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OPTIMAL GOVERNMENT DEBT UNDER DISTORTIONARY TAXATION

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The problem of optimal government debt is analysed in a two-period model with optimizing private agents and a distortionary labour tax. The economic role of government debt is to shift tax distortions over time. The role for debt policy is strengthened if the government treats future generations differently or if account is taken of public investment. Government policy may be time-inconsistent, which calls for the introduction of a credibility constraint. The consequences for deficit financing are shown by comparing the open-loop and the feedback Stackelberg solutions in a model with asymmetric preferences.

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1. Introduction

In the 1980's government budget deficits have been exceptionally large in a number of Western countries. This raises the question if and to what extent these deficits have had a negative effect on economic welfare. In Romer (1988) this question is discussed from various perspectives. One way to deal with the problem is to consider the impact of distortionary taxation assuming that lump-sum taxes are not available as an instrument of economic policy. Large deficits today call for some action in the future to warrant government's solvency. If a cut in government spending is undesirable and seigniorage revenues are excluded an increase in future taxation will be necessary to stabilize government debt. In such a case the time profile of taxes may not be optimal and the costs of extensive deficits can be estimated by specifying a "cost-function" of raising tax revenue.

Instead of discussing the costs of exceptional deficits one could pose the problem of the optimal amount of debt right away. This is the approach taken in a seminal paper by Barro (1979), where the distortionary effects of taxation and the costs of collecting taxes are approximated by a reduced form equation. To obtain quantitative estimates on a macro-economic level this seems the proper route to follow. However, to get a more elaborate picture of the factors determining optimal debt it is desirable to start from microfoundations. The debt issue can then directly be related to the theory of optimal taxation in dynamic macroeconomic models (e.g. Turnovsky and Brock (1980), Lucas and Stokey (1983), Rogers (1987), Chang (1988), Hillier (1989), Persson and Tabellini (1990)).

In models with different generations living at the same moment in time there is a role for government debt even if lump sum taxation is practicable. This holds for overlapping generations models (e.g. Diamond (1965), Atkinson and Stiglitz (1980, Ch. 8)) as well as for models with finitely-lived agents discussed in Blanchard (1985) and analysed more fully in Calvo and Obstfeld (1988). Here we abstract from these questions by pre-supposing infinitely-lived households or a society with finitely-lived but nonoverlapping generations. Instead, the paper focusses on the role of government lending and borrowing in a macroeconomic model with a distortionary tax on labour income. It is convenient to cast the analysis in terms of a two-period model, where the second period can be seen as representing the entire future. The economic
role of public debt in such a model is to shift tax distortions over time. Under well specified conditions this policy implies tax smoothing (uniform tariff across periods). The familiar case of tax smoothing will used as a benchmark to analyse the role of government debt in some special but not unimportant situations. More specifically, attention will be paid to optimal public debt when government spending is productive instead of exhaustive. Moreover, we will consider optimal tax policy tax policy in case the government treats present and future generations alike, while the private sector discounts future consumption of goods and leisure. As will be shown, tax smoothing is not optimal in these cases for purely economic reasons. Political aspects of public debt as discussed in Persson and Svensson (1989) and Alesina and Tabellini (1987) are not taken into account.

Even with a tax on wage income only government policy may be time-inconsistent as shown in Rogers (1987) and Persson and Tabellini (1990). Under these circumstances the introduction of a credibility constraint to attain time-consistency may have consequences for debt policy. Credibility may be another reason to deviate from a policy of tax smoothing. A discussion of the problem of time consistency is therefore appropriate to get a proper understanding of the economic factors which may determine public debt.

The paper is organised as follows. In Section 2 we present the basic model and formulate the conditions for tax-smoothing and its implications for debt. Appyling a specification of the model analytical solutions for the cases mentioned above are derived in Section 3. Time-inconsistency is eliminated in Sections 2 and 3 by assuming that government spending is exogenous for the time being. As a consequence the government has no possibility to cheat consumers in the second period. The issue of time-consistency and debt is discussed in Section 4 using a different specification of the basic model. Numerical examples are presented to illustrate the analytical results. The paper closes with some conclusions.

2. The basic model

Consumers have to decide on consumption and leisure in both periods. They live for two periods or alternatively they live for one period and care about their
heirs in the second period. The utility function of a representative agent is assumed to be linear separable:

\[ U = u_1(c_1) + v_1(l-l_1) + h_1(g_1) + \frac{1}{1+\delta}[u_2(c_2) + v_2(l-l_2) + h_2(g_2)], \]  

(2.1)

where \( c_t, l_t, g_t \) denote respectively consumption, labour supply and public spending in period \( t \), \( l \) denotes the maximum available time and \( \delta \) denotes the discount rate. The felicity functions \( u_t, v_t \) and \( h_t \) are increasing and concave. For the time being we assume that government expenditure is exogenous, so that the terms in \( g_t \) can be ignored. The consumer maximizes equation (2.1) subject to budget constraints for periods 1 and 2. The government impose a tax on wage income in both periods. The real wage rate and the interest rate are given for the individual. Assuming zero initial wealth and setting the real wage rate equal to one, we may write

\[ c_1 + s \leq l_1(1-\tau_1), \]  

(2.2)

\[ c_2 \leq l_2(1-\tau_2) + (1+r)s, \]  

(2.3)

where \( s \) denotes first period savings, \( r \) denotes the interest rate on savings and \( \tau_t \) denotes the tax rate in period \( t \). Savings cum interest payments can be spend in the second period.

Maximization of (2.1) subject to (2.2) and (2.3) results in the first-order conditions (assuming an interior solution):

\[ \frac{u_1'(c_1)}{u_2'(c_2)} = \frac{1+r}{1+\delta}, \]  

(2.4)

\[ v_1'(l-l_1) = (1-\tau_1)u_1'(c_1), \]  

(2.5)

\[ v_2'(l-l_2) = (1-\tau_2)u_2'(c_2), \]  

(2.6)
where first derivatives are indicated by a prime. Equation (2.4) states that the marginal rate of substitution between current and future consumption equals the real interest rate. Equation (2.5) and (2.6) specify the leisure-consumption trade-off in each period. Equations (2.4)-(2.6) together with the budget constraints (2.2) and (2.3) can be solved for $c_1, c_2, l_1, l_2$ and $s$.

The real wage rate and the interest rate are determined in factor markets by equating supply and demand. Aggregate labour supply and aggregate savings follow by summing over individuals. Labour demand and demand for (financial) capital follow from the profit maximizing behaviour of firms. The problem can be solved in an elegant way by postulating a linear technology in labour and capital

\[ f(k_t, l_t) = \eta k_t + \eta l_t, \quad t=1,2. \] (2.7)

Equation (2.7) implies that marginal products are constant, so that factor demands are completely elastic. The real wage rate equals $w = \eta$, which is set equal to unity for convenience. For the gross return to capital we can write $1 + r = \eta$.

The government supplies the public consumption good in both periods. Expenditure in the first period can be financed by taxing labour income or by issuing government bonds, which are perfect substitutes for financial assets issued by the private sector. In the second period the government has to raise taxes to pay for second period expenditure on goods and servicing debt incurred in the first period. It is assumed that the government always honours its debt. The alternative of financing a deficit by printing money is not taken into account. The government's budget constraints can now be written as

\[ g_1 \leq l_1 \tau_1 + b, \] (2.8)
\[ g_2 + (1+r)b \leq l_2 \tau_2. \] (2.9)

where \( b \) denotes the level of government debt\(^2\). It should be recalled that public spending is assumed to be exogenous. This assumption is relaxed in Section 4. The government maximizes the welfare of its citizens, but may apply a different discount rate (\( \delta \)). As observed by Ramsey it may be rational not to discount future streams of goods and leisure. The consequences of such a behaviour for debt are analysed in Section 3. In maximizing the common good the government behaves as a leader in a Stackelberg game with atomistic behaving consumers as followers. It announces the optimal tax rate whereupon the private sector chooses the time-paths of consumption and leisure. The reactions of the private sector are taken into account by the government in determining the optimal tax rate on labour income. Moreover, the government must satisfy its budget constraints (2.8) and (2.9).

As is well-known government borrowing may be applied to smooth tax distortions over time. However, it is difficult to characterise the solution of the problem of optimal taxation in general terms. To clarify this issue the special case of tax smoothing \( \tau_1 = \tau_2 \) may serve as a benchmark for further discussion. As appears from equations (2.4)-(2.6) tax smoothing is optimal if the following conditions are fulfilled: (1) the discount rate of the government and the private sector should be the same (\( \delta = \delta \)); (2) the private discount rate should be equal to the rate of interest (\( \delta = r \)); (3) the felicity functions for leisure should be equal across periods (\( v_1 = v_2 \)). It should be noticed that it is not necessary to assume that consumption is smoothed over time (\( c_1 = c_2 \)). The latter result obtains if the felicity function for the consumption of goods are concave and equal across periods (\( u_1 = u_2 \)).

Under conditions (1)-(3) the marginal trade-off between leisure and consumption is the same in both periods. It is then optimal for the government to impose equal tax rates (\( \tau_1 = \tau_2 \), so that \( l_1 = l_2 \)). The optimal amount of debt can then be derived from equation (2.8) and (2.9) (with equality sign):

\[
\text{---------------------------}
\]

2) Summation of the budget constraints of consumers and the government in both periods yields the resource constraint \( c_1 + g_1 + k = l_1 \) and \( c_2 + g_2 = l_2 + s \) assuming \( s = k + b \). As both financial assets give the same rate of return consumers are indifferent with respect to holding private and public debt.
where \( g_P = (g_1 \cdot \frac{g_2}{1+r}) \) is the permanent level of government spending. Debt policy should be aimed at smoothing peaks and troughs in government spending (e.g., Barro (1979)). In the next section we shall present explicit solutions for some situations in which it is not optimal to smooth tax rates.

3. Optimal debt: some special cases

The special cases considered in this section relate to government investment (productive spending) and to a divergence between private and public discount rates. To obtain tractable results the felicity functions have to be specified. This will be done in such a way that the conditions for tax smoothing will be fulfilled. As observed in section 2 tax smoothing may be used as a benchmark in discussing more complex cases of optimal taxation. In section 3.1 the benchmark model will be specified. Government spending is assumed exogenous and exhaustive. The consequences of productive government spending are discussed in section 3.2. In section 3.3 it is assumed that the government behaves in a non-Pigouvian way by treating present and future generations on the same footing (the egalitarian society). The analytical results obtained are illustrated by presenting a numerical example in section 3.4.

3.1 The benchmark model

It is convenient to work with a linear felicity function for consumption of goods \( u(c_t) = c_t \) for \( t=1,2 \). To avoid corner solutions we further assume that the private discount rate equals the rate of interest (\( \delta = r \)). The felicity function for leisure is given by \( v(l-l_t) = \alpha \ln(l-l_t) \), \( t=1,2 \). As can be easily checked the conditions for tax smoothing are fulfilled in this case. Moreover, consumers are indifferent with respect to present and future consumption, so that savings can be set equal to public borrowing: \( s = b \).
Substitution of these assumptions into equation (2.1) gives:

\[ U = c_1 + \alpha \ln(l-l_1) + \frac{1}{1+\delta}(c_2 + \alpha \ln(l-l_2)). \]  (3.1)

Maximization of equation (3.1) subject to the budget constraints (2.2) and (2.3) and assuming \( \alpha=1 \) for convenience yields:

\[ c_1 = \lambda_1 (1-\tau_1) - b, \]  (3.2)
\[ c_2 = \lambda_2 (1-\tau_2) + (1+r)b, \]  (3.3)
\[ \frac{1}{l-l_1} = (1-\tau_1), \]  (3.4)
\[ \frac{1}{l-l_2} = (1-\tau_2). \]  (3.5)

Substitution of these results into equation (3.1) taking account of \( \alpha=1 \) gives the indirect utility function:

\[ V = \frac{\lambda_1}{l-l_1} + \ln(l-l_1) + \frac{1}{1+\delta} \left[ \frac{\lambda_2}{l-l_2} + \ln(l-l_2) \right]. \]  (3.6)

The government maximizes \( V \) subject to the budget constraints (2.8) and (2.9) which can be written as:

\[ g_1 = \lambda_1 [1 - \frac{1}{l-l_1}] + b, \]  (3.7)
\[ g_2 + (1+r)b = \lambda_2 [1 - \frac{1}{l-l_2}]. \]  (3.8)

The first order conditions for a maximum of \( V \) with respect to \( \lambda_1, \lambda_2 \) and \( b \) are:
where \( \lambda_1 \) and \( \lambda_2 \) denote the Lagrange-multipliers associated with the government budget constraints (3.7) and (3.8). Substitution of equation (3.11) in equation (3.10), taking account of the assumption \( \delta = r \), result in:

\[
l_2 + \lambda_1 (l - l_2)^2 = 0.
\] (3.12)

As appears from equations (3.9) and (3.12) the supply of labour is equal across periods, this result is obtained by imposing uniform tax rates over time: \( \tau_1 = \tau_2 \). The optimal amount of government debt follows from equation (2.10).

3.2. Productive government spending

The government has to invest in social overhead capital to keep the economy going. As discussed in Arrow and Kurz (1970) some forms of social overhead may yield direct utility. In other cases government capital may enhance productivity and therefore consumption in an indirect way. The latter aspect is also analysed in the seminal work by Arrow and Kurz, but here we do not take it into account. In line with the two-period model it is assumed that government capital fully depreciates after having been used in the second period. If there is no initial social overhead capital we can equate the stock of capital available at the beginning of the second period with the amount of government investment in the first period which is denoted by \( g_1 \). In our model social overhead capital is supposed to diminish the disutility of labour, because it improves the environment in which people have to work. This idea is captured by the preference function:
Government spending is again an exogenous variable in the model. To realize the intended effects we assume $g_1 > 1$.

There is a Pigouvian government, which maximizes equation (3.13) subject to the government budget constraints and the behavioural equations of economic agents. The first order conditions for this case read:

$$\lambda_1 + \lambda_1 ((l-l_1)^2-l) = 0,$$
$$\frac{1}{1+\delta} \lambda_2 + \lambda_2 (\frac{1}{g_1} (l-l_2)^2-l) = 0,$$
$$\frac{1+r}{1+\delta} - 1 + \lambda_1 - (1+r) \lambda_2 = 0,$$
$$g_1 = \lambda_1 [1 - \frac{1}{(l-l_1)}] + b,$$
$$(1+r)b = \lambda_2 [1 - \frac{g_1}{(l-l_2)}].$$

These results give rise to the following proposition.

**Proposition:** It is optimal to impose a higher tax rate in the second period compared with that in the first period ($\tau_1 < \tau_2$). Labour supply will nevertheless be larger in the second period ($l_1 < l_2$) and government borrowing exceeds the level which is optimal in case of exhaustive expenditure ($b > \frac{g_1}{2+r}$).

**Proof.** From equations (3.14), (3.15) and (3.16) we get

$$\frac{l_1}{l_2} = \frac{(l-l_1)^2-l}{\frac{1}{g_1} (l-l_2)^2-l} < 1$$

if $\frac{1}{g_1} (l-l_1)^2-l > 1$. The latter equation holds for
$\lambda_1 < \lambda_2$ and $g_1 > 1$. From the first order conditions of consumer behaviour we have:

$$\frac{\tau_1}{\tau_2} = \frac{1-1/(l-l_1)}{1-g_1/(l-l_2)} < 1 \text{ if } (l-l_1) > \frac{1}{g_1} (l-l_2),$$

which also holds for $l_1 < l_2$ and $g_1 > 1$. Combining both results we may write $\tau_2 l_2 > \tau_1 l_1$, so that

$$l_2(1 - \frac{g_1}{l-l_2}) > l_1(1 - \frac{1}{l-l_1}).$$

Applying equation (3.18) we can state

$$(1+r)b > l_1(1 - \frac{1}{l-l_1}),$$

which together with equation (3.17) yields

$$(1+r)b > g_1 - b = b > \frac{g_1}{2+r}.$$ q.e.d.

If there is only non-productive government expenditure in period 1 optimal debt equals $b = \frac{g_1}{2+r}$ as appears from formula (2.10) and it is optimal to smooth tax rates. In the case that government spending is productive a larger amount should be borrowed, because it is optimal to impose a higher tax rate in the second period. To put it differently, the work-leisure margin in the second period shifts in favour of work. The second period tax rate is therefore less distortionary and can be increased vis à vis the first period tax rate. Total tax receipts will nevertheless be higher in the second period than under tax smoothing, and so will be the optimal amount of government debt.

### 3.3. The egalitarian society

The government discount rate is set to zero ($\delta = 0$), while the private discount rate is assumed to be positive ($\delta > 0$). In this case one could speak of an egalitarian society, because the government treats present and future generations alike. The second period optimality conditions can in this case be written as:

$$\lambda_2 + \lambda_2((l-l_2)^2-l) = 0.$$ (3.19)
The first period optimum corresponds to equations (3.7) and (3.9). From these conditions the following proposition can be derived

**Proposition.** It is optimal to tax labour in the first period more heavily \((\tau_1 > \tau_2)\), so that relatively more labour is supplied in the second period \((\ell_1 < \ell_2)\). Even with uniform government expenditure across periods \((g_1 = g_2)\) tax smoothing is not optimal in this case and the government retires debt or lends from the public \((b < 0)\).

**Proof.** We first show \(\lambda_1 > 1\), so that \(\lambda_1 > \lambda_2\) from equation (3.20). Equation (3.9) can be written as

\[
\lambda_1 = \frac{\ell_1}{(\ell - (\ell_1 - \ell))^2} = \left(1 - \frac{1}{1 - \tau_1}\right) > 1 \quad \text{for} \quad (1 - \tau_1) < 1.
\]

Dividing equations (3.9) and (3.19) gives:

\[
\pm \frac{\ell_1}{\ell_2} = \frac{\lambda_1}{\lambda_2} \frac{(\ell - \ell_1)^2 - \ell_1}{(\ell - \ell_2)^2 - \ell_2}.
\]

If \(\frac{\ell_1}{\ell_2} < 1\) and \(\frac{\lambda_1}{\lambda_2} > 1\) then we must have \(\frac{(\ell - \ell_1)^2 - \ell}{(\ell - \ell_2)^2 - \ell} < 1\).

Notice that the nominator and denominator are both negative so that this condition can be reformulated as \((\ell - \ell_1)^2 < (\ell - \ell_2)^2\). However, this can only be true if \(\ell_1 < \ell_2\). With \(\tau_1 > \tau_2\) and \(\ell_1 < \ell_2\) we have \(\tau_1 \ell_1 > \tau_2 \ell_2\) because optimal tax rates lie on the rising branch of the Laffer curve. From equations (3.9) and (3.12) it can be derived that this rising branch is characterised by \((\ell - \ell_t)^2 > \ell\) \((t=1,2)\). For \(g_1 = g_2\) we now get \(b < 0\) as follows directly from equations (2.8) and (2.9).

q.e.d.

The government has a lower discount rate than the private sector and therefore raises aggregate savings in favour of future generations. The additional
savings must come from the government sector ($b < 0$), which taxes the younger generation more heavily than the older one ($\tau_1 > \tau_2$). Optimal taxation is now not only a matter of minimizing the excess burden, but also of inter-generational welfare. These objectives are to be traded off in an optimal manner. An alternative way to describe this trade off between objectives is to argue in terms of the marginal cost of public funds, which are reflected in the present value of the Lagrange multipliers (e.g. Persson and Tabellini (1990)). The marginal cost of public funds is higher in the second period $(\lambda_2 \frac{1}{1+r})$, so that it is optimal to have a higher tax rate in the first period.

3.4. A numerical example

The analytical results obtained are illustrated by numerical examples presented in Table 1. The examples are based on the following parameter values and exogenous variables:

Benchmark model : $l=10$, $r=0.5$, $g_1=2$, $g_2=0$.

Productive spending: $l=10$, $r=0.5$, $g_1=2$, $g_2=0$.

Egalitarian society: $l=10$, $r=0.5$, $g_1=2$, $g_2=0$.

[insert Table 1]

In the benchmark model there is tax smoothing and the amount of government debt ($b$) corresponds to formula (2.10). Exhaustive public spending crowds out private consumption in period 1, so that private savings match the public deficit. With productive government expenditure the tax rate in the second period ($\tau_2$) is higher, while the first period rate ($\tau_1$) is lower compared with the benchmark. The lower tax rate in the first period allows the government to raise the amount of borrowing from the private sector. In the second period consumption ($c_2$) is higher despite a higher tariff on wage income. In the egalitarian society the tax rate in the first period is higher than in the
second period. The lower tax rate in the second period allows a higher consumption level \( c_2 \) than that in the first period. Public debt is lower than in the benchmark model because the government fosters aggregate savings. With uniform government expenditure across periods \( g_1 = g_2 = 2 \) there would be a surplus \( b = -0.179 \).

A strict welfare comparison of the cases considered is not possible, because the social welfare function differs across the examples presented. However, the result with respect to \( V \) are plausible. With productive government spending social welfare is higher than with pure exhaustive spending. In the egalitarian society social welfare is higher than in the benchmark model, as a result of the lower public discount rate, which dominates the negative welfare effect of a change in private spending.

4. Time-inconsistency and debt

If government expenditure is taken into account as an endogenous variable the government has two instruments in the second period. Once this period has been arrived it may then be optimal to change the optimal plan with respect to these instruments. The government may surprise or cheat consumers, so that economic policy is time-inconsistent. In a case of wage taxation cheating may be preferable if the consumption-leisure margin ex-post differs from this margin ex-ante. It is always possible to eliminate the cheating option by a suitable choice of tax instruments. In the terminology of game theory such an escape route is called a feedback Stackelberg solution. However, there is a price to be paid for consistency. A feedback Stackelberg solution usually yields a lower welfare level than an ex-ante solution or open-loop Stackelberg solution (e.g. Fischer (1980), Rogers (1987))\(^3\). It may be interesting to analyse the consequences of time-consistency for government debt policy. Unfortunately the problem cannot be solved in general terms, so that we must rely on a (numerical) example.

3) However see for a counter-example Van der Ploeg and De Zeeuw (1989).
Following a suggestion by Persson and Tabellini (1990, Ch. 8) we introduce the following assumptions: \( u_1(c_1) = c_1, u_2(c_2) = \ln(c_2), v_1(l-\lambda_1) = \ln(l-\lambda_1), \)
\( v_2(l-\lambda_2) = \ln(l-\lambda_2), h_2(g_2) = \ln(g_2) \), \( g_1 = 0 \) and \( \delta = 0 \). The utility function of the representative individual can then be expressed as:

\[
U = c_1 + \ln(l-\lambda_1) + \ln(c_2) + \ln(l-\lambda_2) + \ln(g_2). \tag{4.1}
\]

Consumers maximize \( U \) with respect to \( c_1, c_2, \lambda_1 \) and \( \lambda_2 \) subject to their budget constraints. Assuming \( r = 0 \) the intertemporal budget constraint can be written as

\[
c_1 + c_2 = (1-\tau_1)\lambda_1 + (1-\tau_2)\lambda_2. \tag{4.2}
\]

It should be noticed that the conditions for tax smoothing are fulfilled in this case. From the first order conditions for a maximum of \( U \) subject to equation (4.2) the following solutions can easily be derived:

\[
c_1 = (2-\tau_1-\tau_2)\lambda - 3, \tag{4.3}
\]
\[
c_2 = 1, \tag{4.4}
\]
\[
\lambda_1 = \lambda - \frac{1}{1-\tau_1}, \tag{4.5}
\]
\[
\lambda_2 = \lambda - \frac{1}{1-\tau_2}. \tag{4.6}
\]

It is assumed that \( c_1 \) is always positive. Substitution of equations (4.3) - (4.6) in equation (4.1) gives the indirect utility function:

\[
V = (2-\tau_1-\tau_2)\lambda - 3 + \ln\left(\frac{1}{1-\tau_1}\right) + \ln\left(\frac{1}{1-\tau_2}\right) + \ln(g_2). \tag{4.7}
\]
The government maximizes \( V \) with respect to \( \tau_1, \tau_2 \) and \( g_2 \) subject to the inter-temporal budget constraint:

\[
g_2 = \tau_1 l_1 + \tau_2 l_2. \tag{4.8}
\]

First period government expenditure is set equal to zero, because it is inessential for the problem of time-inconsistency. The first order conditions for a maximum of \( V \) are

\[
-\frac{1}{1-\tau_1} + \lambda \left( \frac{1}{1-\tau_1} - \frac{\tau_1}{(1-\tau_1)^2} \right) = 0, \tag{4.9}
\]

\[
-\frac{1}{1-\tau_2} + \lambda \left( \frac{1}{1-\tau_2} - \frac{\tau_2}{(1-\tau_2)^2} \right) = 0, \tag{4.10}
\]

\[
\lambda = \frac{1}{g_2}, \tag{4.11}
\]

where \( \lambda \) denotes the Lagrange multiplier associated with the government budget constraint (4.8). Equations (4.8) - (4.11) can be solved for \( \tau_1, \tau_2, g_2 \) and \( \lambda \). As appears from (4.9) and (4.10) it is indeed optimal to smooth taxes \((\tau_1 = \tau_2)\). Moreover, it should be noted that government debt is negative \((b = -\tau_1 l_1)\), because taxes are raised in the first period, while government spending is zero in this period.

In the second period when the decisions made in period 1 belong to the past the consumers' problem can be specified as

\[
\max_{\{c_2, l_2\}} U_2 = \ln(c_2) + \ln(l_2 - l_2) + \ln(g_2), \tag{4.12}
\]

subject to \( c_2 = (1-\tau_2) l_2 + s \). \tag{4.13}
Savings \((s)\) are invested in private or public bonds as shown in equation (2.12). Denoting the Lagrange multiplier associated with budget constraint (4.13) by \(\mu_2\) the first order conditions can be written as

\[
\mu_2 = \frac{1}{c_2}, \tag{4.14}
\]

\[
\frac{1}{l - \lambda_2} = \mu_2(1 - \tau_2). \tag{4.15}
\]

Solving equations (4.13)-(4.15) for \(c_2\) and \(\lambda_2\) yields

\[
c_2 = \frac{1}{2}[(1 - \tau_2)l + s], \tag{4.16}
\]

\[
\lambda_2 = \frac{1}{2}[l - \frac{s}{1 - \tau_2}]. \tag{4.17}
\]

It should be observed that equation (4.16) and (4.17) are equivalent to equations (4.4) and (4.6) if \(\tau_2\) is not changed. This can easily be checked. Savings are defined as \(s = (1 - \tau_1)l_1 - c_1\). Substitution of equation (4.3) and (4.5) results in

\[
s = 2 - (1 - \tau_2)\lambda. \tag{4.18}
\]

If this outcome is substituted in equations (4.16) and (4.17) we get the second period consumption level and labour supply as shown in respectively equations (4.4) and (4.6). However, there is no reason to assume that \(\tau_2\) remains the same, because the government can take advantage of the situation.

The effects of a rise in the second period tax rate on the supply of labour differ in the ex-ante situation from that in the ex-post situation as can be

[insert Figure 1]
deduced from equations (4.6) and (4.15). There is no reason to suppose that the ex-post Lagrange multiplier $\lambda_2$ is equal to unity. To put it differently, the income effects of a change in the price of leisure are unequal in both cases considered as shown in Figure 1. The initial equilibrium is given in point A, where the marginal substitution ratio between second period consumption and second period leisure equals the price of leisure, which is equal to $(1-\tau_2)$ in the present example. An increase in $\tau_2$ implies a downward rotation of the budget line and the new equilibrium will be at point B in the ex-post situation. The change in $\lambda_2$ is the result of a positive substitution effect and a negative income effect with respect to leisure. Consumption always declines if both goods are normal because the substitution and income effects are then both negative. In the ex-ante situation this will lead to a rise in capital accumulation because the rate of time-preference now falls below the interest rate\(^4\). An increase in non-human wealth shifts the second period budget line upwards as indicated by the vertical arrow in Figure 1. The new equilibrium in the ex-ante situation is at point B with $c_2$ unchanged as required by equation (4.4). The negative income effect with respect to leisure is more than compensated, so that labour supply falls by a smaller amount than in the ex-post situation. The distortionary effect of change in $\tau_2$ is therefore lower ex-post and the government has an incentive to surprise consumers by raising the wage tax in the second period. In the ex-post situation investment decisions are bygones and the individual is unable to rearrange the consumption profile.

Consumers will be aware of the opportunities the government has to cheat them in the second period. A credible tax strategy requires an elimination of the surprise incentive. This can be done by constructing a feedback solution in the following way. First, the government solves the second period problem for given (first period) consumers' savings

\[^4\] The rate of time-preference is equal to: $-dc_2/dc_1 - 1 = u_1'(c_1)/u_2'(c_2) - 1 = c_2 - 1$.\]
Max. \( V_2 = \ln\left[ \frac{1}{2}\left( (1-\tau_2) l + s \right) \right] + \ln\left[ \frac{1}{2}\left( l - \frac{s}{1-\tau_2} \right) \right] + \ln(g_2) \) \hspace{1cm} (4.19)

subject to the budget constraint

\[ \tau_2\left( \frac{1}{2}\left( l - \frac{s}{1-\tau_2} \right) \right) = b + g_2. \] \hspace{1cm} (4.20)

The first order conditions for a maximum of \( V \) are

\[ \nu_2 = \frac{1}{g_2}, \] \hspace{1cm} (4.21)

\[ -\frac{l}{\lambda(1-\tau_2) + s} + \frac{s}{(1-\tau_2)(1-(1-\tau_2)^2)} + \nu_2\left[ \frac{l}{2} - \frac{s}{2(1-\tau_2)} - \frac{\tau_2}{(2(1-\tau_2)^2)} \right] = 0, \] \hspace{1cm} (4.22)

where \( \nu_2 \) denotes the Lagrange multiplier associated with the government budget constraint (4.20). Second, given the optimal choice of \( \tau_2 \) and \( g_2 \) the government solves the problem:

Max. \( V = (2-\tau_1-\tau_2) l - 3 + \ln\left[ \frac{1}{1-\tau_1} \right], \) \hspace{1cm} (4.23)

subject to \( \tau_1\left( \frac{1}{1-\tau_1} \right) = -b. \) \hspace{1cm} (4.24)

Denoting the Lagrange multiplier associated with the government budget constraint (4.24) by \( \nu_1. \) The first order condition can be written as:

\[ -\lambda + \frac{1}{1-\tau_1} + \nu_1\left( \lambda - \frac{1}{1-\tau_1} - \frac{\tau_1}{(1-\tau_1)^2} \right) = 0. \] \hspace{1cm} (4.25)
Third, decisions should be made in a consistent way, so that total utility $V = V_1 + V_2$ is maximized. This requires that the government satisfies its inter-temporal budget constraint, which implies $\nu_1 = \nu_2 = \nu$. The model now consists of equations (4.18), (4.20), (4.21), (4.22), (4.24) and (4.25) and solves for the endogenous variables $\tau_1$, $\tau_2$, $g$, $b$, $s$ and $\nu$.

The consistent or feedback solution will have a higher $\tau_2$ than the time-inconsistent or open-loop solution for reasons given above. If so, the first period tax rate will be lower in the feedback Stackelberg equilibrium. As a consequence the government will save less than in the open-loop solution. Tax distortions are shifted towards the second period to gain credibility. As a result social welfare will be lower. The shift towards taxation in the future raises the potential for government borrowing.

It may be useful to illustrate these results numerically. The outcomes for the different solution concepts discussed in this section are presented in Table 2. Calculations are based on a parameter value $\ell = 3$, so that corner solutions are avoided. Initial non-human wealth is assumed to be zero.

[insert Table 2]

The open-loop solution shows tax rate smoothing and a reduction of government debt. As appears from the last column in Table 2 it is preferable from a welfare point of view to surprise consumers in the second period by raising the tax rate and government spending at the expense of private consumption. The negative effect on second period leisure is small, which corroborates our earlier conclusion. Time-consistency has its price in terms of welfare as appears from the feedback solution. The second period tax rate on wages is now even higher than under cheating. This can be explained by the lower tax rate in period 1, which raises debt service in period 2. However, because the relatively high distortionary impact in the second period compared with the cheating solution the tax rate $\tau_2$ must be high to generate sufficient revenue to pay for all government outlays. Finally, it should be noted that $c_2$ is equal to unity in the feedback solution, because the rate of time preference must be equal to the rate of interest.
5. Conclusions

If preferences are symmetric across periods and the discount rates of the private and the public sector are equal to the real interest rate it is optimal to smooth taxation over time. The optimal amount of debt then depends on transitory components in government spending and the rate of interest. These more familiar results change if the symmetry is relaxed. More specifically, when the government discount rate lies below the private discount rate tax smoothing is no longer optimal. Welfare can then be raised by increasing aggregate savings, which calls for a shift in taxation from the future to the present and a reduction in public debt. With productive government spending things are just the other way around. Public investment extends the tax base in the future as social overhead capital contributes to private consumption either directly or indirectly. As a consequence taxes can be shifted to the future to a certain degree and there is an additional role for government borrowing.

Asymmetry in the sense defined above is also interesting from another perspective. Even if there is no tax on non-human wealth, but only a tax on labour earnings government policy may be time-inconsistent if the symmetry assumption is given up. Under these circumstances a rational expectations equilibrium with the government as leader in a Stackelberg game requires the introduction of a credibility constraint. Comparing the resulting feedback solution with the time-inconsistent open-loop solution it can be concluded that taxes have to be shifted to the the future to generate credibility. If so, there is again an additional role for deficit financing.

Our analysis of public debt points in the direction interesting areas for future research. Here we will mention two such topics but the list could easily be supplemented with other examples. First, the impact of public investment on the economy and on government policies could be given a more complete treatment along the lines set out in Arrow and Kurz (1970). More specifically, it might be rewarding to extend their model by introducing endogenous labour supply. Second, assuming a non-linear technology factor renumerations would become endogenous. In this case an optimal policy should take account of the effects of tax and debt instruments on factor incomes.
References


Arrow, K.J. and M. Kurz, 1970, Public Investment, the Rate of Return, and Optimal Fiscal Policy (John Hopkins Press, Baltimore).


Romer, D., 1988, What are the Costs of Excessive Deficits, NBER Macroeconomics Annual, 63-98.

TABLE 1. Optimal debt in the different situations.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Benchmark Model</th>
<th>Productive Spending</th>
<th>Egalitarian Society</th>
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<td>2.00</td>
<td>2.00</td>
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<tr>
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<td>0.00</td>
<td>0.00</td>
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<td>0.152</td>
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<tr>
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<td>8.821</td>
</tr>
<tr>
<td>l₂</td>
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<td>9.392</td>
<td>8.874</td>
</tr>
<tr>
<td>c₁</td>
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<td>6.889</td>
<td>6.821</td>
</tr>
<tr>
<td>c₂</td>
<td>8.843</td>
<td>9.392</td>
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<tr>
<td>v</td>
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TABLE 2. Time-inconsistency: an example.

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<th>Feedback Stackelberg</th>
<th>&quot;Cheating&quot;</th>
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