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Abstract

The tax depreciation decision potentially has significant impact on the profitability of firms and projects. Indeed, the depreciation method chosen for tax purposes affects the timing of tax payments, and, as a consequence, it also affects the after-tax net present value of investment projects. Previous research focusses on the optimal choice of depreciation method under the assumption that the depreciation method has to be set ex ante and cannot be changed during the useful life of the asset. In reality however, changes are allowed under certain circumstances. This paper develops a dynamic programming approach to determine the firm's optimal choice with regard to the initial depreciation method, and whether changes of method are proposed in later periods.

Keywords: Tax depreciation, Net Present Value, Dynamic Programming.
JEL codes: C61, M41

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1 Introduction

In order to determine a firm's taxable income, cash flows are reduced with depreciation charges that reflect the decrease in economic value of the firm's assets. For practical purposes, a number of standardized depreciation methods have been designed, and firms can choose a method from this set. Typically, tax authorities allow firms to choose between the straight line depreciation method, which divides the asset value equally over the useful life of the asset, and a specified accelerated method, which assigns higher depreciation charges to earlier periods. The choice of depreciation method potentially has important consequences. Indeed, since the depreciation method affects the timing of tax payments, it affects the net present value of the firm's after-tax revenues. Existing literature shows that tax depreciation can significantly affect the value of the firm and its investment behavior (see, e.g., Sansing, 1998, De Waegenaere et al., 2003, Wielhouwer et al., 2000, and Arkin and Slastnikov, 2007).

The focus in our paper is on the choice of depreciation strategy that minimizes the expected present value of tax payments. The early literature on optimal tax depreciation (e.g., Davidson and Drake, 1961, 1964, Roemich et al., 1978, and Wakeman, 1980) assumes that the cash flow in every future period is high enough to cover the highest possible depreciation charge in that period. Then, it is optimal to choose the most accelerated depreciation method. The reason is that due to the time value of money, a dollar of tax paid *this year* decreases firm value to a larger extent than a dollar of tax paid *in a later year*. Therefore, it is optimal to depreciate as much as possible as early as possible, but never more than the cash flow in the corresponding period. More recent literature extends the early work by allowing for uncertain future cash flows, reinvestments, and/or progressive tax structures (Berg and Moore 1989, Berg et al. 2001, De Waegenaere and Wielhouwer 2002, and Wielhouwer et al., 2002), and shows that it is no longer necessarily the case that the most accelerated method is preferred.

The above described literature assumes that the depreciation method has to be set ex ante and cannot be changed during the lifetime of the asset. However, firms typically have the option to request a change of method (see, e.g., the US Department of Treasury, 2008). It is intuitively clear that the option to change method can reduce the expected net present value (NPV) of tax payments significantly. Ignoring the option value might imply that the firm foregoes positive NPV investment projects. Our goal in this paper is therefore to determine the firm's optimal strategy with regard to the choice of depreciation method in the year of acquisition of an asset, and whether or not a change of method is proposed in later years. Since firms needs to have an accurate motivation for a proposed change, we consider a model where proposing and negotiating a change is costly, and there exists a positive probability that the tax authority will not accept the proposal. We allow this probability to depend on whether changes were proposed in earlier periods, and whether they were accepted.

We formulate the firm's decision problem as a dynamic optimization problem, and determine recursive relationships for the corresponding value functions. Then, we use these recursive relationships to investigate the effect of discounting (i.e., the time value of money) on the optimal depreciation strategy. We show that the key result in the static setting, which states that stronger discounting (i.e. a lower discount factor) works in favor of more accelerated methods (see, e.g., Berg et al., 2001), does not extend to the dynamic setting. Specifically, we identify settings where the opposite holds, in the sense that switching to a *more* accelerated method is preferred only if the discount factor is sufficiently *high*. Finally, in a numerical analysis we show that the option to change method can reduce expected discounted tax payments significantly, and that the value of the option depends crucially on how the probability that proposed changes are accepted depends on whether prior changes were proposed and/or accepted.

The paper is organized as follows. Section 2 presents the model and the optimization problem. Section 3 reformulates the problem as a dynamic optimization problem. Section 4 deals with the effect of discounting on the optimal solution. Section 5 gives a numerical analysis. Section 6 concludes. All proofs are deferred to the Appendix.

2 The model

A firm has an asset of value D at time 0, which can be depreciated for tax purposes over a maximum of N periods. The depreciation charge in period i reflects the reported decrease in value of the asset during that period. We formally distinguish a *depreciation method* and a *depreciation scheme*.

Definition 1

- *A depreciation scheme for an asset of value D consists of a vector $\overline{d} = (d_1, \cdots, d_N) \in \mathbb{R}_+^N$ for some N that satisfies*

$$\sum_{i=1}^{N} d_i = D. \tag{1}$$

- *A depreciation method M is represented by a function $f_M(\cdot,\cdot) : \mathbb{N} \times \mathbb{N} \rightarrow [0,1]$ that satisfies*

$$\sum_{i=1}^{N} f_M(i,N) = 1, \qquad \textit{for all } N. \tag{2}$$

When depreciation method M is used throughout the depreciable life of an asset with initial value D, the corresponding depreciation scheme is given by

$$d_i = f_M(i,N) \cdot D, \qquad \textit{for all } i = 1, \cdots, N.$$

An example of a commonly used depreciation method is the *straight line depreciation method* (SDM), which divides the depreciation charges equally over the depreciable life of the asset, i.e.

$$f_{SDM}(i,N) \;\; = \tfrac{1}{N}, \quad \text{for } i = 1, ..., N. \tag{3}$$

In contrast, the *sum of the years' digits method* (SYD) is an accelerated method in the sense that the depreciation charges decrease over time. Specifically,

$$f_{SYD}(i,N) \;\; = \tfrac{2(N-i+1)}{N(N+1)}, \quad \text{for } i = 1, ..., N. \tag{4}$$

Another example of an accelerated method is the *double declining balance method* (DDB), which yields:

$$\begin{aligned} f_{DDB}(i,N) \;\; &= \tfrac{2}{N}\left(1 - \tfrac{2}{N}\right)^{i-1}, \quad &&\text{for } i = 1, ..., N-1, \\ &= \left(1 - \tfrac{2}{N}\right)^{N-1}, \quad &&\text{for } i = N. \end{aligned} \tag{5}$$

The choice of depreciation scheme potentially has important consequences for tax payments. Indeed, taxable income equals cash flow minus depreciation charges. If the cash flow in a given year exceeds the depreciation charge for that year, so that taxable income is positive, then taxes are paid on the difference between the cash flow and the depreciation charge. If the cash flow is lower than the depreciation charge, taxable income is negative, and no taxes are paid.[1] This implies that the expected present value of all future tax payments is given by:

$$\tau \sum_{i=1}^{N} \alpha^{(i)} E\left[(C_i - d_i)^+\right], \tag{6}$$

where C_i denotes the random cash flow in year i, d_i denotes the depreciation charge in year i, $(C_i - d_i)^+ = \max\{C_i - d_i, 0\}$, $\tau \in (0,1]$ denotes the tax rate, and $\alpha^{(i)}$ denotes the present value of one unit to be paid i years from now.

The existing literature focuses on the choice of depreciation method that minimizes (6) over a given set of acceptable depreciation methods, under the assumption that the depreciation method is chosen at date zero, and never changed afterwards. As stated in the introduction, however, firms typically have the possibility to propose a change of depreciation method in later periods. This implies that, in addition to choosing the initial depreciation method, the firm has to decide whether or not a change is proposed at the beginning of periods $i = 2, \dots, N$. Our goal in this paper is to determine the firm's optimal strategy with regard to the choice of depreciation method, and whether or not changes are proposed. According to common practice, we consider a setting where there are two acceptable depreciation methods, A and B, e.g. an accelerated method and the straight line method, and we use the following notation:

$$\begin{aligned} M^c &= B, \quad \text{if } M = A, \\ &= A, \quad \text{if } M = B. \end{aligned}$$

With two methods and a possible change in each period, there are at most 2^{N-1} possible depreciation schemes. When there is no uncertainty regarding whether a proposal to change will be accepted, the firm's decision problem amounts to choosing the depreciation method that minimizes (6) among a subset of these 2^{N-1} depreciation schemes.

[1] We abstract from carry over possibilities (compensating taxable profits with losses in earlier years, or losses with profits in earlier years). Earlier research (see e.g. Berg et al. 2001) shows that carry over possibilities do not significantly affect the qualitative results.

The subset is determined by the tax authority's policy with regard to acceptance of changes. This is illustrated in the following example.

Example 2 *Suppose that the tax authority accepts the first proposal to change method, and rejects all subsequent proposals. Then, there are* $2N$ *possible depreciation schemes, resulting from using depreciation method* $M \in \{A, B\}$ *in periods* $i = 1, \cdots, k$*, and method* M^c *in periods* $i = k+1, \cdots, N$*, for* $k = 1, \cdots, N$*. The corresponding depreciation charges are given by:*

$$\begin{aligned} d_i &= f_M(i, N) D_0, & i &= 1, \cdots, k, \\ &= f_{M^c}(i-k, N-k) D_k, & i &= k+1, \cdots, N, \end{aligned}$$

where D_0 *denotes the initial asset value, and* D_k *denotes the residual tax base at the start of period* $k+1$*, i.e. the initial asset value reduced with prior depreciation charges.*

In this paper, however, we consider a setting where there is uncertainty as to whether a proposal will be accepted. The probability that a proposal to change will be accepted in a given period may depend on whether changes were proposed in prior periods and/or whether they were accepted. This implies that the firm optimally conditions it's decision to propose a change on whether or not changes were proposed in prior periods and whether they were accepted. In addition, since the tax authority requires detailed motivation, proposing changes is costly. The objective of the firm then is to minimize the sum of the expected discounted tax payments and the expected cost of proposing and negotiating changes through the following decision variables:

(i) the choice of the initial depreciation method M, and

(ii) the choice whether or not to propose a change in period i, depending on whether or not changes were proposed in prior periods and whether they were accepted.

Now let for any given period i, a decision node j represent one possible scenario with respect to whether changes were proposed in prior periods, and whether they were accepted. Since either no change is proposed, a change is proposed but not accepted, or a change is proposed and accepted, there are 3^{i-2} decision nodes in period i. We therefore introduce the decision variables:

$M \in \{A, B\} =$ the initial depreciation method, and, for every $i = 2, \cdots, N$, and $j = 1, \cdots, 3^{i-2}$,

$$\begin{aligned} \xi_{i,j} &= 0, \qquad \text{if no change is proposed in period } i, \text{ decision node } j; \\ &= 1, \qquad \text{if a change is proposed in period } i, \text{ decision node } j, \end{aligned} \tag{7}$$

and the random variables:

$$\begin{aligned} \phi_{i,j} &= 0, \qquad \text{if } \xi_{i,j} = 0; \\ &= 0, \qquad \text{if } \xi_{i,j} = 1, \text{ and the proposed change is rejected;} \\ &= 1, \qquad \text{if } \xi_{i,j} = 1, \text{ and the proposed change is accepted.} \end{aligned} \tag{8}$$

We are now ready to formulate the optimization problem. First, it is clear that if the firm's decision in period i depends on whether changes were proposed and/or accepted in prior periods, uncertainty regarding acceptance of proposals implies that the depreciation charge that will be applied in a given period, as well as whether a change will be proposed, are random variables that depend on the initial choice M and the change strategy ξ. Specifically, for any given (M, ξ), we denote

- $\overline{d}_i(M, \xi)$ for the random variable that yields the depreciation charge in period i, for $i = 1, \cdots, N$;
- $\overline{\xi}_i(M, \xi)$ for the random variable that equals 1 if a change is proposed at the beginning of period i, and 0 otherwise, for $i = 2, \cdots, N$.

Now, let k denote the cost of proposing a change. Then, the amount of tax to be paid at the end of period i equals $\tau(C_i - \overline{d}_i(M, \xi))^+$, and the cost of proposing a change at the beginning of period i is given by k if $\overline{\xi}_i(M, \xi) = 1$, and 0 otherwise. The optimal initial choice M and the optimal change strategy ξ therefore solve the following optimization problem:

$$\begin{aligned} \min \quad & \tau \sum_{i=1}^{N} \alpha^{(i)} E[(C_i - \overline{d}_i(M, \xi))^+] + k \sum_{i=2}^{N} \alpha^{(i-1)} E[\overline{\xi}_i(M, \xi)] \\ \text{s.t.} \quad & \begin{cases} M \in \{A, B\} \\ \xi_i \in \{0, 1\}^{3^{i-2}}, \text{ for } i = 2, \cdots, N. \end{cases} \end{aligned} \tag{9}$$

It is clear that the optimal strategy depends on k and τ only through $\kappa = k/\tau$. Therefore, without loss of generality, we let $\tau = 1$.

In order to solve the optimization problem, the probability distribution of $\overline{d}_i(M, \xi)$ for $i = 1, \cdots, N$ as well as the probability distribution of $\overline{\xi}_i(M, \xi)$ for $i = 2, \cdots, N$, need to be determined for any given change strategy ξ, and any initial method M. These probability distributions clearly depend on how the probability of acceptance of a proposal depends on whether proposals were made in earlier periods, and whether they were accepted.

The following proposition considers the case where at most one proposal will be accepted. First, we introduce some notation. Without loss of generality, we let decision node $(i, 1)$ represent the unique node in period i in which no changes were proposed in prior periods. Moreover, let $d_{(M,k)}$, $k = 1, \cdots, N$, represent the depreciation scheme that results from using method $M \in \{A, B\}$ in periods $i = 1, \cdots, k$, and using method M^c in periods $k + 1, \cdots, N$.

Proposition 3 *Consider a setting where, as long as no prior proposal has been accepted, the probability that a proposal will be accepted is constant and equal to p. As soon as a prior proposal has been accepted, all future proposals will be rejected. Then, under the optimal strategy, the probability distribution of the resulting depreciation scheme is given by:*

$$\begin{aligned} P(\overline{d}(M, \xi) = d_{(M,k)}) \quad &= \xi_{k+1,1} \cdot p \cdot (1-p)^{\sum_{j=2}^{k} \xi_{j,1}}, \quad && k = 1, \cdots, N-1, \\ &= (1-p)^{\sum_{j=2}^{N} \xi_{j,1}}, && k = N, \end{aligned} \tag{10}$$

and the probability that a change is proposed at the beginning of period i is given by:

$$P(\overline{\xi}_i(M, \xi) = 1) = \xi_{i,1} \cdot (1-p)^{\sum_{j=2}^{i-1} \xi_{j,1}}. \tag{11}$$

The above proposition shows that in the case where the probability of acceptance is constant as long as no prior change has been accepted, and drops to zero as soon as a change has been accepted, the number of decision variables in optimization problem (6) reduces from $1 + \sum_{i=2}^{N} 3^{i-2}$ to N. The firm's optimal strategy can then be found by plugging in (10) and (11) in the objective function of optimization problem (9), and determining the optimal value over all $M \in \{A, B\}$, and $\xi_{i,1} \in \{0, 1\}$ for $i = 2, \cdots, N$. We illustrate this in the following example.

Example 4 *Consider an asset with a depreciable lifetime of 5 years, i.e. $N = 5$, and an initial value of $D_0 = 5$. The firm initially has the option to choose either the Straight line Depreciation Method (SDM), or the Sum of the Years Digits method (SYD), and can propose a change of method in later periods. As long as no prior proposal has been accepted, the probability that a proposal will be accepted is constant and equal to $p = 0.9$. As soon as a prior proposal has been accepted, all future proposals will be rejected. The decision tree is displayed in Figure 1.*

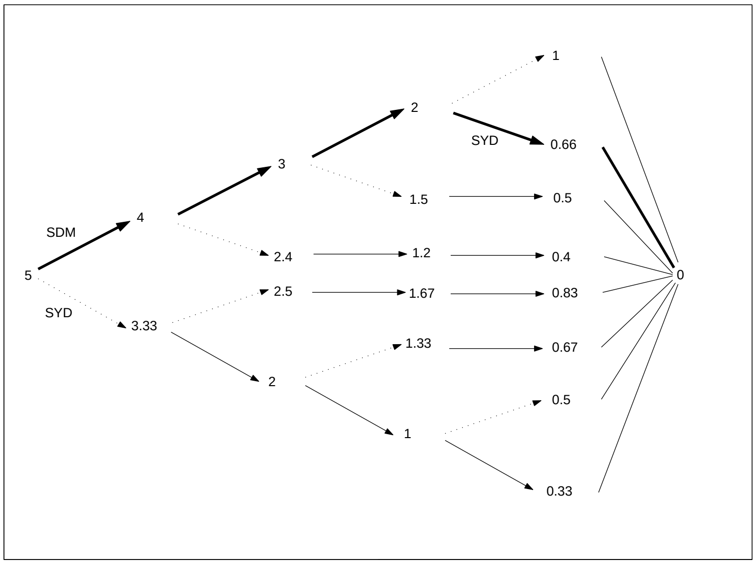

Figure 1: *Decision tree. In every decision node, the upper (lower) arrow corresponds to the use of SDM (SYD); a straight arrow indicates that a change is no longer possible. In each node, the optimal (suboptimal) choice is indicated with a solid (dotted) arrow. The optimal trajectory is indicated with bold arrows. The values at the nodes indicate the residual tax base in that node.*

The only possible depreciation schemes are $d_{(M,k)}$, for $M \in \{SDM, SYD\}$ and $k = 1, \cdots, N$, with depreciation charges given by:

$$\begin{aligned} d_{(M,k),i} &= f_M(i,5) \cdot 5, && i = 1, \cdots, k, \\ &= f_{M^c}(i-k, 5-k) \left(5 - \textstyle\sum_{j=1}^{k} d_{(M,k),j}\right), && i = k+1, \cdots, 5. \end{aligned}$$

Now, we let $\alpha^{(i)} = \alpha^i$ with $\alpha = 0.95$, and $\kappa = k/\tau = 0.01$. The cash flow in period $i = 1, \cdots, 5$ is normally distributed with mean μ_i and standard deviation σ_i as given in *the following table:*

period i	1	2	3	4	5
μ_i	1.5	1	2	3	0.5
σ_i	1	2	2	2	3

Without the possibility to change depreciation method, the expected discounted tax payments using SDM and SYD are 5.36 *and* 5.34*, respectively, so that SYD is the preferable method. Allowing for the possibility to change, however, implies that it is optimal to start with SDM, and propose a change of method in period* 4*. This strategy reduces the expected tax costs (tax payments plus costs of proposing changes) to* 5.26*. The possibility to change depreciation method therefore reduces expected tax costs by 1.5%.* ◇

In the setting described in Proposition 3, it is possible to determine the probability distribution of the depreciation charges in each period for any given strategy. In more general cases, however, this becomes complicated. Then, an efficient method to solve optimization problem (9) is dynamic programming. In the next section, we therefore develop a dynamic optimization approach to determine the firm's optimal strategy.

3 Dynamic optimization

The basic idea of the dynamic optimization approach is to recursively determine the *value function* in period i, for $i = N, N-1, \cdots, 1$. This value function yields the minimal expected present value of the sum of tax payments and costs of proposing changes in periods $i, \cdots, N$, for any possible value of the *state variables* at the beginning of period i. Given the analysis in the previous section, we let the state variables in period i be given by $s_i = (M_1, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$, where:

- $M_1 \in \{A, B\}$ denotes the method that is used in period 1;
- $\overline{\xi}_{i-1} = (\xi_1, \cdots, \xi_{i-1}) \in \{0,1\}^{i-1}$ denotes whether a change was proposed at the beginning of periods $j = 1, \cdots, i-1$;
- $\overline{\phi}_{i-1} = (\phi_1, \cdots, \phi_{i-1}) \in \{0,1\}^{i-1}$ denotes whether a change was accepted in periods $j = 1, \cdots, i-1$.

In order to determine the optimal decision at the start of period i for any given value of s_i, the following has to be known:

- the expected tax payments in period i, as a function of s_i and the decision at the beginning of period i;

- the probability distribution of s_{i+1}, as a function of s_i and the decision at the beginning of period i.

It therefore remains to: i) specify the probability distribution of the state variables, and, ii) show that $(M_1, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$ is sufficient to determine the expected tax payments in period i. This will be dealt with in subsections 3.1 and 3.2, respectively. Subsection 3.3 presents the recursive evaluation of the value functions.

3.1 Probability distribution of state variables

In period $i = 1$, there is no proposal to change, and so $\xi_1 = \phi_1 = 0$. Therefore, $s_2 = (M_1, 0, 0)$ with probability 1. For periods $i = 2, \cdots, N$, we let:

$$p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) = P(\phi_i = 1 | \xi_i = 1; \xi_2, ..., \xi_{i-1}; \phi_2, ..., \phi_{i-1}),$$

denote the probability that a proposed change will be accepted at the beginning of period i, as a function of whether or not changes were proposed in periods prior to period i (i.e. $\xi_2, ..., \xi_{i-1}$), and whether or not they were accepted (i.e. $\phi_2, ..., \phi_{i-1}$).

Now let the state variables at the beginning of period i be given by $s_i = (M_1, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$. Then, the probability distribution of the state variables in period $i + 1$ depends on whether a change is proposed at the beginning of period i. Specifically,

- when no change is proposed, $\xi_i = \phi_i = 0$, and so $s_{i+1} = (M_1, (\overline{\xi}_{i-1}, 0), (\overline{\phi}_{i-1}, 0))$ with probability 1;

- when a change is proposed, $\xi_i = 1$, and with probability $p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1})$ it is accepted ($\phi_i = 1$), so that $s_{i+1} = (M_1, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 1))$. With probability $(1 - p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}))$ it is not accepted ($\phi_i = 0$), so that $s_{i+1} = (M_1, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 0))$.

3.2 Expected tax payments

The following lemma shows that the depreciation charge to be used in period i when method M is used in that period depends only on: the period i, the residual tax base at the beginning of period i, and the last period before period i in which method M was not used.

Lemma 5 *Suppose method M is used in periods $k = j, \cdots, i-1$, method M^c is used in period $j-1$, and the residual tax base at the beginning of period i equals D. Then, there exists a $q_{i,j,N} \in [0,1]$ such that the depreciation charge to be used in period i if method M is used is given by:*

$$\begin{aligned} d_i &= \frac{f_M(i-j+1, N-j+1)}{1-\sum_{k=1}^{i-j} f_M(k, N-j+1)} D \\ &= q_{i,j,N} \cdot D. \end{aligned}$$

Moreover, $q_{N,\cdot,N} = 1$.

The above lemma shows that, for any given depreciation method, the fraction of the residual tax base to be depreciated in period i is a function of the current period i, the useful life of the asset N, and the last period before period i in which the method was not used ($j-1$). For example, it is easily verified that in case of SDM, SYD and DDB, as defined in (3), (4), and (5), respectively, the fractions $q_{i,j,N}$ are given by:

$$\begin{aligned} q_{i,j,N,SDM} &= \tfrac{1}{N+1-i}, && \text{for } i = 1, ..., N, \\ q_{i,j,N,SYD} &= \tfrac{2}{N+2-i}, && \text{for } i = 1, ..., N, \\ q_{i,j,N,DDB} &= \tfrac{2}{N-j+1}, && \text{for } i = 1, ..., N-1, \\ &= 1, && \text{for } i = N. \end{aligned} \tag{12}$$

Note that in case of SDM and SYD, $q_{i,j,N}$ is independent of j. For notational convenience, we restrict to depreciation methods for which the fraction of the residual tax base to be depreciated in a given period i is independent of j.[2] Moreover, since the maximum depreciable lifetime of the asset N is given and fixed, we omit the index N, and we denote $q_{i,M} := q_{i,\cdot,N}$ for the fraction of the residual tax base to be depreciated in period i if method M is used. This yields the following lemma.

[2] Allowing the fractions to depend on j increases notational complexity, but it does not otherwise affect the results.

Lemma 6 *The expected tax to be paid in period i if the residual tax base at the beginning of period i equals D, and method M is used in period i, is given by:*

$$Tax(i, M, D) = \int_{q_{i,M} \cdot D}^{\infty} (1 - F_i(x))dx,$$

where $F_i(\cdot)$ denotes the distribution function of the cash flow in period i.

It now remains to show that $(M_1, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$ is sufficient to determine the expected tax payments in period i for any possible decision. First, since a change of method only occurs when $\xi_i = 1$ and $\phi_i = 1$, it follows immediately that the method M_i used in period i is determined recursively as follows:

$$\begin{aligned} M_j &= M_{j-1}^c, \quad \text{if } \xi_j = 1 \text{ and } \phi_j = 1, \\ &= M_{j-1}, \quad \text{otherwise,} \end{aligned} \tag{13}$$

and, given Lemma 5, it follows that the residual tax base D_{i-1} at the beginning of period i is determined recursively by:

$$\begin{aligned} D_j &= (1 - q_{j,A}) \cdot D_{j-1}, \quad \text{if } M_j = A, \\ &= (1 - q_{j,B}) \cdot D_{j-1}, \quad \text{if } M_j = B, \end{aligned} \tag{14}$$

where D_0 denotes the initial asset value.

Now let $(M_1, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$ be a decision node in period i. Then,

- if no change is proposed at the beginning of period i, the expected tax payments in period i are given by $Tax(i, M_{i-1}, D_{i-1})$;
- if a change is proposed at the beginning of period i, then the expected tax payments in period i are given by $p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot Tax(i, M_{i-1}^c, D_{i-1}) + (1 - p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1})) \cdot Tax(i, M_{i-1}, D_{i-1})$.

Therefore, expected tax payments in period i can be determined from $(M_1, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$, where M_{i-1} and D_{i-1} follow from (13) and (14).

3.3 Recursive evaluation

While $(M_1, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$ yields sufficient information to determine the optimal decision in period i, it follows from Lemma 6, (13) and (14), that it is convenient to use the following, equivalent, set of state variables $(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$, where M denotes the method that was used in period $i-1$, and D denotes the residual tax base at the end of period $i-1$.

We now define the value functions as follows. The function

$$V_i(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1}) : \{A, B\} \times \mathbb{R} \times \{0,1\}^{i-1} \times \{0,1\}^{i-1} \to \mathbb{R},$$

for $i = 2, \cdots, N$, yields the minimal expected value of the sum of the costs of proposing changes and the present value of future tax payments for periods $i, \cdots, N$, given $(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$. The function

$$V_1(D) : \mathbb{R} \to \mathbb{R},$$

yields the minimal expected value of the sum of the costs of proposing changes and the present value of future tax payments for all periods, as a function of the initial value of the asset D. The following proposition provides a recursive relationship for the value functions, where

$$\alpha_i := \alpha^{(i)}/\alpha^{(i-1)} \in [0, 1],$$

denotes the value at date $i-1$ of one unit at date i.

Proposition 7 *Let :*

$$V_{N+1}(\cdot, \cdot, \cdot, \cdot) := 0.$$

Then, for all $i = 2, \cdots, N$,

$$V_i(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1}) =$$
$$\min \left\{ \begin{array}{l} Tax(i, M, D) + \alpha_i \cdot V_{i+1}(M, (1-q_{i,M})D, (\overline{\xi}_{i-1}, 0), (\overline{\phi}_{i-1}, 0)), \\ \kappa + p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot \left[Tax(i, M^c, D) + \alpha_i \cdot V_{i+1}(M^c, (1-q_{i,M^c})D, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 1))\right] \\ +(1 - p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1})) \cdot \left[Tax(i, M, D) + \alpha_i V_{i+1}(M, (1-q_{i,M})D, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 0))\right] \end{array} \right\},$$

and

$$V_1(D) = \min_{M \in \{A,B\}} \left\{ Tax(1, M, D) + \alpha_1 \cdot V_2(M, (1-q_{1,M})D, 0, 0) \right\}.$$

Note that the choice of ξ_i not only affects the depreciation method and the residual tax base in the next period, but also the probability that a future change will be accepted. When no change is proposed in period i, the probability of acceptance in the next period is given by $p_{i+1}((\overline{\xi}_{i-1}, 0), (\overline{\phi}_{i-1}, 0))$; when a change is proposed and accepted, it is given by $p_{i+1}((\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 1))$; when a change is proposed but not accepted, it is given by $p_{i+1}((\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 0))$.

The following corollary follows immediately from Proposition 7.

Corollary 8 *It holds that:*

i) It is optimal to choose method M in period 1 iff

$$\begin{aligned} & Tax(1, M, D) + \alpha_1 \cdot V_2(M, (1 - q_{1,M})D, 0, 0) \\ \leq \; & Tax(1, M^c, D) + \alpha_1 \cdot V_2(M^c, (1 - q_{1,M^c})D, 0, 0). \end{aligned} \tag{15}$$

ii) In periods $i = 2, \ldots, N$, a change should be proposed iff

$$\kappa \leq p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)] + \alpha_i \cdot S_{i+1}(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1}),$$

where

$$\begin{aligned} S_{i+1}(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \;=\; & V_{i+1}(M, D, (\overline{\xi}_{i-1}, 0), (\overline{\phi}_{i-1}, 0)) \\ & - p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot V_{i+1}(M^c, (1 - q_{i,M^c})D, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 1)) \\ & - (1 - p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1})) \cdot V_{i+1}(M, (1 - q_{i,M})D, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 0)). \end{aligned} \tag{16}$$

The above proposition shows that it is optimal to propose a change in period i iff the costs of the proposal are lower than the expected benefit from the proposal. This expected benefit consists of two parts: i) the expected reduction in tax payments in period i, which is equal to $p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)]$, and ii) the present value of the expected reduction in tax payments in all future periods, which is given by $\alpha_i \cdot S_{i+1}(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$. It is clear that the discount factor α_i can play a crucial role in whether or not it is optimal to propose a change. In the next section, we investigate the effect of discounting on the optimal decision in each period.

4 Effect of discounting

Berg et al. (2001) consider the case where $\alpha^{(i)} = \alpha^i$, for all i (i.e., a flat term structure of interest rates), and where a change of method is never allowed. They then show that less discounting (a higher value of α) works in favor of the least accelerated method. In our setting, however, a decision has to be made in every period. We allow the discount factor to be different for different periods, i.e. $\alpha^{(i)} = \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_i \neq \alpha^i$, and consider a method to be more accelerated *in period i* if the depreciation charge in that period exceeds the depreciation charge of the alternative method.

It can be verified easily that for the first period decision, as in Berg et al. (2001), there exists a critical value of the first period discount factor $\widetilde{\alpha}_1$ such that the most (least) accelerated method is optimal if $\alpha_1 < \widetilde{\alpha}_1$ ($\alpha_1 > \widetilde{\alpha}_1$). This, however, is not necessarily the case for later periods.We first show that the decision whether or not to propose a change in the last period is independent of discount factors.

Proposition 9 *It is never optimal to propose a change of method in period N.*

Next, the following proposition shows that, in contrast to Berg et al. (2001), it is possible that a higher discount rate works in favor of the *more* accelerated method.

Proposition 10 *For any given period $i \in \{2, \cdots, N-1\}$, and state $(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$, the following holds:*

i) If $q_{i,M} > q_{i,M^c}$ or $\kappa \leq p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)]$, then there exists a critical value $\tilde{\alpha}_i \leq 1$ such that the most (least) accelerated method in period i is preferable if $\alpha_i < \tilde{\alpha}_i$ ($\alpha_i > \tilde{\alpha}_i$),

ii) If $q_{i,M} < q_{i,M^c}$ and $\kappa > p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)]$, there exists a critical value $\tilde{\alpha}_i \leq 1$ such that the most (least) accelerated method in period i is preferable if $\alpha_i > \tilde{\alpha}_i$ ($\alpha_i < \tilde{\alpha}_i$).

The intuition that more discounting works in favor of more accelerated methods is that these methods typically lower taxable income in earlier periods. Proposition 10, however, shows that *higher values* of α_i can make it optimal for firms to change to the more accelerated method in cases where costs are relatively high. The intuition is as follows. Suppose the firm used the least accelerated method in period $i-1$ (possibly due to refusal of earlier proposals to change). Suppose furthermore that $\kappa > p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1})\,[Tax(i, M, D) - Tax(i, M^c, D)]$, and $S_{i+1}(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1}) > 0$. Then, the expected benefit in *the current period* of changing to the more accelerated method is negative, because the expected reduction in tax payments $(p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)])$ is lower than the cost κ of proposing the change. However, since $S_{i+1}(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1}) > 0$, the expected benefit in *future periods* from changing to the more accelerated method is positive. Now, it is only worth the relatively large cost of proposing a change (κ) if the potential benefits of using the more accelerated method in later periods are sufficiently high to compensate the loss in the current period. When the discount factor is low (low α_i), the firm prefers not to have high costs now since the benefit in future periods is discounted heavily. If α_i is sufficiently high, however, the expected gain in future periods can dominate the expected cost in the current period. A higher discount factor can then make it optimal for the firm to propose a change to the more accelerated method.

5 Numerical illustration

In this section we illustrate the effect of the cost κ and the probability structure $p_i(\cdot, \cdot)$ on the optimal strategy in a numerical example. Specifically, we consider a setting where:

- The acceptable depreciation methods are SDM and SYD, as defined in (12).
- $N = 10$, and the initial amount to depreciate equals $D_0 = 10$.
- Cash flows are normally distributed. The means and standard deviations of the cash flow distributions are given in the following table:

period i	1	2	3	4	5	6	7	8	9	10
$E[C_i]$	1	2	2	3	3	4	4	3	0	2
$\sigma[C_i]$	2	2	2	2	2	2	2	2	2	2

This reflects a setting where the variance is constant over the periods, but expected income generated by the asset grows in early periods, and starts to decrease as of period 8. The expected revenue in the last period includes the salvage value of the asset.

- $\alpha_i = \alpha = 0.95$ for all i, so that the present value of money in period i equals α^i.

With regard to the probability of acceptance, we consider settings where the probability of acceptance decreases with $v \cdot 100\%$ after each accepted proposal. The probability of acceptance of the first proposal is denoted p. Then, for a given change strategy $\overline{\xi}$, the probability that a change proposed in period i will be accepted is given by:

$$\begin{aligned} p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) &= p, && \text{if } \overline{\xi}_{i-1} = (\xi_1, \ldots, \xi_{i-1}) = (0, \ldots, 0), \text{ and} \\ p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) &= (1-v)^{\sum_{k=1}^{i-1} \phi_k} p, && \text{otherwise.} \end{aligned} \tag{17}$$

We first illustrate the effect of κ and p on the expected value of future tax payments and costs, as well as on the value of the option to change in the case where $v = 0$. Then, we investigate the effect of v for given values of κ and p.

Effect of p and κ on initial choice

When $v = 0$, the probability of acceptance does not change over time and is independent of earlier decisions, so that:

$$p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) = p, \qquad \text{for all } i = 2, \cdots, 10.$$

The following table presents the minimum (over all possible change strategies) of the expected present value of costs and tax payments when the initial method is SDM and SYD, respectively, for three different values of κ and five different values of p. The optimal value is indicated in bold. For each value of κ, the third column presents the expected percentage reduction in tax costs due to the possibility to change.

	$\kappa = 0.01$			$\kappa = 0.05$			$\kappa = 0.2$		
p	SDM	SYD		SDM	SYD		SDM	SYD	
0	14.28	**14.10**	0%	14.28	**14.10**	0%	14.28	**14.10**	0%
0.3	**14.01**	14.08	0.6%	14.11	**14.10**	0%	14.28	**14.10**	0%
0.5	**13.95**	14.03	1.1%	**14.02**	14.10	0.6%	14.24	**14.10**	0%
0.8	**13.89**	13.93	1.5%	**13.95**	14.03	1.1%	14.11	**14.10**	0%
1	**13.87**	13.88	1.6%	**13.92**	13.97	1.3%	**14.05**	14.10	0.4%

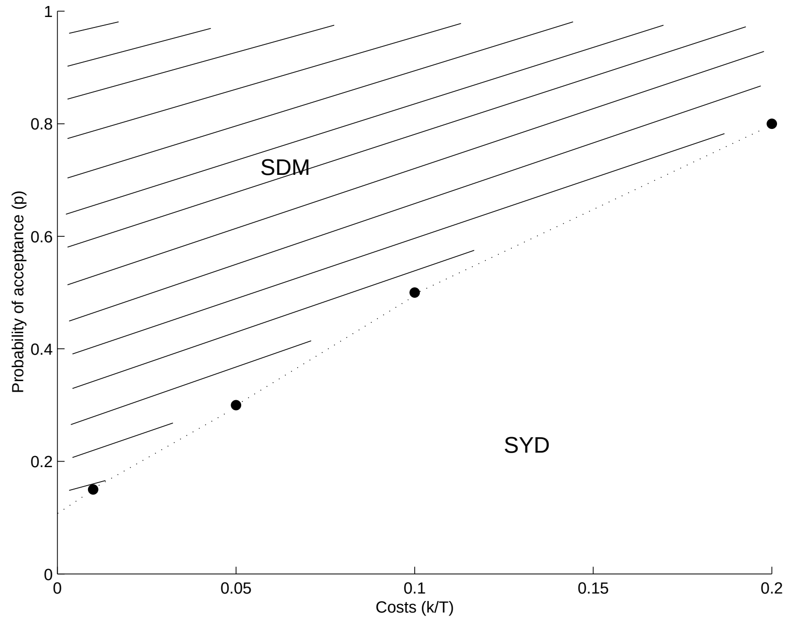

Figure 2: *The choice between SDM and SYD as a function of* p *and* $\kappa = k/\tau$.

Without the possibility to change the depreciation method, i.e. when $p = 0$, the expected discounted tax payments equal 14.28 for SDM, and 14.10 for SYD, so that SYD is the optimal method.

Now, consider $p > 0$ and $\kappa = 0.01$. Then, it can be verified that the expected discounted tax costs (tax payments plus cost of proposed changes) are equal for both methods when $p = 0.155$. When $p < 0.155$, the possibility of changing does not affect the optimal strategy, and the optimal initial choice is SYD. However, when $p > 0.155$, the possibility of changing the depreciation method results in a change of strategy. Then, the firm optimally starts with SDM, and later on proposes a change to SYD. As soon as the probability that proposed changes are accepted is higher than 0.5 the firm expects to save more than 1% on its tax costs.

When proposing and negotiating a change becomes more costly, the value of the option to change reduces. The initial choice of the depreciation method as a function of $\kappa = k/\tau$, and the probability p of acceptance is illustrated in Figure 2. There is a sizeable range of values of p and κ where SDM is the optimal initial choice, whereas when ignoring the possibility of changing, the conclusion would have been that SYD is the optimal method.

Effect of v

We now consider the case where the probability of acceptance of a next proposal decreases with every accepted proposal, i.e. $v > 0$. For $v > 0$, proposing a change not only brings along costs κ, but also some opportunity costs, since it reduces the opportunity for strategic behavior in the future. The results for different values of v and p are presented in the following table, for $\kappa = 0.01$.

	$v = 5\%$		$v = 10\%$		$v = 20\%$		$v = 50\%$	
p	SDM	SYD	SDM	SYD	SDM	SYD	SDM	SYD
0.5	**13.95**	14.04	**13.95**	14.05	**13.95**	14.05	**13.96**	14.08
1	**13.87**	13.88	**13.87**	13.89	**13.87**	13.90	**13.88**	13.96

Again, the table presents the minimum over all possible change strategies of the sum of the expected present value of costs and tax payments, when the initial method is SDM and SYD, respectively. When SYD is chosen initially, the decrease in probability of acceptance has significant effect, because the optimal change strategy then involves multiple proposals to change. For example, expected tax costs equal 13.88 when $v = 0.05$, and increase to 13.96 when $v = 0.5$. However, in all the above cases, SDM is the optimal initial choice. The effects of decreasing probabilities is not significant in this case.

6 Conclusion

The paper extends the literature on optimal tax depreciation by incorporating the option to negotiate a change of depreciation method. It develops a dynamic programming model to determine the firm's optimal strategy with regard to the initial choice of depreciation method, and whether or not to propose changes in later periods. Since negotiating a change is costly, and a proposed change might not be accepted by the tax authority, the optimal choice reflects a trade-off between the costs of the proposal and the expected reduction in future tax payments.

Efficiency in the negotiation process and flexibility of the tax authority with respect to proposed changes create value for the firm that can be important in investigating the profitability of projects. Our analysis indicates that the value of the option to change can be significant. Furthermore, we show that, in contrast to the static case where changes are not allowed, a lower discount factor can make the accelerated method less attractive. Specifically, if the cost associated with negotiating a change exceeds a threshold value, a change from straight line depreciation to an accelerated method is optimal only if the discount factor is sufficiently high. This occurs because, for the change to be optimal, the net present value of future tax savings due to the change must outweigh the costs. A higher discount factor increases this net present value.

Appendix

Proof of Proposition 3. First, it is clear that it is optimal to set $\xi_{i,k} = 0$ in all decision nodes in which a change of method was implemented in a predecessor node. Indeed, proposing a change then (i.e. setting $\xi_{i,k} = 1$) weakly increases the expected cost of proposing changes, without affecting the probability distribution of the resulting depreciation scheme. Moreover, since changes that are proposed but not accepted do not affect the probability distribution of the resulting depreciation scheme, it is optimal to set $\xi_{i,k} = \xi_{i,1}$ in every decision node k in which no changes were accepted in prior nodes.

Second, it is seen immediately that the only depreciation schemes that can occur with non-zero probability are $d_{(M,k)}$, for $k = 1, \cdots, N$, and $M \in \{A, B\}$.

Now it remains to determine the probability that each of these depreciation schemes will occur, as well as the probability that a change will be proposed in a given period.

- Depreciation scheme $d_{(M,k)}$ for $k < N$ will result if the following three conditions are satisfied:
  - a change is proposed at the beginning of period $k+1$, i.e. $\xi_{k+1,1} = 1$;
  - all prior proposals, if any, were rejected, which occurs with probability $(1-p)^{\sum_{j=2}^{k} \xi_{j,1}}$; and
  - the proposed change in period $k+1$ is accepted, which, given that prior proposals were rejected, occurs with probability p.
- Depreciation scheme $d_{(M,N)}$ results if all proposals to change were rejected, which occurs with probability $(1-p)^{\sum_{j=2}^{N} \xi_{j,1}}$.

This implies that (10) holds true. Next, a change will be proposed in period i iff $\xi_{i,1} = 1$, and all prior proposals, if any, were rejected. Therefore, (11) holds true.

This concludes the proof. ■

Proof of Lemma 5. Suppose method $f_M(\cdot, \cdot)$ is used in periods $j, \cdots, i-1$, and method M^c is used in period $j-1$. Then, it follows that

$$d_{j+k-1} = f_M(k, \widetilde{N}) \cdot \widetilde{D}, \qquad k = i-j, \cdots, 1,$$

where $\widetilde{N} = N - j + 1$ denotes the remaining depreciable lifetime of the asset at the beginning of period j, and $\widetilde{D}$ denotes the residual tax base at the beginning of period j. Then, it holds that:

$$\widetilde{D} = D + \sum_{k=j}^{i-1} d_k = D + \sum_{k=1}^{i-j} f_M(k, \widetilde{N}) \cdot \widetilde{D}.$$

Therefore, if method M is used in period i, the depreciation charge in that period is given by:

$$\begin{aligned} d_i &= f_M(i - j + 1, \widetilde{N}) \cdot \widetilde{D} \\ &= q_{i,j,N} \cdot D, \end{aligned}$$

where

$$q_{i,j,N} = \frac{f_M(i - j + 1, N - j + 1)}{1 - \sum_{k=1}^{i-j} f_M(k, N - j + 1)}.$$

Note that it follows from (2) that

$$\begin{aligned} q_{N,\cdot,N} &= \frac{f_M(N - j + 1, N - j + 1)}{1 - \sum_{k=1}^{N-j} f_M(k, N - j + 1)} \\ &= 1. \end{aligned}$$

This concludes the proof. ■

Proof of Lemma 6. Follows immediately from Lemma 5, and the fact that for any given random variable X with cumulative distribution function $F(\cdot)$, and any $d \in \mathbb{R}$, it holds that

$$E[(X - d)^+] = \int_d^\infty (1 - F(x))dx.$$

■

Proof of Proposition 7. At any given period $i = 2, \cdots, N$, the firm decides whether or not to propose a switch of depreciation method, i.e. whether to stick with method M (i.e. $\xi_i = 0$) or to propose a switch to method M^c (i.e. $\xi_i = 1$). For both options, the present value of future tax payments and costs is given by the sum of expected value of tax payments and costs in period i, and the expected present value of future tax payments and costs in periods $i + 1, \cdots, N$.

- If $\xi_i = 0$, then method M is used in period i, so that the expected tax payments in period i are given by $Tax(i, M, D)$, and the residual tax base in period $i+1$ is given by $(1-q_{i,M})D$. Because no change was proposed this implies that the value function in period $i+1$ is given by $V_{i+1}(M, (1-q_{i,M})D, (\overline{\xi}_{i-1}, 0), (\overline{\phi}_{i-1}, 0))$.

- If $\xi_i = 1$, then costs κ are incurred. With probability $p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1})$, the proposal will be accepted, so that the expected tax payments in period i are given by $Tax(i, M^c, D)$, and the residual tax base in period $i+1$ is given by $(1-q_{i,M^c})D$. Because a change was proposed and accepted, the value function in period $i+1$ is given by $V_{i+1}(M^c, (1-q_{i,M^c})D, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 1))$. With probability $1-p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1})$, the proposal is not accepted, so that the expected tax payments in period i are given by $Tax(i, M, D)$, and the residual tax base in period $i+1$ is given by $(1-q_{i,M})D$. Because a change was proposed but not accepted, the value function in period $i+1$ is given by $V_{i+1}(M, (1-q_{i,M})D, (\overline{\xi}_{i-1}, 1), (\overline{\phi}_{i-1}, 0))$.

In period $i = 1$, both methods are accepted with probability 1, and there is no proposal to change, so that the expected tax to be paid in period 1 is given by $Tax(1, M, D_0)$, and the value function at date 2 is given by $V_2(M, (1-q_{1,M})D_0)$. ■

Proof of Proposition 9. Since $q_{N,M} = 1$ for all methods M, it holds that:

$$Tax(N, M, D) = Tax(N, M^c, D) = \int_D^\infty (1 - F_N(x))dx.$$

Therefore, since $\kappa \geq 0$ and $V_{N+1}(\cdot, \cdot, \cdot, \cdot) = 0$, it holds that:

$$\begin{aligned} V_N(M, D, \overline{\xi}_N, \overline{\phi}_N) &= \min\left\{ \int_D^\infty (1 - F_N(x))dx, \kappa + \int_D^\infty (1 - F_N(x))dx \right\} && (18) \\ &= \int_D^\infty (1 - F_N(x))dx, && (19) \end{aligned}$$

so that it is never optimal to propose a change in period N. ■

Proof of Proposition 10. Let $i \in \{1, \cdots, N-1\}$, $M \in \{A, B\}$, $D \in [0, D_0]$, $\overline{\xi}_{i-1} \in \{0,1\}^{i-2}$, and $\overline{\phi}_{i-1} \in \{0,1\}^{i-2}$ be given, and denote $G_i(\alpha_i)$ for the expected benefit of proposing a change in period i, given the current method M, the residual tax base D, and $\overline{\xi}_{i-1}, \overline{\phi}_{i-1}$. It then follows from Proposition 7 that:

$$\begin{aligned} G_i(\alpha_i) &= -\kappa + p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)] \\ &\quad + \alpha_i \cdot S_{i+1}(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1}), && (20) \end{aligned}$$

where $S_{i+1}(M, D, \overline{\xi}_{i-1}, \overline{\phi}_{i-1})$ is as defined in (16). It is optimal to propose a change in period i iff $G_i(\alpha_i) > 0$. Note that $G_i(\alpha_i)$ is linear in α_i, and

$$G_i(0) = -\kappa + p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)] \, .$$

We now show that i) and ii) are satisfied with $\widetilde{\alpha}_i \in [0, 1]$ defined as follows:

$$\begin{aligned} \widetilde{\alpha}_i &= 1, && \text{if } \nexists \ \alpha \in [0,1] \text{ s.t. } G_i(\alpha) = 0, \\ &= G_i^{-1}(0), && \text{otherwise.} \end{aligned}$$

We then distinguish the following three cases:

- If the current method M is the most accelerated method, i.e. $q_{i,M} > q_{i,M^c}$, then $G_i(0) = -\kappa + p_i \int_{q_{i,M}D}^{q_{i,M^c}D} (1 - F_i(x))dx \leq 0$. Therefore,
  - If $\widetilde{\alpha}_i = 1$, then $G_i(\alpha) \leq 0$ for all $\alpha \in [0, 1]$.
  - If $\widetilde{\alpha}_i < 1$, then $G_i(\alpha) < 0$ for all $\alpha < \widetilde{\alpha}_i$ and $G_i(\alpha) > 0$ for all $\alpha > \widetilde{\alpha}_i$.

  Since $q_{i,M} > q_{i,M^c}$, this implies that the most (least) accelerated method is preferable for $\alpha < \widetilde{\alpha}_i$ ($\alpha > \widetilde{\alpha}_i$).

- If the current method M is the least accelerated method, i.e. $q_{i,M} < q_{i,M^c}$, and $\kappa < p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)]$, then $G_i(0) > 0$. Therefore,
  - If $\widetilde{\alpha}_i = 1$, then $G_i(\alpha) \geq 0$ for all $\alpha \in [0, 1]$.
  - If $\widetilde{\alpha}_i < 1$, then $G_i(\alpha) > 0$ for all $\alpha < \widetilde{\alpha}_i$ and $G_i(\alpha) < 0$ for all $\alpha > \widetilde{\alpha}_i$.

  Since $q_{i,M} < q_{i,M^c}$, this implies that the most (least) accelerated method is preferable for $\alpha < \widetilde{\alpha}_i$ ($\alpha > \widetilde{\alpha}_i$).

This concludes the proof of i).

- Finally, if the current method M is the least accelerated method, i.e. $q_{i,M} < q_{i,M^c}$, and $\kappa > p_i(\overline{\xi}_{i-1}, \overline{\phi}_{i-1}) \cdot [Tax(i, M, D) - Tax(i, M^c, D)]$, then $G_i(0) < 0$. Therefore,
  - If $\widetilde{\alpha}_i = 1$, then $G_i(\alpha) \leq 0$ for all $\alpha \in [0, 1]$.
  - If $\widetilde{\alpha}_i < 1$, then $G_i(\alpha) < 0$ for all $\alpha < \widetilde{\alpha}_i$ and $G_i(\alpha) > 0$ for all $\alpha > \widetilde{\alpha}_i$.

Since $q_{i,M} < q_{i,M^c}$, this implies that the least (most) accelerated method is preferable for $\alpha < \widetilde{\alpha}_i$ ($\alpha > \widetilde{\alpha}_i$).

This concludes the proof of ii). ■

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