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A generalization of the Higman-Sims technique

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In the present note we announce a result on the eigenvalues of partitioned matrices, together with some applications to graphs and designs. Proofs and further details will be contained in the author's thesis.

Let A denote a Hermitian $n \times n$ matrix over $\mathbb{C}$. Let A be partitioned into m^2 block matrices A_{ij} such that all A_{ii} are square matrices:

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mm} \end{bmatrix}.$$

Let B denote the $m \times m$ matrix whose ij^{th} entry equals the average row sum of A_{ij}, for all $i, j \in [1, m]$. The eigenvalues of A and B are real. It is known [1] that the eigenvalues of B are bounded by the largest and the smallest eigenvalue of A. This fact is often used in combinatorial situations under the name "Higman-Sims technique". Now the following more general theorem holds:

THEOREM. The eigenvalues $\alpha_1 \geqslant \dots \geqslant \alpha_n$ of A and the eigenvalues $\beta_1 \geqslant \dots \geqslant \beta_m$ of B satisfy $\alpha_{n-m+i} \leqslant \beta_i \leqslant \alpha_i$, for all $i \in [1, m]$.
If, for some $k \in [0, m]$, $\beta_i = \alpha_i$ for all $i \in [1, k]$ and $\beta_i = \alpha_{n-m+i}$ for all $i \in [k+1, m]$, then A_{ij} has constant row and column sum for all $i, j \in [1, m]$.

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This theorem has a variety of applications, some of which are indicated below. Let G be a regular graph on v vertices of degree k. Let H be an induced subgraph of G on v_1 vertices of average degree k_1. Let $\alpha_1 \geqslant \ldots \geqslant \alpha_v$ be the eigenvalues of the adjacency matrix of G, so $\alpha_1 = k$. Then

$$(1) \qquad v_1 k - v k_1 \leqslant -\alpha_v (v - v_1).$$

A similar result holds for non-regular graphs G, but the general formula is complicated. However, in the special case when the subgraph is a coclique (has no edges) the formula is nice. Let G denote any graph on v vertices, with eigenvalues $\alpha_1 \geqslant \ldots \geqslant \alpha_v$, and let k_m be the smallest degree in G. Then the size v_1 of a coclique in G satisfies

$$(2) \qquad v_1 \leqslant v \frac{-\alpha_1 \alpha_v}{k_m^2 - \alpha_1 \alpha_v}.$$

This implies that the coloring number of any graph is bounded below by $1 - k_m^2/\alpha_1 \alpha_v$. If G is regular, then both (1) and (2) imply a formula due to A. J. Hoffman (unpublished) for the size v_1 of a void subgraph in G:

$$(3) \qquad v_1 \leqslant v \frac{-\alpha_v}{k - \alpha_v}.$$

For an arbitrary partitioned matrix N, our theorem applies to

$$\begin{bmatrix} 0 & N \\ N^{\mathrm{H}} & 0 \end{bmatrix}$$

and analogous results are obtained in terms of the singular values of N. Here is a sample of such results.

Suppose $\sigma_1 \geqslant \sigma_2 \geqslant \ldots > 0$ are the singular values of the incidence matrix of a 1-design with parameters (v, b, k, r), so $\sigma_1 = \sqrt{rk}$. For a sub-1-design with parameters (v_1, b_1, k_1, r_1) we have

$$(4) \qquad (v r_1 - b_1 k)(b k_1 - v_1 r) \leqslant \sigma_2^2 (v - v_1)(b - b_1).$$

For an arbitrary incidence structure we give an inequality in case the subdesign is the void design $(v_1, b_1, 0, 0)$:

$$(5) \qquad r_m^2 k_m^2 v_1 b_1 \leqslant \sigma_1^2 \sigma_2^2 (v - v_1)(b - b_1),$$

where r_m and k_m denote the minimal row and column sum of the incidence matrix.

As a consequence of the second part of our theorem we infer the following. If equaity holds in (1) ... (5), then all four submatrices of the adjacency [incidence] matrix have constant row and column sum.

The above formulas become easy to apply when the eigenvalues or singular values are expressible in terms of the parameters of the graph or incidence structure. This applies to strongly regular graphs, block designs and partial geometries.

REFERENCE

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