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Abstract
Let $k$ be a natural number and let $G$ be a graph with at least $k$ vertices. A.E. Brouwer conjectured that the sum of the $k$ largest Laplacian eigenvalues of $G$ is at most $e(G) + \left(\frac{k+1}{2}\right)$, where $e(G)$ is the number of edges of $G$. We prove this conjecture for $k = 2$. We also show that if $G$ is a tree, then the sum of the $k$ largest Laplacian eigenvalues of $G$ is at most $e(G) + 2k - 1$.

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1 Introduction

Let $G$ be a simple graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$. The degree of a vertex $v \in V(G)$, denoted by $d(v)$, is the number of neighbors of $v$. The Laplacian matrix of $G$ is the $n \times n$ matrix $L(G) = [\ell_{ij}]$ that records the vertex degrees $d(v_1), \ldots, d(v_n)$ on its diagonal and for any $i \neq j$, $1 \leq i, j \leq n$, $\ell_{ij} = -1$ if $v_i$ and $v_j$ are adjacent and $\ell_{ij} = 0$, otherwise. It is well-known that $L(G)$ is positive semi-definite and so its eigenvalues are nonnegative real numbers. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of $G$ and are denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$. Note that each row sum of $L(G)$ is 0 and therefore, $\mu_n(G) = 0$.

In this paper, we investigate the sum $S_k(G) = \sum_{i=1}^{k} \mu_i(G)$ for $1 \leq k \leq n$. We denote the edge set of $G$ by $E(G)$ and we let $e(G) = |E(G)|$. In [2], A.E. Brouwer has conjectured the following.

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Conjecture 1 Let $G$ be a graph with $n$ vertices. Then $S_k(G) \leq e(G) + \binom{k+1}{2}$ for $k = 1, \ldots, n$.

In [2], Brouwer points out that, by the use of computer, he has checked Conjecture 1 for all graphs with at most 10 vertices. For $k = 1$, the conjecture follows from the well-known inequality $\mu_1(G) \leq |V(G)|$ (see [6, p. 281]). Here, we prove Conjecture 1 for $k = 2$. We also show that $S_k(T) \leq e(T) + 2k - 1$ for any tree $T$ and any $1 \leq k \leq n$ from which the conjecture follows for trees.

Some results and conjectures related to $S_k(G)$ can be found in the literature. First we state the Grone-Merris conjecture [7]. Let $d_i^T = |\{v \in V(G) \mid d(v) \geq i\}|$ for $i = 1, \ldots, n$. The numbers $d_1^T \geq d_2^T \geq \cdots \geq d_n^T$ are called the conjugate degrees of $G$. The Grone-Merris conjecture asserts that $S_k(G) \leq \sum_{i=1}^k d_i^T$ for $k = 1, \ldots, n$. This inequality for $k = 1$ is immediate from $\mu_1(G) \leq |V(G)|$ and the equality obviously occurs for $k = n - 1, n$. Moreover, the conjecture has been proved whenever $k = 2$ [3, Theorem 7.1] or $G$ is a tree [9]. Next, we note that the upper bound

$$S_k(G) \leq \frac{2mk + \sqrt{mk(n - k - 1)(n^2 - n - 2m)}}{n - 1},$$

is obtained in [10], where $1 \leq k < n$ and $m = e(G)$.

2 Notation and Preliminaries

We first present some notation and definitions. For a subset $X$ of $V(G)$, $N(X)$ denotes the set of vertices which have at least one neighbor in $X$. An independent set in $G$ is a subset $Y$ of $V(G)$ such that no two distinct vertices in $Y$ are adjacent. Two distinct edges of $G$ are called independent if they have no common endpoint. A set of pairwise independent edges in $G$ is called a matching. The maximum size of a matching in $G$ is known as the matching number of $G$, denoted by $m(G)$. For two graphs $G_1$ and $G_2$, the union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$. If $V(G_1) \cap V(G_2) = \emptyset$, then the union of $G_1$ and $G_2$ is denoted by $G_1 + G_2$. We denote the complete graph, star and path with $n$ vertices by $K_n$, $S_n$ and $P_n$, respectively. The complete bipartite graph with the part sizes $m$ and $n$ is denoted by $K_{m,n}$.

Brouwer has checked Conjecture 1 for all graphs with at most 10 vertices. For our purpose we only need the following statement.

Lemma 1 [2] For any graph $G$ with at most 8 vertices, $S_2(G) \leq e(G) + 3$.

We next state some lemmas and theorems which will be used in the subsequent sections.

Lemma 2 Let $n$ be a natural number.

(i) The Laplacian eigenvalues of $K_n$ are $n$ with multiplicity $n - 1$, and 0.

(ii) The Laplacian eigenvalues of $S_n$ are 1 with multiplicity $n - 2$, and 0.
The following lemma gives an affirmative answer to Conjecture 1 for $k = 1$.

**Lemma 3** [6, p. 281] If $G$ is a graph with $n$ vertices, then $\mu_1(G) \leq n$.

**Theorem 1** [6, p. 291] Let $G$ be a graph with $n$ vertices and let $G'$ be a graph obtained from $G$ by inserting a new edge into $G$. Then the Laplacian eigenvalues of $G$ and $G'$ interlace, that is,

$$
\mu_1(G') \geq \mu_1(G) \geq \cdots \geq \mu_n(G') = \mu_n(G) = 0.
$$

**Theorem 2** [8] Let $G$ be a graph. Then $\mu_1(G) \leq \max\{d(v) + m(v) \mid v \in V(G)\}$, where $m(v)$ is the average of the degrees of the vertices of $G$ adjacent to the vertex $v$.

**Theorem 3** [1] Let $G$ be a graph with $n$ vertices and vertex degrees $d_1 \geq \cdots \geq d_n$. If $G$ is not $K_s + (n - s)K_1$, then $\mu_s(G) \geq d_s - s + 2$ for $1 \leq s \leq n$.

The following theorem from matrix theory plays a key role in our proofs. We denote the eigenvalues of a symmetric matrix $M$ by $\lambda_1(M) \geq \cdots \geq \lambda_n(M)$.

**Theorem 4** [4] (see also [5]) Let $A$ and $B$ be two real symmetric matrices of size $n$. Then for any $1 \leq k \leq n$,

$$
\sum_{i=1}^{k} \lambda_i(A + B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).
$$

An immediate consequence of Theorem 4 is the following corollary which will be used frequently.

**Corollary 1** Let $G_1, \ldots, G_r$ be some edge disjoint graphs. Then $S_k(G_1 \cup \cdots \cup G_r) \leq \sum_{i=1}^{r} S_k(G_i)$ for any $k$.

The following Lemma asserts that to prove Conjecture 1 for $k = 2$, it suffices to consider connected graphs.

**Lemma 4** Let $G$ be a graph. Then either $S_2(G) = S_2(H)$ for a connected component $H$ of $G$ or $S_2(G) \leq e(G) + 2$.

**Proof.** If the first statement does not hold, then $G$ has two connected components $H_1$ and $H_2$ such that $\mu_1(G) = \mu_1(H_1)$ and $\mu_2(G) = \mu_1(H_2)$. By Lemma 3, we have $\mu_1(H_i) \leq |V(H_i)| \leq e(H_i) + 1$ for $i = 1, 2$. Therefore, $S_2(G) \leq (e(H_1) + 1) + (e(H_2) + 1) \leq e(G) + 2$. □

The next lemma is the key to our approach. It gives a sufficient condition for the truth of Conjecture 1 with $k = 2$, that holds for almost all graphs.
Lemma 5 If $G$ is a graph with a subgraph $H$ for which $S_2(H) \leq e(H)$, then $S_2(G) \leq e(G) + 3$.

Proof. Assume that $G$ is a counterexample with a minimum possible number of edges. By Corollary 1, we have $e(G) + 3 < S_2(G) \leq S_2(H) + S_2(G - H)$. This implies that $S_2(G - H) > e(G - H) + 3$, which contradicts the minimality of $e(G)$. □

Lemma 6 Let $G$ be a graph with $n$ vertices. Suppose that there exist two non-adjacent vertices $u, v \in V(G)$ such that $\mu_k(G) \geq d(u) + d(v) + 2$ for some integer $k$, $1 \leq k \leq n$. If $G'$ is the graph obtained from $G$ by inserting edge $e = \{u, v\}$ into $G$, then $S_k(G') \leq S_k(G) + 1$.

Proof. For $i = 1, \ldots, n$, define $\epsilon_i = \mu_i(G') - \mu_i(G)$. By Theorem 1, $\epsilon_i \geq 0$ for any $i$. Let $d_1 \geq \cdots \geq d_n$ and $d'_1 \geq \cdots \geq d'_n$ be vertex degrees of $G$ and $G'$, respectively. Recall that for any graph $\Gamma$, considering the trace of the matrix $L(\Gamma)^2$, we have
\[
\sum_{i=1}^{\lvert V(\Gamma) \rvert} \mu_i(\Gamma)^2 = \sum_{v \in V(\Gamma)} d(v)^2 + 2e(\Gamma).
\]
Applying this fact, we have
\[
\sum_{i=1}^{n} \mu_i(G')^2 = \sum_{i=1}^{n} d_i'^2 + 2e(G')
\]
\[
= \sum_{i=1}^{n} d_i^2 + 2e(G) + 2d(u) + 2d(v) + 4
\]
\[
= \sum_{i=1}^{n} \mu_i(G)^2 + 2(d(u) + d(v) + 2).
\]
This yields that
\[
2\mu_k(G) \sum_{i=1}^{k} \epsilon_i \leq \sum_{i=1}^{k} 2\epsilon_i \mu_i(G)
\]
\[
\leq \sum_{i=1}^{n} \mu_i(G')^2 - \sum_{i=1}^{n} \mu_i(G)^2
\]
\[
= 2(d(u) + d(v) + 2).
\]
Since $\mu_k(G) \geq d(u) + d(v) + 2$, $S_k(G') - S_k(G) = \sum_{i=1}^{k} \epsilon_i \leq 1$ and the assertion follows. □

3 Trees and threshold graphs

In the following, we obtain an upper bound for the sum of the $k$ largest Laplacian eigenvalues of a tree which implies Conjecture 1 for trees.
Theorem 5 Let $T$ be a tree with $n$ vertices. Then $S_k(T) \leq e(T) + 2k - 1$ for $1 \leq k \leq n$.

Proof. We prove the assertion by induction on $|V(T)|$. If $T$ is a star, then by Lemma 2(ii), $S_k(T) = n + k - 1$ for $1 \leq k < n$, and we are done. Thus assume that $T$ is not a star. Then $T$ has an edge whose removing leaves a forest $F$ consisting of two trees $T_1$ and $T_2$, both having at least one edge. Suppose that $k_i$ of the $k$ largest eigenvalues of $F$ comes from the Laplacian spectrum of $T_i$ for $i = 1, 2$, where $k_1 + k_2 = k$. If one of $k_i$, say $k_2$, is zero, then by $|V(T_2)| \geq 2$, Corollary 1, and the induction hypothesis, we conclude that $S_k(T) = S_k(F \cup K_2) \leq S_{k_1}(T_1) + S_{k_2}(K_2) \leq (e(T_1) + 2k_1 - 1) + 2 \leq n + 2k - 2 = e(T) + 2k - 1$. Otherwise, using Corollary 1 and the induction hypothesis, we have $S_k(T) = S_k(T_1 \cup T_2 \cup K_2) \leq S_{k_1}(T_1) + S_{k_2}(T_2) + S_{k_3}(K_2) \leq (e(T_1) + 2k_1 - 1) + (e(T_2) + 2k_2 - 1) + 2 = e(T) + 2k - 1$. This completes the proof. \qed

A threshold graph is a graph obtained from $K_1$ by a sequence of operations of the form (i) adding an isolated vertex or (ii) taking the complement. It is clear that adding isolated vertices to a graph only increases the multiplicity of the Laplacian eigenvalue 0. This observation and the next theorem shows that Conjecture 1 is valid for threshold graphs.

Theorem 6 Let $G$ be a graph with $n$ vertices and $1 \leq k \leq n - 2$. If $S_k(G) \leq e(G) + \binom{k + 1}{2}$, then $S_{n-k-1}(\overline{G}) \leq e(\overline{G}) + \binom{n-k}{2}$, where $\overline{G}$ is the complement of $G$.

Proof. From [6, p. 280], we have $\mu_i(\overline{G}) = n - \mu_{n-i}(G)$ for $i = 1, \ldots, n - 1$. Therefore,

\[
S_{n-k-1}(\overline{G}) = n(n - k - 1) - (\mu_{k+1}(G) + \cdots + \mu_{n-1}(G)) \\
= n(n - k - 1) - 2e(G) + (\mu_1(G) + \cdots + \mu_k(G)) \\
= n(n - k - 1) - \left(\binom{n}{2}\right) + e(\overline{G}) + (\mu_1(G) + \cdots + \mu_k(G)) - e(G) \\
\leq e(\overline{G}) + n(n - k - 1) - \left(\binom{n}{2}\right) + \binom{k + 1}{2} \\
= e(\overline{G}) + \left(\binom{n-k}{2}\right),
\]

as desired. \qed

4 The case $k = 2$

In this section, we prove Conjecture 1 for $k = 2$. First we establish the conjecture for graphs with matching number at most three and then we conclude the assertion using Lemma 5.

Lemma 7 Let $G$ be a graph with $m(G) = 1$. Then $S_2(G) \leq e(G) + 3$.

Proof. Let $n = |V(G)|$. Since $m(G) = 1$, it is easily checked that either $G = S_m + (n - m)K_1$ for some $m$, $1 \leq m \leq n$ or $G = K_3 + (n - 3)K_1$. By Lemma 2, the assertion holds. \qed
We say that a connected graph has the form $\triangle$ if it has a subgraph $H$ isomorphic to $K_3$ such that every edge is incident with some vertex of $H$.

Lemma 8 Let $G$ be a graph of the form $\triangle$. Then $S_2(G) \leq e(G) + 3$.

Proof. Let $n = |V(G)|$ and $d_1^T \geq \cdots \geq d_n^T$ be the conjugate degrees of $G$. If $t$ is the number of vertices of degree 1 in $G$, then it is not hard to see that $2(n - t - 3) \leq e(G) - t - 3$. This implies that $d_2^T = n - t \leq e(G) - n + 3$. Since $d_1^T = n$, $d_1^T + d_2^T \leq e(G) + 3$. By [3, Theorem 7.1], the Grone-Merris conjecture is true for $k = 2$. Therefore, $S_2(G) \leq d_1^T + d_2^T \leq e(G) + 3$. □

Lemma 9 Let $n \geq 3$ and let $G$ be a connected spanning subgraph of $K_{2,n-2}$. Then $S_2(G) \leq e(G) + 3$.

Proof. Assume that $\{\{v, w\}, B\}$ is the partition of $V(G)$. For simplicity, we write $\mu_i(G) = \mu_i$ for $1 \leq i \leq n$. Let $d_1 \geq \cdots \geq d_n$ be the vertex degrees of $G$ and let $r$ and $s$ be the number of vertices of degree 1 and 2 in $B$, respectively. By Theorem 5, we can assume that $G$ is not a tree. Hence $s \geq 2$ and the degrees $d_1, d_2 \geq 2$ are the degrees of $v$ and $w$. It is easily seen that $s$ rows of $2I - \mathcal{L}(G)$ are identical and therefore the multiplicity of 2 as an eigenvalue of $\mathcal{L}(G)$ is at least $s - 1$. Similarly, the multiplicity of 1 as eigenvalues of $\mathcal{L}(G)$ is at least $r - 2$. If $\mu_2 \leq 2$, then Lemma 3 implies that $\mu_1 + \mu_2 \leq n + 2 < e(G) + 3$. Hence we may assume that $\mu_2 > 2$ and so $\mu_1 \geq \mu_2 \geq \mu_a \geq \mu_b \geq \mu_n = 0$ are the five remaining eigenvalues. By $\text{trace}(\mathcal{L}(G)) = \sum_{i=1}^n \mu_i = \sum_{i=1}^n d_i$, we have $\mu_1 + \mu_2 + \mu_a + \mu_b \leq d_1 + d_2 + 4$. Finally, by the interlacing theorem [6, p. 193] for the $(n - 2) \times (n - 2)$ submatrix $D = \text{diag}(1, \ldots, 1, 2, \ldots, 2)$ of $\mathcal{L}(G)$, we find that $\mu_a \geq \mu_{n-2} \geq \lambda_{n-2}(D) \geq 1$. Hence $\mu_1 + \mu_2 \leq d_1 + d_2 + 4 - \mu_a - \mu_b \leq d_1 + d_2 + 3 = e(G) + 3$. □

Lemma 10 Let $G$ be a graph with $m(G) = 2$. Then $S_2(G) \leq e(G) + 3$.

Proof. By Lemmas 1 and 4, we may assume that $G$ is a connected graph with at least 7 vertices. First suppose that $G$ has a subgraph $H = K_3$ with $V(H) = \{u, v, w\}$. If every edge of $G$ has at least one endpoint in $V(H)$, then by Lemma 8, we are done. Hence assume that there exists an edge $e = \{a, b\}$ whose endpoints are in $V(G) \setminus V(H)$. Let $M = V(G) \setminus \{a, b, u, v, w\}$. Since $m(G) = 2$, there are no edges between $V(H)$ and $M$. Since $|M| \geq 2$, it is easily seen that all vertices in $M$ are adjacent to one of the endpoints of $e$, say $a$. Hence there are no edges between $b$ and $V(H)$. Now by ignoring the edges between $a$ and $V(H)$, we find a subgraph $K$ of $G$ which is a disjoint union of $K_3$ and a star with the center $a$. Since the graph $L = G - E(K)$ is a star, Corollary 1 yields that $S_2(G) \leq S_2(K) + S_2(L) \leq (e(K) + 1) + (e(L) + 2) = e(G) + 3$, as required.

Next assume that $G$ has no $K_3$ as a subgraph. Suppose that $e_1 = \{a_1, b_1\}$ and $e_2 = \{a_2, b_2\}$ are two independent edges in $G$. Since $G$ contains no $3K_2$ and $K_3$ as subgraphs, $M = V(G) \setminus \{a_1, b_1, a_2, b_2\}$ is an independent set and at least one of the two endpoints of $e_i$ has no neighborhood in $M$ for $i = 1, 2$. Assume those endpoints to be $b_1$ and $b_2$. If $b_1$ and $b_2$
are adjacent, then \(|M| \geq 2\) yields that all vertices in \(M\) are adjacent to only one of the two vertices \(a_1\) and \(a_2\), say \(a_1\). This implies that \(G\) is a bipartite graph with the vertex set partition \(\{a_1, b_2\}, V(G) \setminus \{a_1, b_2\}\) and so Lemma 9 yields the assertion. Now assume that \(b_1\) and \(b_2\) are not adjacent. If \(a_1\) and \(a_2\) are adjacent, then \(G\) is a tree and we are done by Theorem 5. Otherwise, \(G\) is a bipartite graph with the vertex set partition \(\{a_1, a_2\}, V(G) \setminus \{a_1, a_2\}\) and using Lemma 9, the proof is complete.

**Lemma 11** Let \(G\) be a graph with \(m(G) = 3\). Then \(S_2(G) \leq e(G) + 3\).

**Proof.** By Lemmas 1 and 4, we may assume that \(G\) has a connected graph with at least 9 vertices. Using Lemma 5, we may suppose that \(G\) has no subgraph \(H\) with \(S_2(H) \leq e(H)\). In particular, Lemma 2 implies that \(G\) has no subgraph \(3S_3\). Suppose that \(G\) has a subgraph \(K = K_3 + 2K_2\). Let \(x \in V(G) \setminus V(K)\). Since \(m(G) = 3\), the vertex \(x\) is not incident with the subgraph \(K_3\) of \(K\) and so \(G\) has a subgraph \(H = K_3 + S_3 + K_2\). Now by Lemma 2, we have \(S_2(H) = e(H)\) and therefore \(G\) has no subgraph \(K_3 + 2K_2\).

Let \(e_1 = \{a_1, b_1\}, e_2 = \{a_2, b_2\}\) and \(e_3 = \{a_3, b_3\}\) be three independent edges in \(G\). Since \(m(G) = 3\), \(M = V(G) \setminus V(\{e_1, e_2, e_3\})\) is an independent set. Since \(G\) has no 4\(K_2\) and \(K_3 + 2K_2\) as subgraphs, either \(N(a_i) \cap M = \emptyset\) or \(N(b_i) \cap M = \emptyset\), for \(i = 1, 2, 3\). With no loss of generality, we may assume that \(N(M) \subseteq \{a_1, a_2, a_3\}\). We consider the following three cases.

**Case 1.** \(|N(M)| = 3\). We have \(N(M) = \{a_1, a_2, a_3\}\). Since \(G\) has no \(3S_3\), the bipartite subgraph \(G - \{b_1, b_2, b_3\}\) has no perfect matching. By Hall’s Theorem, there exists a subset of \(\{a_1, a_2, a_3\}\) with 2 elements, say \(\{a_2, a_3\}\), such that \(|N(\{a_2, a_3\}) \cap M| = 1\). This means that there exists exactly one vertex \(y \in M\) which is adjacent to both \(a_2\) and \(a_3\). If \(d(b_1) \geq 2\), then we clearly find a subgraph isomorphic to \(3S_3\) in \(G\), a contradiction. Therefore, \(d(b_1) = 1\). Suppose that \(H\) is the star with center \(a_1\) and \(V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3, y\}\). Then \(G - E(H)\) is a disjoint union of a star \(S\) with center \(a_1\) and a graph \(K\) containing \(P_3\) with the vertex set \(\{a_2, a_3, b_2, b_3, y\}\). Using Theorem 2, we have \(\mu_1(P_3) \leq 4\) and by Lemma 2, we obtain that \(\mu_1(K) \leq e(K)\). This yields that \(S_2(G - E(H)) \leq \mu_1(S) + \mu_1(K) \leq e(G - E(H)) + 1\). Thus \(S_2(G) \leq S_2(H) + S_2(G - E(H)) \leq e(G) + 3\), as desired.

**Case 2.** \(|N(M)| = 2\). Without loss of generality, assume that \(N(M) = \{a_1, a_2\}\). Since \(m(G) = 3\), \(b_1\) is not adjacent to \(b_2\). If \(b_1\) is adjacent to \(a_3\) or \(b_3\), then changing the role of \(e_1, e_2, e_3\) by three independent edges \(\{a_1, z\}, e_2, e_3\) for some vertex \(z \in M \cap N(a_1)\), we have Case 1. Therefore, we may assume that \(b_1\), and similarly \(b_2\), is adjacent to none of the vertices \(a_3\) and \(b_3\). Let \(H\) be the induced subgraph on \(\{a_1, a_2, a_3, b_3\}\).

First assume that \(H\) has a subgraph \(L = K_3\). If \(\{a_1, a_2\}\) is an edge of \(L\), then clearly any edge of \(G\) is incident with \(L\) and by Lemma 8, there is nothing to prove. Now assume that exactly one of the two vertices \(a_1\) and \(a_2\), say \(a_1\), is a vertex in \(L\). Let \(K\) be the disjoint union of \(L\) and the induced subgraph of \(G\) on \(\{a_2, b_2\}, N(a_2) \cap M\) which is a star with at least three vertices. Note that \(G - E(K)\) is a star or a disjoint union of two stars. Now, by Lemma 2 and Corollary 1, \(S_2(G) \leq S_2(K) + S_2(G - E(K)) = (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3\), as required.
Next suppose that \( H \) has no \( K_3 \) as a subgraph. Let \( t = d(a_3) + d(b_3) \). We have \( t = 3, 4 \). It is not hard to see that \( G - e_3 \) contains two disjoint stars \( S_i \) with centers \( a_1 \) and \( a_2 \). Therefore, by Theorem 1, \( \mu_2(G - e_3) \geq \mu_2(2S_t) = t \). Using Lemmas 6 and 10, we find that \( S_2(G) \leq S_2(G - e_3) + 1 \leq (e(G - e_3) + 3) + 1 = e(G) + 3 \), as required.

**Case 3.** \( |N(M)| = 1 \). Without loss of generality, assume that \( N(M) = \{a_1\} \). If \( d(b_1) \geq 2 \), then we clearly find three independent edges \( e'_1, e'_2, e'_3 \) in \( G \) such that the set \( M' = V(G) \setminus V(\{e'_1, e'_2, e'_3\}) \) is an independent set and \( |N(M')| \geq 2 \) which is dealt with as the previous cases. Hence we assume that \( d(b_1) = 1 \). Suppose that \( H \) is the star with center \( a_1 \) and the vertex set \( V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3\} \). Then \( G - E(H) \) is a disjoint union of a star \( S \) with center \( a_1 \) and a graph \( L \) containing \( 2K_2 \) with \( V(L) = \{a_2, a_3, b_2, b_3\} \). First assume that \( L \neq P_4 \). Using Lemma 2(i) and Lemma 3, we have \( \mu_1(L) \leq e(L) \). This yields that \( S_2(G - E(H)) \leq \mu_1(S) + \mu_1(L) \leq e(G - E(H)) + 1 \). Thus \( S_2(G) \leq S_2(H) + S_2(G - E(H)) \leq e(G) + 3 \), as desired. Next assume that \( L = P_4 \). With no loss of generality, suppose that \( L \) is the path \( a_2 - b_2 - b_3 - a_3 \). If \( |N(a_1) \cap L| = 1 \), then \( G \) is a tree and the assertion follows from Theorem 5. If \( a_1 \) is adjacent to both \( b_2 \) and \( b_3 \), then by Lemma 8, there is nothing to prove. Suppose that \( a_1 \) is adjacent to none of \( b_2 \) and \( b_3 \). If we let \( K \) be the disjoint union of the star \( G - V(L) \) and the edges \( \{a_2, b_2\} \) and \( \{a_3, b_3\} \), then the graph \( G - E(K) \) is a disjoint union of a star with the center \( a_1 \) and the edge \( \{b_2, b_3\} \). Now, by Lemma 2 and Corollary 1, we have \( S_2(G) \leq S_2(K) + S_2(G - E(K)) \leq (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3 \). If none of the above cases occurs, then \( G \) is one of the following forms:

If \( G = G_1 \), then by Theorem 3, we have \( \mu_2(G) \geq 3 \). Since \( d(a_3) + d(b_3) = 3 \), applying Lemma 6 for the graph \( G - e_3 \) and using Lemma 10, we find that \( S_2(G) \leq S_2(G - e_3) + 1 \leq (e(G - e_3) + 3) + 1 = e(G) + 3 \), as required. Hence assume that \( G = G_2 \) or \( G = G_3 \). First suppose that \( \mu_2(G) \geq 4 \). Since \( d(a_3) + d(b_3) = 4 \), applying Lemma 6 for the graph \( G - e_3 \) and using Lemma 10, the result follows. Now suppose that \( \mu_2(G) < 4 \). By Theorem 2, we have \( \mu_1(G_2) \leq |V(G_2)| - 1 = e(G_2) - 1 \) and by Lemma 3, \( \mu_1(G_3) \leq |V(G_3)| = e(G_3) - 1 \). Therefore, \( S_2(G) < (e(G) - 1) + 4 = e(G) + 3 \). This completes the proof.

We now present the main theorem of the paper.

**Theorem 7** Let \( G \) be a graph with at least two vertices. Then \( S_2(G) \leq e(G) + 3 \).
Proof. Using Lemmas 7, 10 and 11, we may assume that $G$ has a subgraph $H = 4K_2$, which satisfies $S_2(H) = e(H)$. So the result follows by Lemma 5.

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References


