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A note on the stability number of an orthogonality graph

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Abstract

We consider the orthogonality graph $\Omega(n)$ with 2^n vertices corresponding to the vectors $\{0, 1\}^n$, two vertices adjacent if and only if the Hamming distance between them is $n/2$. We show that, for $n = 16$, the stability number of $\Omega(n)$ is $\alpha(\Omega(16)) = 2304$, thus proving a conjecture of V. Galliard [Classical pseudo telepathy and coloring graphs, Diploma Thesis, ETH Zurich, 2001. Available at <http://math.galliard.ch/Cryptography/Papers/PseudoTelepathy/SimulationOfEntanglement.pdf>]. The main tool we employ is a recent semidefinite programming relaxation for minimal distance binary codes due to A. Schrijver [New code upper bounds from the Terwilliger algebra, IEEE Trans. Inform. Theory 51 (8) (2005) 2859–2866].

Also, we give a general condition for a Delsarte bound on the (co)cliques in graphs of relations of association schemes to coincide with the ratio bound, and use it to show that for $\Omega(n)$ the latter two bounds are equal to $2^n/n$.

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1. Introduction

The graph $\Omega(n)$ and its properties

Let $\Omega(n)$ be the graph on 2^n vertices corresponding to the vectors $\{0, 1\}^n$, such that two vertices are adjacent if and only if the Hamming distance between them is $n/2$. Note that $\Omega(n)$ is k -regular, where $k = \binom{n}{\frac{n}{2}}$.

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Fig. 1. The connected component of $\Omega(4)$ corresponding to vertices of even Hamming weight.

It is known that $\Omega(n)$ is bipartite if $n \equiv 2 \pmod{4}$, and empty if n is odd. We will therefore assume throughout that n is a multiple of 4. The graph owes its name to another description, in terms of ± 1 -vectors. Then the orthogonality of vectors corresponds to the Hamming distance $n/2$.

Moreover, $\Omega(n)$ consists of two isomorphic connected components, $\Omega_0(n)$ and $\Omega_1(n)$, containing all the vertices of even and odd, respectively, Hamming weight (see Fig. 1). For a detailed discussion of the properties of $\Omega(n)$, see Godsil [9], the Ph.D. Thesis of Newman [14], and [10].

In this note we study upper bounds on the stability number $\alpha(\Omega(n))$.

Galliard [7] pointed out the following way of constructing maximal stable sets in $\Omega(n)$. Consider the component $\Omega_\epsilon(n)$ of $\Omega(n)$, for $1 - \epsilon \equiv \frac{n}{4} \pmod{2}$, and take all vertices of Hamming weight $\epsilon, \epsilon + 2, \dots, \epsilon + 2\ell, \dots, n/4 - 1$. Obviously, these vertices form a stable set S of $\Omega(n)$ of size

$$\sum_{i=\epsilon}^{\lfloor n/8 \rfloor} \binom{n}{2i - \epsilon}.$$

We can double the size of S by adding the bit-wise complements of the vertices in S , and double it again by taking the union with the corresponding stable set in $\Omega_{1-\epsilon}(n)$. Thus we find that

$$\alpha(\Omega(n)) \geq 4 \sum_{i=\epsilon}^{\lfloor n/8 \rfloor} \binom{n}{2i - \epsilon} := \underline{\alpha}(n). \quad (1)$$

For $n = 16$ this evaluates to $\alpha(\Omega(n)) \geq 2304$. Galliard et al. [8] were able to show that $\alpha(\Omega(16)) \leq 3912$. In this note we will show that, in fact, $\alpha(\Omega(16)) = 2304$. This was conjectured by Galliard [7], and Newman [14] has recently conjectured that the value (1) actually equals $\alpha(\Omega(n))$ whenever n is a multiple of 4.

A quantum information game

One motivation for studying the graph $\Omega(n)$ comes from quantum information theory. Consider the following game from [8].

Let $r \geq 1$ and $n = 2^r$. Two players, A and B, are asked the questions x_A and x_B , coded as n -bit strings satisfying

$$d_H(x_A, x_B) \in \left\{0, \frac{1}{2}n\right\}$$

where d_H denotes the Hamming distance. A and B win the game if they give answers y_A and y_B , coded as binary strings of length r such that

$$y_A = y_B \iff x_A = x_B.$$

A and B are not allowed any communication (except a priori deliberation).

It is known that A and B can always win the game if their r output bits are *maximally entangled quantum bits* [2] (see also [14]).

For classical bits, it was shown by Galliard et al. [8] that the game cannot always be won if $r = 4$. The authors proved this by pointing out that whether or not the game can always be won is equivalent to the question

$$\chi(\Omega(n)) \leq n?$$

Indeed, if $\chi(\Omega(n)) \leq n$ then A and B may color $\Omega(n)$ a priori using n colors. The questions x_A and x_B may then be viewed as two vertices of $\Omega(n)$, and A and B may answer their respective questions by giving the colors of the vertices x_A and x_B respectively, coded as binary strings of length $\log_2 n = r$.

Galliard et al. [8] showed that $\chi(\Omega(16)) > 16$, i.e. that the game cannot be won for $n = 16$. They proved this by showing that $\alpha(\Omega(16)) \leq 3912$ which implies

$$\chi(\Omega(16)) \geq \left\lceil \frac{2^{16}}{\alpha(\Omega(16))} \right\rceil \geq \left\lceil \frac{2^{16}}{3912} \right\rceil = 17.$$

In this note we sharpen their bound by showing that $\alpha(\Omega(16)) = 2304$, which implies $\chi(\Omega(16)) \geq 29$.

Our main tool will be a semidefinite programming bound on $\alpha(\Omega(n))$ that is due to Schrijver [17], where it is formulated for minimal distance binary codes.

2. Upper bounds on $\alpha(\Omega(n))$

In this section we give a review of known upper bounds on $\alpha(\Omega(n))$ and their relationship.

2.1. The ratio bound

The following discussion is condensed from Godsil [9].

Theorem 1. Let $G = (V, E)$ be a k -regular graph with adjacency matrix $A(G)$, and let $\lambda_{\min}(A(G))$ denote the smallest eigenvalue of $A(G)$. Then

$$\alpha(G) \leq \frac{|V|}{1 - \frac{k}{\lambda_{\min}(A(G))}}. \quad (2)$$

This bound is called the *ratio bound*, and was first derived by Delsarte [4] for graphs in association schemes (see Section 2.2 for more on the latter).

Recall that $\Omega(n)$ is k -regular with $k = \binom{n}{\frac{1}{2}n}$. Ignoring multiplicities, the spectrum of $\Omega(n)$ is given by

$$\lambda_m = \frac{2^{\frac{1}{2}n}}{(\frac{1}{2}n)!} (m-1)(m-3) \cdots (m-n+1) \quad (m = 1, \dots, n). \quad (3)$$

The minimum is reached at $m = 2$, and we get

$$\lambda_{\min}(A(\Omega(n))) = \frac{2^{\frac{1}{2}n}}{(\frac{1}{2}n)!} (1)(-1)(-3) \cdots (-n+3) = -\frac{\binom{n}{\frac{1}{2}n}}{n-1}. \quad (4)$$

The ratio bound therefore becomes

$$\alpha(\Omega(n)) \leq \frac{2^n}{n}. \quad (5)$$

This is the best known upper bound on $\alpha(\Omega(n))$, but it is known that this bound is not tight: Frankl and Rödl [6] showed that there exists some $\epsilon > 0$ such that $\alpha(\Omega(n)) \leq (2 - \epsilon)^n$. For specific (small) values of n one can improve on the bound (5), as we will show for $n \leq 32$.

2.2. The Delsarte bound and ϑ'

Here we are going to use more linear algebra that naturally arises around $\Omega(n)$. We recall the following definitions; cf. e.g. Bannai and Ito [1].

Association schemes

An association scheme $\mathcal{A}$ is a commutative subalgebra of the full $v \times v$ -matrix algebra with a distinguished basis $(A_0 = I, A_1, \dots, A_n)$ of 0–1 matrices, with an extra property that $\sum_i A_i$ equals the all-ones matrix. One often views A_j , $j \geq 1$, as the adjacency matrix of a graph on v vertices; A_j is often referred to as the j -th *relation* of $\mathcal{A}$. As the A_j 's commute, they have $n+1$ common eigenspaces V_i . Then $\mathcal{A}$ is isomorphic, as an algebra, to the algebra of diagonal matrices $\text{diag}(P_{0,j}, \dots, P_{n,j})$, where P_{ij} denotes the eigenvalue of A_j on V_i . The matrix $P = (P_{ij})$ is called the *first eigenvalue matrix* of $\mathcal{A}$. The set of A_j 's is closed under taking transpositions: for each $0 \leq j \leq n$ there exists j' so that $A_j = A_{j'}^T$. In particular, $P_{ij} = \overline{P_{ij'}}$. An association scheme with all A_j 's symmetric is called *symmetric*, and here we shall consider only such schemes. There is a matrix Q (called the *second eigenvalue matrix*) satisfying $PQ = QP = vI$. In what follows it is assumed (as is customary in the literature) that the eigenspace V_0 corresponds to the eigenvector $(1, \dots, 1)$; then the 0-th row of P consists of the degrees v_j of the graphs A_j . It is remarkable that the 0-th row of Q consists of dimensions of V_i .

Let ϑ' denote the Schrijver ϑ' -function [16]:

$$\vartheta'(G) = \max\{\text{Tr}(JX) : \text{Tr}(AX) = 0, \text{Tr}(X) = 1, X \geq 0, X \geq 0\}.$$

For any graph G one has $\alpha(G) \leq \vartheta'(G)$. Moreover, $\vartheta'(G)$ is smaller than or equal to the ratio bound (2) for regular graphs, as noted by Godsil [9, Sect. 3.7].

For graphs with adjacency matrices of the form $\sum_{j \in \mathcal{M}} A_j$, with $\mathcal{M} \subset \{1, \dots, n\}$ and A_j 's from the 0–1 basis of an association scheme $\mathcal{A}$, the bound ϑ' coincides, as was proved by

Schrijver [16], with the following bound due to Delsarte [3,4]:

$$\max 1^T w \text{ subject to } w \geq 0, Q^T w \geq 0, w_0 = 1, w_j = 0 \quad \text{for } j \in \mathcal{M}, \quad (6)$$

where Q is the second eigenvalue matrix of $\mathcal{A}$.

The bound (6) is often stated for (and was originally developed for) bounding the maximal size of a q -ary code of length n and minimal distance d ; then the association scheme $\mathcal{A}$ becomes the Hamming distance association scheme $H(n, q)$ and $\mathcal{M} = \{1, \dots, d-1\}$. The relations of $H(n, q)$ can be viewed as graphs on the vertex set of n -strings on $\{0, \dots, q-1\}$: the j -th graph of $H(n, q)$ is given by

$$(A_j)_{XY} = \begin{cases} 1 & \text{if } d_H(X, Y) = j \\ 0 & \text{otherwise.} \end{cases}$$

For $H(n, q)$ the first and the second eigenvalue matrices P and Q coincide, and are given by $P_{ij} = K_j(i)$, where K_k is the Krawtchouk polynomial

$$K_k(x) := \sum_{j=0}^k (-1)^j (q-1)^{k-j} \binom{x}{j} \binom{n-x}{k-j}.$$

For $\Omega(n)$, the bound (6) is as above with $\mathcal{A} = H(n, 2)$ and $\mathcal{M} = \{\frac{n}{2}\}$. Newman [14] has shown computationally that $\vartheta'(\Omega(n)) = 2^n/n$ if $n \leq 64$, i.e. the ratio and ϑ' bounds coincide for $\Omega(n)$ if $n \leq 64$. We show that this is the case for all n , as an easy consequence of the following.

Proposition 1. *Let $\mathcal{A}$ be an association scheme with the 0–1 basis $(A_0, \dots, A_n)$ and eigenvalue matrices P and Q . Let A_r have the least eigenvalue $\tau = P_{\ell r}$ and assume*

$$v_r P_{\ell i} \geq v_i \tau, \quad 0 \leq i \leq n.$$

Then the Delsarte bound (6), with $\mathcal{M} = \{r\}$, and the ratio bound (2) for A_r coincide.

Proof. Let P_j denote the j -th row of P .

As we already mentioned, the bound (2) for regular graphs always majorates (6). Thus it suffices to present a feasible vector for the LP in (6) that gives the objective value the same as (2).

We claim that

$$a = \frac{-\tau}{v_r - \tau} P_0^T + \frac{v_r}{v_r - \tau} P_\ell^T$$

is such a vector. It is straightforward to check that $a_0 = 1$ and $a_r = 0$, as required. By the assumption of the proposition, $a \geq 0$. As $PQ = vI$, any non-negative linear combination z of the rows of P satisfies $Q^T z^T \geq 0$. As a^T is such a combination, we obtain $Q^T a \geq 0$.

Finally, to compute $1^T a$, note that $1^T P_0^T = v$ and $1^T P_\ell^T = 0$. $\square$

Corollary 1. *The bounds (6) and (2) coincide for $\Omega(n)$.*

Proof. We apply Proposition 1 to $\mathcal{A} = H(n, 2)$ and $r = \frac{n}{2}$. Then the eigenvalues of $A_r = \Omega(n)$ given in (3) comprise the r -th column of P ; in particular the least eigenvalue τ equals $P_{2,r}$, by

(4) above. The assumption of the proposition translates into¹

$$\binom{n}{\frac{n}{2}} K_i(2) - \binom{n}{i} K_{\frac{n}{2}}(2) = \frac{2^{\frac{n}{2}+2} (n-2)! (n-1)! (\frac{n}{2} - i)^2}{i! (\frac{n}{2})! (n-i)!} \geq 0,$$

as claimed. $\square$

2.3. Schrijver's improved SDP-based bound

Recently, Schrijver [17] has suggested a new SDP-based bound for minimal distance codes that is at least as good as the ϑ' bound, and still of size polynomial in n . It is given as the optimal value of a semidefinite programming (SDP) problem.

In order to introduce this bound (as applied to $\alpha(\Omega(n))$) we require some notation.

For $i, j, t \in \{0, 1, \dots, n\}$ and $X, Y \in \{0, 1\}^n$ define the matrices

$$(M_{i,j}^t)_{X,Y} = \begin{cases} 1 & \text{if } |X| = i, |Y| = j, d_H(X, Y) = i + j - 2t \\ 0 & \text{otherwise.} \end{cases}$$

The upper bound is given as the optimal value of the following semidefinite program:

$$\bar{\alpha}(n) := \max \sum_{i=0}^n \binom{n}{i} x_{i,0}^0$$

subject to

$$x_{0,0}^0 = 1$$

$$0 \leq x_{i,j}^t \leq x_{i,0}^0 \quad \text{for all } i, j, t \in \{0, \dots, n\}$$

$$x_{i,j}^t = x_{i',j'}^{t'} \quad \text{if } \{i', j', i' + j' - 2t'\} \text{ is a permutation of } \{i, j, i + j - 2t\}$$

$$x_{i,j}^t = 0 \quad \text{if } \{i, j, i + j - 2t\} \cap \left\{ \frac{1}{2}n \right\} \neq \emptyset,$$

as well as

$$\sum_{i,j,t} x_{i,j}^t M_{i,j}^t \geq 0, \quad \sum_{i,j,t} (x_{i+j-2t,0}^0 - x_{i,j}^t) M_{i,j}^t \geq 0.$$

The matrices $M_{i,j}^t$ are of order 2^n and therefore too large to compute with in general. Schrijver pointed out that these matrices form a basis of the Terwilliger algebra of the Hamming scheme, and worked out the details for computing the irreducible block diagonalization of this (non-commutative) matrix algebra of dimension $O(n^3)$.

Thus, analogously to the ϑ' -case, the constraint $\sum_{i,j,t} x_{i,j}^t M_{i,j}^t \geq 0$ is replaced by

$$\sum_{i,j,t} x_{i,j}^t Q^T M_{i,j}^t Q \geq 0$$

where Q is an orthogonal matrix that gives the irreducible block diagonalization. For details the reader is referred to Schrijver [17]. Since SDP solvers can exploit block diagonal structure, this reduces the sizes of the matrices in question to the extent that computation is possible in the range $n \leq 32$.

¹ Here $m!! = m(m-2)(m-4)\dots$, the double factorial.

2.4. Laurent's improvement

In Laurent [13] one finds a study placing the relaxation [17] into the framework of *moment sequences* of [11,12]. This study also explains the relationship with known lift-and-project methods for obtaining hierarchies of upper bounds on $\alpha(G)$.

Moreover, Laurent [13] suggests a refinement of the Schrijver relaxation that takes the following form:

$$l_+(n) := \max 2^n x_{0,0}^0$$

subject to

$$\begin{aligned} 0 &\leq x_{i,j}^t \leq x_{i,0}^0 \quad \text{for all } i, j, t \in \{0, \dots, n\} \\ x_{i,j}^t &= x_{i',j'}^{t'} \quad \text{if } \{i', j', i' + j' - 2t'\} \text{ is a permutation of } \{i, j, i + j - 2t\} \\ x_{i,j}^t &= 0 \quad \text{if } \{i, j, i + j - 2t\} \cap \left\{\frac{1}{2}n\right\} \neq \emptyset, \end{aligned}$$

as well as

$$\sum_{i,j,t} x_{i,j}^t M_{i,j}^t \succeq 0$$

and

$$\begin{pmatrix} 1 - x_{0,0}^0 & c^T \\ c & \sum_{i,j,t} (x_{i+j-2t,0}^0 - x_{i,j}^t) M_{i,j}^t \end{pmatrix} \succeq 0,$$

where $c := \sum_{i=0}^n (x_{0,0}^0 - x_{0,i}^0) \chi_i$, and χ_i is defined by

$$(\chi_i)_X := \begin{cases} 1 & \text{if } |X| = i \\ 0 & \text{else.} \end{cases}$$

This SDP problem may be block-diagonalized as before to obtain an SDP of size $O(n^3)$.

3. Computational results

To summarize, the bounds we have mentioned satisfy

$$\underline{\alpha}(n) \leq \alpha(\Omega(n)) \leq l^+(n) \leq \bar{\alpha}(n) \leq \vartheta'(\Omega(n)) = 2^n/n.$$

In Table 1 we show the numerical values for $\bar{\alpha}(n)$ and $l_+(n)$ that were obtained using the SDP solver SeDuMi by Sturm [18], with Matlab 7 on a Pentium IV machine with 1 GB of memory. Matlab routines that we have written to generate the corresponding SeDuMi input are available online [15].

Note that the lower and upper bounds coincide for $n = 16$, proving that $\alpha(\Omega(16)) = 2304$. The best previously known upper bound, obtained by an *ad hoc* method, was $\alpha(\Omega(16)) \leq 3912$ [8].

The value $\bar{\alpha}(20) = 20,166.98$ implies that

$$\alpha(\Omega(20)) \in \{20144, 20148, 20152, 20156, 20160, 20164\}$$

Table 1

Lower and upper bounds on $\alpha(\Omega(n))$

n	$\underline{\alpha}(n)$	$l_+(n)$	$\bar{\alpha}(n)$	$\vartheta'(\Omega(n)) = \lfloor 2^n/n \rfloor$
16	2304	2304	2304	4096
20	20,144	20,166.62	20,166.98	52,428
24	178,208	183,373	184,194	699,050
28	1,590,376	1,848,580	1,883,009	9,586,980
32	14,288,896	21,103,609	21,723,404	134,217,728

since $\alpha(\Omega(n))$ is always a multiple of 4. Another implication is that $n = 20$ is the smallest value of n where the upper bounds $\bar{\alpha}(n)$ and $l_+(n)$ are not tight.

It is worth noticing that the Schrijver and Laurent bounds ($\bar{\alpha}(n)$ and $l_+(n)$ respectively) give relatively big improvements over the Delsarte bound $\frac{2^n}{n}$. This is in contrast to the relatively small improvements that these bounds give for binary codes; cf. [17,13]. We also note that these relaxations are numerically ill-conditioned for $n \geq 24$. This makes it difficult to solve the corresponding SDP problems to high accuracy. The recent study by De Klerk, Pasechnik, and Schrijver [5] suggests a different way to solve such SDP problems, leading to larger SDP instances, but which may avoid the numerical ill-conditioning caused by performing the irreducible block factorization.

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