot e^{-t} dt.
\]

1) This class is sometimes called the class of generalized extreme value distributions.
Recall that a Poisson process on \([0,1]XR\) is completely analogous to a Poisson process on \(R\). Here the realizations occur independently and have coordinates \((T(z), \epsilon(z))\), respectively. The probability that there is a point within

\[
(t, t+dt) \times (c, c+dc)
\]

is (approximately) equal to

\[
\lambda(dt)e^{-\epsilon}dc
\]

and the expected number of points within an area \(A \subset [0,1]XR\) is given by

\[
\Lambda(A) = \int_A \lambda(dt)e^{-\epsilon}dc.
\]

Let us now introduce the choice context. Assume that the choice universe is \(R\), and let \(K \subset R\) denote a choice set. Let \(\{U(x), x \geq 0\}\) be the individual's utility assigned to \(x\). \(U\) is perceived as a stochastic process by the observer due to unobserved heterogeneity in preferences and in opportunities across consumers. We shall assume that \(U\) is continuous in probability (see appendix, Lemma A2). In that case it follows that we may without loss of generality define the utility process by

\[
U(x) = \max_U U(x,z)
\]

where

\[
(2.2) \quad U(x,z) = v(x, T(z)) + \epsilon(z)/a
\]

and \((T(z), \epsilon(z))\) represents the points of the Poisson process defined above. Furthermore we assume that \(v(x,t)\) is continuous in \(x\) for given \(t\). The representation (2.2) has an interesting behavioral interpretation. We may think of \(z\) as an indexation of a set of latent countable choice alternatives where \(T(z)\) is the attribute value assigned to \(z\) and \(\epsilon(z)\) represent unobservables that affect tastes. Specifically, for given \(z\), \(\epsilon(z)\) is random because consumers differ in their taste for alternative \(z\). The collection
of feasible $T$-values is perceived as random because consumers face different (latent) opportunity sets.

There are many examples that allow this interpretation of unobservable choice sets. In consumer demand for example, the choice of quantity is often related to the choice among different variants of a product or is affected by the choice of job-type as well as of non-market activities.

From (2.2) we can now derive the finite-dimensional distributions of the utility function, $U$.

**Theorem 2:** The finite-dimensional distributions of $U$ have the form

$$ P\left\{ \bigcap_{j=1}^{m} (U(x_j) \leq u_j) \right\} = \exp\left\{ - \int_{0}^{\infty} \exp\left[ \max(\alpha v(x_j, t) - \alpha u_j) \right] \lambda(dt) \right\}. $$

**Proof:** We shall present the proof for the bivariate case, $m=2$, since the general case is completely analogous.

Let

$$ M = \{(t, \varepsilon) | \varepsilon > \max(\alpha v(x_1, t) - \alpha u_1, \alpha v(x_2, t) - \alpha u_2) \} $$

and let $\tilde{M}$ denote the complement of $M$. Evidently

$$ P\{U(x_1) \leq u_1, U(x_2) \leq u_2\} = P\{\max(\alpha v(x_1, T(z)) + \varepsilon(z)) \leq \alpha u_1, \max(\alpha v(x_2, T(z)) + \varepsilon(z)) \leq \alpha u_2\} $$

$$ = P\{(T(z), \varepsilon(z)) \in \tilde{M}, \forall z\} $$

$$ = P\{\text{There are no points of the process in } M\}. $$

The expected number of points in $M$ is given by

$$ \Lambda(M) = \int_{M} \lambda(dt) e^{-\varepsilon} d\varepsilon $$

$$ = \int_{0}^{1} \exp[a \max(\nu(x_1, t) - u_1, \nu(x_2, t) - u_2)] \lambda(dt). $$
By the Poisson law the probability of no points within $M$ is given by $\exp(-\Lambda(M))$ which yields the above theorem.

Q.E.D.

3. Choice probabilities

Let $x^*(K)$ be the random variable that maximizes utility subject to $x \in K$, i.e., $x^*(K)$ is determined by

$$U(x^*(K)) = \sup_{x \in K} U(x)$$

where $K$ is a Borel set in $\mathbb{R}$. Let $\Phi(A|K)$ be the probability measure

$$\Phi(A|K) = P\{ \sup_{x \in A} U(x) = \sup_{x \in K} U(x) \}$$

where $A$ is a Borel set, $A \subseteq K$. Our first concern is whether $\Phi$ is a well defined choice probability.

**Theorem 3:** If the family $\{v(x,t)\}$ (indexed by $t$) is equicontinuous and the choice set $K$ is compact then $x^*(K) \in K$ with probability one.

Moreover the choice probability measure is given by

$$P\{x^*(K) \in A\} = \Phi(A|K)$$

if $A$ and $K$ are compact, $A \subseteq K$.

**Proof:** By Lemma A2 in the Appendix equicontinuity implies that the sample paths of $U$ are continuous with probability one. Since $K$ is compact it follows that maximum is attained within $K$ with probability one.

Q.E.D.

Note that equicontinuity follows if $v(x,t)$ is differentiable in $x$ and
This follows immediately since

\[ |v(y,t) - v(x,t)| \leq |y-x| \sup_t \left| \frac{\partial v(x,t)}{\partial x} \right|.
\]

Now let

\[ v(t,K) = \sup_{x \in K} v(x,t). \]

When \( K \) is compact, then since \( v \) is continuous there exists a point within \( K \) at which the supremum is attained, say \( x(t,K) \). Thus

\[ v(x(t,K),t) = v(t,K). \]

We are now ready to state the following theorem.

**Theorem 4:** The choice probability measure is given by

\[
\Phi(A|K) = \frac{\int_{Q(A,K)} \exp(\dot{v}(t,K))\lambda(dt)}{\int_{Q(A,K)} \exp(\dot{v}(t,K))\lambda(dt)}
\]

where \( A \) and \( K \) are Borel sets, \( A \subset K \), and

\[ Q(A,K) = \{ t \mid v(t,A) = v(t,K) \} \]

The proof of this theorem is given in the appendix.

**Corollary 1:** Assume that \( x(t,K) \) is continuously differentiable and one-to-one as a function of \( t \) for \( t \) in the interior of \( Q(K,K) \). Furthermore let \( \lambda \) be absolutely continuous with respect to the Lebesgue measure. Then the density of \( \Phi \) exists and is given by

\[
\phi(x|K) = \frac{\exp(\dot{v}(x,t(x,K))\lambda'(t(x,K))\dot{t}(x,K)/\lambda)\dot{t}(x,K)}{\int_{0}^{1} \exp(\dot{v}(t,K))\lambda'(t)dt}
\]
where \( t(x, K) \) is the inverse function of \( x(t, K) \) and \( x \) is the interior of \( K \).

**Proof:** The conditions imply that \( t \) is welldefined and is continuously differentiable. Put \( B = [x + \Delta x, x] \). Then by Theorem 4

\[
\Phi(B|K) = \Delta x \exp(av(x, t(x, K))) \lambda'(t(x, K)) \frac{\dot{a}(x, K)}{a_x} \int_0^1 \exp(a v(t, K)) \lambda'(t) dt + o(\Delta x)
\]

since

\[
\dot{v}(t(x, K), K) = v(x, t(x, K)).
\]

Hence dividing both sides of (3.3) by \( \Delta x \) and letting \( \Delta x \to 0 \) the corollary follows.

Q.E.D.

Since the opportunity variable \( T(z) \) here is typically unobserved the measure \( \lambda \) is not identified. Consequently there is no loss of generality by letting \( \lambda(t) = t \).

Note that if there exists a Borel set \( A \) for which \( v(t, A) < v(t, K) \) almost everywhere (\( \lambda \)) then \( Q(A, K) \) becomes empty and thus \( \Phi(A|K) = 0 \). This corresponds to "shadowing" in the terminology of Cosslett: A point in \( A \) is (almost certainly) never chosen because \( A \) is in the "shadow" of some nearby peak of the function \( v \).

An interesting question is whether the continuous GEV satisfies "revealed preference type" of properties similarly to the Luce model. The next result specifies a nonparametrically testable property.

**Theorem 5:** Let \( K_1 \) and \( K_2 \) be compact Borel sets and let \( A_1 \) and \( A_2 \) be disjoint compact Borel sets that lie in the interior of \( K_1 \cap K_2 \). Then if \( v(x, t) \) is strictly quasiconcave in \( x \) for fixed \( t \)
\[ \frac{\Phi(A_1|K_1)}{\Phi(A_2|K_1)} = \frac{\Phi(A_1|K_2)}{\Phi(A_2|K_2)}. \]

Since \( v(x,t) \) is strictly quasiconcave in \( x \) then if \( x(t,K) \) lies in the interior of \( K \) it is determined by local criteria. Therefore \( x(t,K) \) as well as \( v(t,K) \) are independent of \( K \) and it follows that also \( \Omega(A,K) \) is independent of \( K \) if \( A \) lies in the interior of \( K \). Consequently, by Theorem 4 the ratio
\[ \frac{\Phi(A_1|K_1)}{\Phi(A_2|K_1)} \]
is independent of \( K_1 \).

Q.E.D.

This Theorem represents a relaxation of the IIA property. Specifically it states that IIA holds for all choices that are not "corner solutions", that is, for alternatives that are contained in the boundary of the choice set.

4. The continuous Luce model

In this section we demonstrate that the continuous Luce model is consistent with the maximization of a random utility function without applying a finite choice set type of approximation.

Let now \( T(z) \) be a (continuous) observable attribute assigned to the alternative \( z \) in the choice universe. This universe is assumed to consist of a countable collection of alternatives. The utility function is defined by

\[ U(z) = v(T(z)) + \epsilon(z)/a \]

where \( v(\cdot) \) is continuous and \( \{T(z),\epsilon(z)\} \) are the points in the Poisson process on \([0,1]\times \mathbb{R} \) with intensity measure
\[ \lambda(dt) = e^{-\xi \lambda} \, dc. \]
The corresponding choice probability is

\[ P\{T'(K) \in A\} \]

where \( T'(K) \) is the optimal choice of \( T \) from \( K \) and \( A, K \) are Borel sets. 
\( A \subset K \subset [0,1] \). Analogously to the preceding section define

\[ \Phi(A|K) = P\{ \sup_{T(z) \in A} (a(T(z)) + c(z)) = \sup_{T(z) \in K} (a(T(z)) + c(z)) \} \]

**Theorem 6**: We have

\[ P\{T'(K) \in A\} = \Phi(A|K) \]

and

\[ \Phi(A|K) = \frac{\int_A \exp(a(t)) \lambda(dt)}{\int_K \exp(a(t)) \lambda(dt)} \]

where \( A \) and \( K \) are Borel sets, \( A \subset K \subset [0,1] \).

**Proof**: By substituting the domain \([0,1]\) of \( \lambda \) by a (Borel) set \( B \) we get by Theorem 2 with \( m=1 \) that

\[ (4.2) \quad P\{ \max_{T(z) \in B} U(z) \leq u \} = \exp(-e^{-a} \int_B \exp(a(t)) \lambda(dt)) \]

Since \( A \) and \( K-A \) are disjoint sets and the realizations of the Poisson process are stochastically independent we have that

\[ \sup_{T(z) \in A} U(z) \quad \text{and} \quad \sup_{T(z) \in K-A} U(z) \]

are independent. Since they by (4.2) also are extreme value distributed it
follows readily (see for instance Maddala, 1983) that
\[
P\{\max_{T(z)\in A} U(z) > \max_{T(z)\in K-A} U(z)\} = \frac{\int_{A} \exp(\alpha v(t)) \lambda(dt)}{\int_{A} \exp(\alpha v(t)) \lambda(dt) + \int_{K-A} \exp(\alpha v(t)) \lambda(dt)}.
\]
Q.E.D.

From Theorem 5 it follows immediately

**Corollary 2**: Assume that \( \lambda \) is absolutely continuous with respect to the Lebesgue measure. Then the probability density of \( \Phi \) exists and is given by
\[
\varphi(t|K) = \frac{e^{\alpha v(t)'} \lambda'(t)}{\int_{K} e^{\alpha v(t)} \lambda'(t) dt}, \quad t \in K.
\]

**Corollary 3**: The choice probabilities of Theorem 6 satisfy the property independence from irrelevant alternatives.

Let
\[
g(t) = \frac{\lambda'(t)}{\lambda(1)}.
\]
The function \( g \) represents the density of the choice opportunities that a (randomly selected) decision-maker faces. The probability density \( g \) can be expressed as
\[
\varphi(t|K) = \frac{e^{\alpha v(t)} g(t)}{\int_{K} e^{\alpha v(t)} g(t) dt}.
\]

With separate information about the distribution of the opportunity variables \( T(z) \) we realize that it is possible to identify \( g \) (nonparametrically).
5. Justification of the max-stable utility function from theoretical assumptions

Here we present assumptions that imply the max-stable utility structure. In addition to the continuous choice variable, assume a countable space of latent alternatives. Also suppose that there exists a mapping $T: S \rightarrow [0,1]$, $z \sim T(z)$. The interpretation is that $T(z)$ is an index that summarizes (unobservable) qualitative characteristics of alternative $z$. Let $([0,1], \mathcal{B}, \mathbb{P}_B(\cdot|x))$ be a probability space where $\mathcal{B}$ is the Borel field and

$$\mathbb{P}_B(A|x) = \mathbb{P}\{\sup_{T(z) \in A} U(x,z) = \sup_{T(z) \in B} U(x,z)\}$$

for $A, B \in \mathcal{B}$, $A \subset B$ and where $U(x,z)$ is the utility function of $(x,z)$. The interpretation of $\mathbb{P}_B(A|x)$ is as the probability of choosing the latent alternative $z$ with $T(z) \in A$ when $x$ is given and the choice set for $T$ is $B$.

**Assumption 1**: For each $x$, $U(x,z)$, $z = 1, 2$, are i.i.d. Furthermore $U(x',z')$ and $U(x,z)$ are independent when $z \neq z'$.

Assumption 1 states that to the observer all latent alternatives are "orthonormal" and "look the same" apart from purely random disturbances. That is, there are no hierarchical difference between the latent alternatives.

**Assumption 2**: If $A_1, A_2, B \in \mathcal{B}$, $A_1 \subset A_2 \subset B$ then

$$\mathbb{P}_B(A_1|x) = \mathbb{P}_B(A_2|x) \mathbb{P}_{A_2}(A_1|x)$$

Moreover $\mathbb{P}_{[0,1]}(A|x)$ is absolutely continuous with respect to a finite measure $\lambda$ (say).

We recognize this assumption as a version of IIA. In other words $\mathbb{P}_B(A|x)$ is assumed to be a conditional probability measure.

Let us now consider the implications of Assumptions 1 and 2 for the structure of the utility function.

Let $\{A_j\}$ be a finite partition of $[0,1]$, $A_j \in \mathcal{B}$. From Assumption 2 it
follows that there exists a measure, \( \mu(\cdot, x) \), that is proportional to 
\[ P_B(A_j | x) \] such that
\[
P_B(A_j | x) = \frac{\mu(A_j, x)}{\sum_{A_i \in B} \mu(A_i, x)}.
\]

By Assumption 1 and from Yellott (1977) it follows that
\[
(5.1) \quad \max_{T(z) \in A_j} U(x, z) \overset{D}{=} \log \mu(A_j, x) + n_j(x)
\]
where \( \overset{D}{=} \) means equality in distribution and \( n_j, j=1,2,\ldots \), are independent draws from \( \exp(-e^{-\lambda}) \). Since \( \mu(\cdot, x) \) is absolutely continuous with respect to \( \lambda \) we have
\[
\mu(A_j, x) = \int_{A_j} \exp(\alpha v(x, t)) \lambda(\text{d}t)
\]
for some function \( v(x, t) \). Thus (5.1) implies that
\[
\{ \max_{T(z) \in A} U(x, z) \leq u \} = \exp(-e^{-\alpha u} \int A \exp(\alpha v(x, t)) \lambda(\text{d}t))
\]
for \( A \in B \). But this means that we have
\[
(5.2) \quad \max_{T(z) \in A} U(x, z) \overset{D}{=} \max_{T(z) \in A} \{ v(x, T(z)) + \frac{c(z)}{\alpha} \}
\]
where \( \{ T(z), c(z) \}, z=1,2,\ldots \), is an enumeration of the points in a Poisson process on \([0,1] \times \mathbb{R} \) with intensity measure \( \lambda(\text{d}t) = e^{-C} \text{d}c \). To see that (5.2) holds let us for notational convenience define
\[
m(t) = \begin{cases} 
    u - v(x, t) & \text{for } t \in A \\
    -\infty & \text{otherwise.}
\end{cases}
\]

Using this notation we get
\[ P\{\max_{T(z) \in A} \left( v(x, T(z)) + \frac{e(z)}{a} \right) \leq u \} \]
\[ = P\{c(z) \leq au - av(x, T(z)), \forall z, T(z) \in A \} \]
\[ = P\{ \text{There are no points of the Poisson process above the graph of } m(\cdot) \} \]
\[ = \exp\left(- \int_{m(t) \leq c} ae^{-e \text{d}c} \cdot \lambda(\text{d}t) \right) = \exp\left(-e^{-au} \int_{A} \exp(av(x, t)) \lambda(\text{d}t) \right), \]

which proves (5.2). Eq. (5.2) implies that

\[ (5.3) \quad U(x, z) = v(x, T(z)) + \frac{e(z)}{a} \]

We have thus provided a set of assumptions that are consistent with (2.2). We state this below.

**Theorem 7**: If Assumptions 1 and 2 hold then the utility function is distributed as (5.3).

6. Application of the continuous GEV framework to consumer demand. The two-good case.

Let \( U^*(x, y, z) \) be the utility function of the three goods \((x, y, z)\), where \((x, y) \in \mathbb{R}^2_+\) are continuous and \(z\) is countable and unobservable. The variable \(z\) is interpreted as an indexation of all unobserved choice alternatives that affect the preferences for the observable goods \((x, y)\). Assume that

\[ U^*(x, y, z) = v^*(x, y, T(z)) + c(z)/a \]

where \(\{T(z), c(z)\}\) are the points of the bivariate Poisson process as defined above. Let the budget constraint be given by

\[ px + qy = 1 \]
where \( p \) and \( q \) are the respective normalized prices.

Define

\[
U(x,z) = U'(x,(1-px)/q,z)
\]

and

\[
v(x,t) = v'(x,(1-px)/q,t).
\]

Let us assume that \( v'(x,y,t) \) for fixed \( t \) is increasing, strictly quasiconcave and continuously differentiable. Then the demand function for \( x \) conditional on \( z \) is well defined and it is given by \( x(T(z),K) \). The unconstrained demand function conditional on \( z \), \( x(T(z),R_+^t) \), can be found by the application of Roy's identity to the indirect utility function \( v(T(z),R_+) \) and the distribution of the unconditional demand, \( x^*(R_+) \), is found by the application of Theorem 4. Under the assumption above it follows that the corresponding constrained demand distribution satisfies Theorem 5.

From Theorem 4 it also follows that when \( a = 0 \) then the distribution \( \Phi \) reduces to the "conventional" demand distribution which is expressed as

\[
\lim_{a \to 0} \Phi(A|K) = \int_{Q(A,K)^{(0.1)}} \lambda(dt)
\]

i.e., the demand distribution equals the probability mass of all \( \{T(z)\} \) for which the Marshallian demand \( x(T(z),K) \in A \).

The intuition is as follows. From (2.2) we see that when \( a \) is small then the effect of the "systematic" part \( v(x,T(z)) \) will be small. Thus for a value of \( a \) near zero the optimal value of \( z \) would be determined by the sequence \( \{c(z)\} \) which is purely random. Given \( z, x^* \) is determined by maximizing \( v(x,T(z)) \) with respect to \( x \). But this is a standard textbook utility maximization problem when \( T(z) \) is given. Since the determination of \( z \) is independent of the rest of the variables that enter the utility function it follows that \( \{T(z)\} \) can be viewed as a draw from the same distribution as the one that generated the sequence \( \{T(z)\} \).

Thus the GEV framework contains the standard textbook consumer
demand econometric model as a special case where now \( T(z) \) may be interpreted as a conventional heterogeneity parameter with distribution function

\[
\frac{\lambda(t)}{\lambda([0,1])}
\]

An example

Let the conditional (unconstrained) indirect utility have the form

\[
v(t,R_t) = -tb(p/q) + c(1/q)
\]

where \( b'(x) > 0, c'(x) > 0, b''(x) < 0 \).

By the application of Roy's identity we get

\[
x(t) = \frac{t}{\beta(p,q)}
\]

where

\[
\beta(p,q) = \frac{c'(1/q)}{b'(p/q)}.
\]

Hence

\[
\exp(v(t(x),R_t)) = \exp(-bx\beta(p,q) + c(1/q))
\]

From Corollary 1 we get the density of the demand:

\[
\varphi(x|R_t) = \frac{\exp(-abx\beta(p,q))\lambda'(x\beta(p,q))\beta(p,q)}{\int_0^1 \exp(-abt)\lambda(dt)}
\]
Appendix

Lemma A1: Assume that $f$ and $g$ are two bounded functions on $[0,1]$. Let

$$U_1 = \max_z (f(T(z)) + \varepsilon(z)), \quad U_2 = \max_z (g(T(z)) + \varepsilon(z))$$

where $\{T(z), \varepsilon(z)\}$ is an enumeration of a Poisson process on $[0,1] \times \mathbb{R}$ with intensity measure

$$\lambda(dt) = e^{-\varepsilon} dc.$$

If $f > g$ almost everywhere $\lambda$ then $U_1 > U_2$ with probability one. If $f \leq g$, a.e., then $U_1 > U_2$ with probability zero.

Proof: From Theorem 2 it follows that when $f > g$

$$P(U_1 \leq u_1, U_2 \leq u_2) = P(U_1 \leq u_1)$$

for $u_1, u_2$ which is equivalent to

$$P(U_1 \leq u_1, U_2 > u_2) = 0.$$

But then

$$P(U_1 > U_2) = 1.$$

Assume next that $f \leq g$ a.e. $\lambda$. Let

$$A = \{t| f(t) = g(t)\}$$

and define

$$U_{11} = \max_{T(z) \in \tilde{A}} (f(T(z)) + \varepsilon(z)), \quad U_{21} = \max_{T(z) \in \tilde{A}} (g(T(z)) + \varepsilon(z))$$

$$U_0 = \max_{T(z) \in \mathbb{A}} (f(T(z)) + \varepsilon(z)).$$
Where \( \hat{A} \) is the complement of \( A \). We have

\[
P(U_1 > U_2) = P\{ (U_{11} > \max (U_{21}, U_0)) \land (U_0 > \max (U_{21}, U_0)) \}
\]

\[
= P(U_{11} > \max (U_{21}, U_0)) \leq P(U_{11} > \max (U_{21}, U_0))
\]

\[
\leq P(U_{11} > U_{21}) = 1 - P(U_{11} < U_{21}).
\]

On \( \hat{A} \) f\( \geq g \) a.e., which implies that \( U_{11} < U_{21} \) with probability one. Thus \( U_1 > U_2 \) with probability zero.

Q.E.D.

**Lemma A2:** If the family of functions \( \{v(x,t), t \in [0,1]\} \) (indexed by \( t \)) is equicontinuous almost everywhere \( \lambda \) then

\[
U(x) = \max_{z}(v(x,T(z)) + \epsilon(z))
\]

is continuous with probability one.

**Proof:** Consider the event

\[
|U(x) - U(y)| < \delta
\]

which holds if

\[
\max_{z}(v(x,T(z)) + \delta + \epsilon(z)) > \max_{z}(v(y,T(z)) + \epsilon(z))
\]

and

\[
\max_{z}(v(y,T(z)) + \delta + \epsilon(z)) > \max_{z}(v(x,T(z)) + \epsilon(z)).
\]

By applying Lemma A1 we realize that for \( x \) and \( y \) sufficiently close the two inequalities above hold with probability one since by equicontinuity there exists a \( \kappa > 0 \) such that
\[ |v(x,t) - v(y,t)| < \delta \]

when \(|x-y|<\kappa\) except on a set of \(\lambda\) measure zero.

Q.E.D.

**Proof of Theorem 4.**

Let

(A.1.) \[ f_1(t) = \max_{x \in A} v(x,t) = av(t,A). \]

(A.2.) \[ f_2(t) = \max_{x \in K-A} v(x,t) = av(t,K-A), \quad A \subset K, \]

and

\[ E = \{ t | f_1(t) > f_2(t) \}. \]

The event

\[ \{ \max_{T(z)} (f_1(t) + \epsilon(z)) > \max_{T(z)} (f_2(t) + \epsilon(z)) \} \]

is equivalent to

(A.3.) \[ \{ \max_{T(z)} (f_1(t) + \epsilon(z)) > \max_{T(z)} (f_2(t) + \epsilon(z)) \} \]

\[ T(z) \in \bar{E} \]

\[ U \{ \max_{T(z)} (f_1(t) + \epsilon(z)) > \max_{T(z)} (f_2(t) + \epsilon(z)) \} \]

\[ T(z) \in \bar{E} \]

where \(\bar{E}\) is the complement of \(E\). Moreover

\[ \{ \max_{T(z)} (f_1(t) + \epsilon(z)) > \max_{T(z)} (f_2(t) + \epsilon(z)) \} \]

\[ T(z) \in \bar{E} \]
By Lemma A1 the last event has probability zero since \( f_1(t) \leq f_2(t) \) when \( t \in \tilde{E} \). Similarly

\[
\{ \max(f_1(T(z)) + \varepsilon(z)) > \max(f_2(T(z)) + \varepsilon(z)) \}
\]

\[ T(z) \in \tilde{E} \]

\[ z \]

because

\[
\{ \max(f_1(T(z)) + \varepsilon(z)) > \max(f_2(T(z)) + \varepsilon(z)) \}
\]

\[ T(z) \in \tilde{E} \]

\[ T(z) \in \tilde{E} \]

has probability one by Lemma A1. Since \( E \cap \tilde{E} = \emptyset \) it follows directly from theorem 2 that

\[
\max(f_1(T(z)) + \varepsilon(z)) \text{ and } \max(f_2(T(z)) + \varepsilon(z))
\]

\[ T(z) \in E \]

\[ T(z) \in \tilde{E} \]

are stochastically independent with joint distribution function

\[
\exp\left\{-e^{-u_1} \int_{E} \exp(f_1(t)) \lambda(dt) - e^{-u_2} \int_{\tilde{E}} \exp(f_2(t)) \lambda(dt)\right\}.
\]

Hence by straightforward calculus and (A.1.) to (A.4.)

\[
P\{ \max(f_1(T(z)) + \varepsilon(z)) > \max(f_2(T(z)) + \varepsilon(z)) \}
\]

\[ z \]

\[ z \]

\[
= \frac{\int_{E} \exp(f_1(t)) \lambda(dt)}{\int_{E} \exp(f_1(t)) \lambda(dt) + \int_{\tilde{E}} \exp(f_2(t)) \lambda(dt)} = \frac{\int_{E} \exp(av(t,K)) \lambda(dt)}{\int_{0}^{1} \exp(av(t,K)) \lambda(dt)}.
\]

Q.E.D.
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