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On the sum of Laplacian eigenvalues of graphs

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Abstract

Let \(k\) be a natural number and let \(G\) be a graph with at least \(k\) vertices. A.E. Brouwer conjectured that the sum of the \(k\) largest Laplacian eigenvalues of \(G\) is at most \(e(G) + (k+1)\), where \(e(G)\) is the number of edges of \(G\). We prove this conjecture for \(k = 2\). We also show that if \(G\) is a tree, then the sum of the \(k\) largest Laplacian eigenvalues of \(G\) is at most \(e(G) + 2k - 1\).

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JEL code: C0.

1 Introduction

Let \(G\) be a simple graph with the vertex set \(V(G) = \{v_1, \ldots, v_n\}\). The degree of a vertex \(v \in V(G)\), denoted by \(d(v)\), is the number of neighbors of \(v\). The Laplacian matrix of \(G\) is the \(n \times n\) matrix \(L(G) = [\ell_{ij}]\) that records the vertex degrees \(d(v_1), \ldots, d(v_n)\) on its diagonal and for any \(i \neq j\), \(1 \leq i, j \leq n\), \(\ell_{ij} = -1\) if \(v_i\) and \(v_j\) are adjacent and \(\ell_{ij} = 0\), otherwise. It is well-known that \(L(G)\) is positive semi-definite and so its eigenvalues are nonnegative real numbers. The eigenvalues of \(L(G)\) are called the Laplacian eigenvalues of \(G\) and are denoted by \(\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)\). Note that each row sum of \(L(G)\) is 0 and therefore, \(\mu_n(G) = 0\).

In this paper, we investigate the sum \(S_k(G) = \sum_{i=1}^{k} \mu_i(G)\) for \(1 \leq k \leq n\). We denote the edge set of \(G\) by \(E(G)\) and we let \(e(G) = |E(G)|\). In [2], A.E. Brouwer has conjectured the following.

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Conjecture 1 Let \( G \) be a graph with \( n \) vertices. Then \( S_k(G) \leq e(G) + \binom{k+1}{2} \) for \( k = 1, \ldots, n \).

In [2], Brouwer points out that, by the use of computer, he has checked Conjecture 1 for all graphs with at most 10 vertices. For \( k = 1 \), the conjecture follows from the well-known inequality 
\[ \mu_1(G) \leq |V(G)| \] (see [6, p. 281]). Here, we prove Conjecture 1 for \( k = 2 \). We also show that \( S_k(T) \leq e(T) + 2k - 1 \) for any tree \( T \) and any \( 1 \leq k \leq n \) from which the conjecture follows for trees.

Some results and conjectures related to \( S_k(G) \) can be found in the literature. First we state the Grone-Merris conjecture [7]. Let \( d_i^T = \{|v \in V(G) | d(v) \geq i\} \) for \( i = 1, \ldots, n \). The numbers \( d_1^T \geq d_2^T \geq \cdots \geq d_n^T \) are called the conjugate degrees of \( G \). The Grone-Merris conjecture asserts that \( S_k(G) \leq \sum_{i=1}^{k} d_i^T \) for \( k = 1, \ldots, n \). This inequality for \( k = 1 \) is immediate from \( \mu_1(G) \leq |V(G)| \) and the equality obviously occurs for \( k = n - 1, n \). Moreover, the conjecture has been proved whenever \( k = 2 \) [3, Theorem 7.1] or \( G \) is a tree [9]. Next, we note that the upper bound
\[ S_k(G) \leq 2mk + \sqrt{mk(n-k-1)(n^2-n-2m)} \]
\[ \frac{n-1}{n-1}, \]

is obtained in [10], where \( 1 \leq k < n \) and \( m = e(G) \).

## 2 Notation and Preliminaries

We first present some notation and definitions. For a subset \( X \) of \( V(G) \), \( N(X) \) denotes the set of vertices which have at least one neighbor in \( X \). An independent set in \( G \) is a subset \( Y \) of \( V(G) \) such that no two distinct vertices in \( Y \) are adjacent. Two distinct edges of \( G \) are called independent if they have no common endpoint. A set of pairwise independent edges in \( G \) is called a matching. The maximum size of a matching in \( G \) is known as the matching number of \( G \), denoted by \( m(G) \). For two graphs \( G_1 \) and \( G_2 \), the union of \( G_1 \) and \( G_2 \), denoted by \( G_1 \cup G_2 \), is the graph whose vertex set is \( V(G_1) \cup V(G_2) \) and whose edge set is \( E(G_1) \cup E(G_2) \). If \( V(G_1) \cap V(G_2) = \emptyset \), then the union of \( G_1 \) and \( G_2 \) is denoted by \( G_1 + G_2 \). We denote the complete graph, star and path with \( n \) vertices by \( K_n \), \( S_n \) and \( P_n \), respectively. The complete bipartite graph with the part sizes \( m \) and \( n \) is denoted by \( K_{m,n} \).

Brouwer has checked Conjecture 1 for all graphs with at most 10 vertices. For our purpose we only need the following statement.

**Lemma 1** [2] For any graph \( G \) with at most 8 vertices, \( S_2(G) \leq e(G) + 3 \).

We next state some lemmas and theorems which will be used in the subsequent sections.

**Lemma 2** Let \( n \) be a natural number.

(i) The Laplacian eigenvalues of \( K_n \) are \( n \) with multiplicity \( n - 1 \), and 0.

(ii) The Laplacian eigenvalues of \( S_n \) are \( n, 1 \) with multiplicity \( n - 2 \), and 0.
The following lemma gives an affirmative answer to Conjecture 1 for $k = 1$.

**Lemma 3** [6, p. 281] If $G$ is a graph with $n$ vertices, then $\mu_1(G) \leq n$.

**Theorem 1** [6, p. 291] Let $G$ be a graph with $n$ vertices and let $G'$ be a graph obtained from $G$ by inserting a new edge into $G$. Then the Laplacian eigenvalues of $G$ and $G'$ interlace, that is,

$$\mu_1(G') \geq \mu_1(G) \geq \ldots \geq \mu_n(G') = \mu_n(G) = 0.$$ 

**Theorem 2** [8] Let $G$ be a graph with $n$ vertices and let $G'$ be a graph obtained from $G$ by inserting a new edge into $G$. Then the Laplacian eigenvalues of $G$ and $G'$ interlace, that is,

$$\mu_1(G) \leq \mu_1(G') \leq \ldots \leq \mu_n(G) = \mu_n(G') = 0.$$ 

**Theorem 3** [1] Let $G$ be a graph with $n$ vertices and vertex degrees $d_1 \geq \ldots \geq d_n$. If $G$ is not $K_s + (n-s)K_1$, then $\mu_s(G) \geq d_s - s + 2$ for $1 \leq s \leq n$.

The following theorem from matrix theory plays a key role in our proofs. We denote the eigenvalues of a symmetric matrix $M$ by $\lambda_1(M) \geq \ldots \geq \lambda_n(M)$.

**Theorem 4** [4] (see also [5]) Let $A$ and $B$ be two real symmetric matrices of size $n$. Then for any $1 \leq k \leq n$,

$$\sum_{i=1}^{k} \lambda_i(A + B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

An immediate consequence of Theorem 4 is the following corollary which will be used frequently.

**Corollary 1** Let $G_1, \ldots, G_r$ be some edge disjoint graphs. Then $S_k(G_1 \cup \cdots \cup G_r) \leq \sum_{i=1}^{r} S_k(G_i)$ for any $k$.

The following Lemma asserts that to prove Conjecture 1 for $k = 2$, it suffices to consider connected graphs.

**Lemma 4** Let $G$ be a graph. Then either $S_2(G) = S_2(H)$ for a connected component $H$ of $G$ or $S_2(G) \leq e(G) + 2$.

**Proof.** If the first statement does not hold, then $G$ has two connected components $H_1$ and $H_2$ such that $\mu_1(G) = \mu_1(H_1)$ and $\mu_2(G) = \mu_1(H_2)$. By Lemma 3, we have $\mu_1(H_i) \leq |V(H_i)| \leq e(H_i) + 1$ for $i = 1, 2$. Therefore, $S_2(G) \leq (e(H_1) + 1) + (e(H_2) + 1) \leq e(G) + 2$. \hfill $\square$

The next lemma is the key to our approach. It gives a sufficient condition for the truth of Conjecture 1 with $k = 2$, that holds for almost all graphs.
Lemma 5 If $G$ is a graph with a subgraph $H$ for which $S_2(H) \leq \varepsilon(H)$, then $S_2(G) \leq \varepsilon(G) + 3$.

Proof. Assume that $G$ is a counterexample with a minimum possible number of edges. By Corollary 1, we have $\varepsilon(G) + 3 < S_2(G) \leq S_2(H) + S_2(G - H)$. This implies that $S_2(G - H) > \varepsilon(G - H) + 3$, which contradicts the minimality of $\varepsilon(G)$. □

Lemma 6 Let $G$ be a graph with $n$ vertices. Suppose that there exist two non-adjacent vertices $u, v \in V(G)$ such that $\mu_k(G) \geq d(u) + d(v) + 2$ for some integer $k$, $1 \leq k \leq n$. If $G'$ is the graph obtained from $G$ by inserting edge $e = \{u, v\}$ into $G$, then $S_k(G') \leq S_k(G) + 1$.

Proof. For $i = 1, \ldots, n$, define $\epsilon_i = \mu_i(G') - \mu_i(G)$. By Theorem 1, $\epsilon_i \geq 0$ for any $i$. Let $d_1 \geq \cdots \geq d_n$ and $d'_1 \geq \cdots \geq d'_n$ be vertex degrees of $G$ and $G'$, respectively. Recall that for any graph $\Gamma$, considering the trace of the matrix $L(\Gamma)^2$, we have

$$\sum_{i=1}^{\varepsilon(V(\Gamma))} \mu_i(\Gamma)^2 = \sum_{v \in V(\Gamma)} d(v)^2 + 2\varepsilon(\Gamma).$$

Applying this fact, we have

$$\sum_{i=1}^{n} \mu_i(G')^2 = \sum_{i=1}^{n} d'_i^2 + 2\varepsilon(G')$$

$$= \sum_{i=1}^{n} d_i^2 + 2\varepsilon(G) + 2d(u) + 2d(v) + 4$$

$$= \sum_{i=1}^{n} \mu_i(G)^2 + 2(d(u) + d(v) + 2).$$

This yields that

$$2\mu_k(G) \sum_{i=1}^{k} \epsilon_i \leq \sum_{i=1}^{k} 2\epsilon_i \mu_i(G)$$

$$\leq \sum_{i=1}^{n} \mu_i(G')^2 - \sum_{i=1}^{n} \mu_i(G)^2$$

$$= 2(d(u) + d(v) + 2).$$

Since $\mu_k(G) \geq d(u) + d(v) + 2$, $S_k(G') - S_k(G) = \sum_{i=1}^{k} \epsilon_i \leq 1$ and the assertion follows. □

3 Trees and threshold graphs

In the following, we obtain an upper bound for the sum of the $k$ largest Laplacian eigenvalues of a tree which implies Conjecture 1 for trees.
Theorem 5 Let $T$ be a tree with $n$ vertices. Then $S_k(T) \leq e(T) + 2k - 1$ for $1 \leq k \leq n$.

Proof. We prove the assertion by induction on $|V(T)|$. If $T$ is a star, then by Lemma 2(iii), $S_k(T) = n + k - 1$ for $1 \leq k < n$, and we are done. Thus assume that $T$ is not a star. Then $T$ has an edge whose removing leaves a forest $F$ consisting of two trees $T_1$ and $T_2$, both having at least one edge. Suppose that $k_i$ of the $k$ largest eigenvalues of $F$ comes from the Laplacian spectrum of $T_i$ for $i = 1, 2$, where $k_1 + k_2 = k$. If one of $k_i$, say $k_2$, is zero, then by $|V(T_2)| \geq 2$, Corollary 1, and the induction hypothesis, we conclude that $S_k(T) = S_k(F \cup K_2) \leq S_k(T_1) + S_k(T_2) \leq e(T_1) + 2k_1 - 1 + 2 \leq n + 2k - 2 = e(T) + 2k - 1$. Otherwise, using Corollary 1 and the induction hypothesis, we have $S_k(T) = S_k(T_1 \cup T_2 \cup K_2) \leq S_k(T_1) + S_k(T_2) + S_k(K_2) \leq (e(T_1) + 2k_1 - 1) + (e(T_2) + 2k_2 - 1) + 2 = e(T) + 2k - 1$. This completes the proof. □

A threshold graph is a graph obtained from $K_1$ by a sequence of operations of the form (i) adding an isolated vertex or (ii) taking the complement. It is clear that adding isolated vertices to a graph only increases the multiplicity of the Laplacian eigenvalue 0. This observation and the next theorem shows that Conjecture 1 is valid for threshold graphs.

Theorem 6 Let $G$ be a graph with $n$ vertices and $1 \leq k \leq n - 2$. If $S_k(G) \leq e(G) + \binom{k+1}{2}$, then $S_{n-k-1}(\overline{G}) \leq e(\overline{G}) + \binom{n-k}{2}$, where $\overline{G}$ is the complement of $G$.

Proof. From [6, p. 280], we have $\mu_i(\overline{G}) = n - \mu_{n-i}(G)$ for $i = 1, \ldots, n - 1$. Therefore,

\[
S_{n-k-1}(\overline{G}) = n(n - k - 1) - \left(\mu_{k+1}(G) + \cdots + \mu_{n-1}(G)\right)
= n(n - k - 1) - 2e(G) + \left(\mu_1(G) + \cdots + \mu_k(G)\right)
= n(n - k - 1) - \binom{n}{2} + e(\overline{G}) + \left(\mu_1(G) + \cdots + \mu_k(G)\right) - e(G)
\leq e(\overline{G}) + n(n - k - 1) - \binom{n}{2} + \binom{k+1}{2}
= e(\overline{G}) + \binom{n-k}{2},
\]
as desired. □

4 The case $k = 2$

In this section, we prove Conjecture 1 for $k = 2$. First we establish the conjecture for graphs with matching number at most three and then we conclude the assertion using Lemma 5.

Lemma 7 Let $G$ be a graph with $m(G) = 1$. Then $S_2(G) \leq e(G) + 3$.

Proof. Let $n = |V(G)|$. Since $m(G) = 1$, it is easily checked that either $G = S_m + (n - m)K_1$ for some $m$, $1 \leq m \leq n$ or $G = K_3 + (n - 3)K_1$. By Lemma 2, the assertion holds. □
We say that a connected graph has the form $\triangle$ if it has a subgraph $H$ isomorphic to $K_3$ such that every edge is incident with some vertex of $H$.

**Lemma 8** Let $G$ be a graph of the form $\triangle$. Then $S_2(G) \leq e(G) + 3$.

**Proof.** Let $n = |V(G)|$ and $d_1^T \geq \cdots \geq d_n^T$ be the conjugate degrees of $G$. If $t$ is the number of vertices of degree 1 in $G$, then it is not hard to see that $2(n - t - 3) \leq e(G) - t - 3$. This implies that $d_1^T = n - t \leq e(G) - n + 3$. Since $d_1^T = n$, $d_1^T + d_2^T \leq e(G) + 3$. By [3, Theorem 7.1], the Grone-Merris conjecture is true for $k = 2$. Therefore, $S_2(G) \leq d_1^T + d_2^T \leq e(G) + 3$. \hfill $\Box$

**Lemma 9** Let $n \geq 3$ and let $G$ be a connected spanning subgraph of $K_{2,n-2}$. Then $S_2(G) \leq e(G) + 3$.

**Proof.** Assume that $\{\{v, w\}, B\}$ is the partition of $V(G)$. For simplicity, we write $\mu_i(G) = \mu_i$ for $1 \leq i \leq n$. Let $d_1 \geq \cdots \geq d_n$ be the vertex degrees of $G$ and let $r$ and $s$ be the number of vertices of degree 1 and 2 in $B$, respectively. By Theorem 5, we can assume that $G$ is not a tree. Hence $s \geq 2$ and the degrees $d_1, d_2 \geq 2$ are the degrees of $v$ and $w$. It is easily seen that $s$ rows of $2I - L(G)$ are identical and therefore the multiplicity of 2 as an eigenvalue of $L(G)$ is at least $s - 1$. Similarly, the multiplicity of 1 as eigenvalues of $L(G)$ is at least $r - 2$. If $\mu_2 \leq 2$, then Lemma 3 implies that $\mu_1 + \mu_2 \leq n + 2 < e(G) + 3$. Hence we may assume that $\mu_2 > 2$ and so $\mu_1 \geq \mu_2 \geq \mu_a \geq \mu_b \geq \mu_n = 0$ are the five remaining eigenvalues. By $\text{trace}(L(G)) = \sum_{i=1}^n \mu_i = \sum_{i=1}^n d_i$, we have $\mu_1 + \mu_2 + \mu_a + \mu_b \leq d_1 + d_2 + 4$. Finally, by the interlacing theorem [6, p. 193] for the $(n - 2) \times (n - 2)$ submatrix $D = \text{diag}(1, \ldots, 1, 2, \ldots, 2)$ of $L(G)$, we find that $\mu_a \geq \mu_{n-2} \geq \lambda_{n-2}(D) \geq 1$. Hence $\mu_1 + \mu_2 \leq d_1 + d_2 + 4 - \mu_a - \mu_b \leq d_1 + d_2 + 3 = e(G) + 3$. \hfill $\Box$

**Lemma 10** Let $G$ be a graph with $m(G) = 2$. Then $S_2(G) \leq e(G) + 3$.

**Proof.** By Lemmas 1 and 4, we may assume that $G$ is a connected graph with at least 7 vertices. First suppose that $G$ has a subgraph $H = K_3$ with $V(H) = \{u, v, w\}$. If every edge of $G$ has at least one endpoint in $V(H)$, then by Lemma 8, we are done. Hence assume that there exists an edge $e = \{a, b\}$ whose endpoints are in $V(G) \setminus V(H)$. Let $M = V(G) \setminus \{a, b, u, v, w\}$. Since $m(G) = 2$, there are no edges between $V(H)$ and $M$. Since $|M| \geq 2$, it is easily seen that all vertices in $M$ are adjacent to one of the endpoints of $e$, say $a$. Hence there are no edges between $b$ and $V(H)$. Now by ignoring the edges between $a$ and $V(H)$, we find a subgraph $K$ of $G$ which is a disjoint union of $K_3$ and a star with the center $a$. Since the graph $L = G - E(K)$ is a star, Corollary 1 yields that $S_2(G) \leq S_2(K) + S_2(L) \leq (e(K) + 1) + (e(L) + 2) = e(G) + 3$, as required.

Next assume that $G$ has no $K_3$ as a subgraph. Suppose that $e_1 = \{a_1, b_1\}$ and $e_2 = \{a_2, b_2\}$ are two independent edges in $G$. Since $G$ contains no $3K_2$ and $K_3$ as subgraphs, $M = V(G) \setminus \{a_1, b_1, a_2, b_2\}$ is an independent set and at least one of the two endpoints of $e_i$ has no neighborhood in $M$ for $i = 1, 2$. Assume those endpoints to be $b_1$ and $b_2$. If $b_1$ and $b_2$
are adjacent, then $|M| \geq 2$ yields that all vertices in $M$ are adjacent to only one of the two vertices $a_1$ and $a_2$, say $a_1$. This implies that $G$ is a bipartite graph with the vertex set partition $\{\{a_1, b_2\}, V(G) \setminus \{a_1, b_2\}\}$ and so Lemma 9 yields the assertion. Now assume that $b_1$ and $b_2$ are not adjacent. If $a_1$ and $a_2$ are adjacent, then $G$ is a tree and we are done by Theorem 5. Otherwise, $G$ is a bipartite graph with the vertex set partition $\{\{a_1, a_2\}, V(G) \setminus \{a_1, a_2\}\}$ and using Lemma 9, the proof is complete.

Lemma 11 Let $G$ be a graph with $m(G) = 3$. Then $S_2(G) \leq e(G) + 3$.

Proof. By Lemmas 1 and 4, we may assume that $G$ has no subgraph $H$ with $S_2(H) \leq e(H)$. In particular, Lemma 2 implies that $G$ has no subgraph $3S_3$. Suppose that $G$ has a subgraph $K = K_3 + 2K_2$. Let $x \in V(G) \setminus V(K)$. Since $m(G) = 3$, the vertex $x$ is not incident with the subgraph $K_3$ of $K$ and so $G$ has a subgraph $H = K_3 + S_3 + K_2$. Now by Lemma 2, we have $S_2(H) = e(H)$ and therefore $G$ has no subgraph $K_3 + 2K_2$.

Let $e_1 = \{a_1, b_1\}$, $e_2 = \{a_2, b_2\}$ and $e_3 = \{a_3, b_3\}$ be three independent edges in $G$. Since $m(G) = 3$, $M = V(G) \setminus V(\{e_1, e_2, e_3\})$ is an independent set. Since $G$ has no $4K_2$ and $K_3 + 2K_2$ as subgraphs, either $N(a_i) \cap M = \emptyset$ or $N(b_i) \cap M = \emptyset$, for $i = 1, 2, 3$. With no loss of generality, we may assume that $N(M) \subseteq \{a_1, a_2, a_3\}$. We consider the following three cases.

Case 1. $|N(M)| = 3$. We have $N(M) = \{a_1, a_2, a_3\}$. Since $G$ has no $3S_3$, the bipartite subgraph $G - \{b_1, b_2, b_3\}$ has no perfect matching. By Hall’s Theorem, there exists a subset of $\{a_1, a_2, a_3\}$ with 2 elements, say $\{a_2, a_3\}$, such that $|N(\{a_2, a_3\}) \cap M| = 1$. This means that there exists exactly one vertex $y \in M$ which is adjacent to both $a_2$ and $a_3$. If $d(b_1) \geq 2$, then we clearly find a subgraph isomorphic to $3S_3$ in $G$, a contradiction. Therefore, $d(b_1) = 1$. Suppose that $H$ is the star with center $a_1$ and $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3, y\}$. Then $G - E(H)$ is a disjoint union of a star $S$ with center $a_1$ and a graph $K$ containing $P_5$ with the vertex set $\{a_2, a_3, b_2, b_3, y\}$. Using Theorem 2, we have $\mu_1(P_5) \leq 4$ and by Lemma 2, we obtain that $\mu_1(K) \leq e(K)$. This yields that $S_2(G - E(H)) \leq \mu_1(S) + \mu_1(K) \leq e(G - E(H)) + 1$. Thus $S_2(G) \leq S_2(H) + S_2(G - E(H)) \leq e(G) + 3$, as desired.

Case 2. $|N(M)| = 2$. Without loss of generality, assume that $N(M) = \{a_1, a_2\}$. Since $m(G) = 3$, $b_1$ is not adjacent to $b_2$. If $b_1$ is adjacent to $a_3$ or $b_3$, then changing the role of $e_1, e_2, e_3$ by three independent edges $\{a_1, z\}$, $e_2$, $e_3$ for some vertex $z \in M \cap N(a_1)$, we have Case 1. Therefore, we may assume that $b_1$, and similarly $b_2$, is adjacent to none of the vertices $a_3$ and $b_3$. Let $H$ be the induced subgraph on $\{a_1, a_2, a_3, b_3\}$.

First assume that $H$ has a subgraph $L = K_3$. If $\{a_1, a_2\}$ is an edge of $L$, then clearly any edge of $G$ is incident with $L$ and by Lemma 8, there is nothing to prove. Now assume that exactly one of the two vertices $a_1$ and $a_2$, say $a_1$, is a vertex in $L$. Let $K$ be the disjoint union of $L$ and the induced subgraph of $G$ on $\{a_2, b_2\} \cup (N(a_2) \cap M)$ which is a star with at least three vertices. Note that $G - E(K)$ is a star or a disjoint union of two stars. Now, by Lemma 2 and Corollary 1, $S_2(G) \leq S_2(K) + S_2(G - E(K)) = (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3$, as required.
Next suppose that $H$ has no $K_3$ as a subgraph. Let $t = d(a_3) + d(b_3)$. We have $t = 3, 4$. It is not hard to see that $G - e_3$ contains two disjoint stars $S_i$ with centers $a_1$ and $a_2$. Therefore, by Theorem 1, $\mu_2(G - e_3) \geq \mu_2(2S_t) = t$. Using Lemmas 6 and 10, we find that $S_2(G) \leq S_2(G - e_3) + 1 \leq (e(G - e_3) + 3) + 1 = e(G) + 3$, as required.

**Case 3.** $|N(M)| = 1$. Without loss of generality, assume that $N(M) = \{a_1\}$. If $d(b_1) \geq 2$, then we clearly find three independent edges $e_1', e_2', e_3'$ in $G$ such that the set $M' = V(G) \setminus V(\{e_1', e_2', e_3'\})$ is an independent set and $|N(M')| \geq 2$ which is dealt with as the previous cases. Hence we assume that $d(b_1) = 1$. Suppose that $H$ is the star with center $a_1$ and the vertex set $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3\}$. Then $G - E(H)$ is a disjoint union of a star $S$ with center $a_1$ and a graph $L$ containing $2K_2$ with $V(L) = \{a_2, a_3, b_2, b_3\}$. First assume that $L \neq P_4$. Using Lemma 2(i) and Lemma 3, we have $\mu_1(L) \leq e(L)$. This yields that $S_2(G - E(H)) \leq \mu_1(S) + \mu_1(L) \leq e(G - E(H)) + 1$. Thus $S_2(G) \leq S_2(H) + S_2(G - E(H)) \leq e(G) + 3$, as desired. Next assume that $L = P_4$. With no loss of generality, suppose that $L$ is the path $a_2 \rightarrow b_2 \rightarrow b_3 \rightarrow a_3$. If $|N(a_1) \cap L| = 1$, then $G$ is a tree and the assertion follows from Theorem 5. If $a_1$ is adjacent to both $b_2$ and $b_3$, then by Lemma 8, there is nothing to prove. Suppose that $a_1$ is adjacent to none of $b_2$ and $b_3$. If we let $K$ be the disjoint union of the star $G - V(L)$ and the edges $\{a_2, b_2\}$ and $\{a_3, b_3\}$, then the graph $G - E(K)$ is a disjoint union of a star with the center $a_1$ and the edge $\{b_2, b_3\}$. Now, by Lemma 2 and Corollary 1, we have $S_2(G) \leq S_2(K) + S_2(G - E(K)) \leq (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3$. If none of the above cases occurs, then $G$ is one of the following forms:

If $G = G_1$, then by Theorem 3, we have $\mu_2(G) \geq 3$. Since $d(a_3) + d(b_3) = 3$, applying Lemma 6 for the graph $G - e_3$ and using Lemma 10, we find that $S_2(G) \leq S_2(G - e_3) + 1 \leq (e(G - e_3) + 3) + 1 = e(G) + 3$, as required. Hence assume that $G = G_2$ or $G = G_3$. First suppose that $\mu_2(G) \geq 4$. Since $d(a_3) + d(b_3) = 4$, applying Lemma 6 for the graph $G - e_3$ and using Lemma 10, the result follows. Now suppose that $\mu_2(G) < 4$. By Theorem 2, we have $\mu_1(G_2) \leq |V(G_2)| - 1 = e(G_2) - 1$ and by Lemma 3, $\mu_1(G_3) \leq |V(G_3)| = e(G_3) - 1$. Therefore, $S_2(G) < (e(G) - 1) + 4 = e(G) + 3$. This completes the proof.

We now present the main theorem of the paper.

**Theorem 7** Let $G$ be a graph with at least two vertices. Then $S_2(G) \leq e(G) + 3$. 

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Proof. Using Lemmas 7, 10 and 11, we may assume that $G$ has a subgraph $H = 4K_2$, which satisfies $S_2(H) = e(H)$. So the result follows by Lemma 5. □

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