Tilburg University

A complement on pole placement

Schumacher, J.M.

Published in:
IEEE Transactions on Automatic Control

Publication date:
1980

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
Schumacher, J. M. (1980). A complement on pole placement. *IEEE Transactions on Automatic Control*, 25(2), 281-282.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Then (A_{22}, B_2) is controllable if and only if $\langle A|\mathfrak{B} + \mathcal{V} \rangle = \mathfrak{X}$.

Proof: This is easily seen from the rank criterion, applied to A and $\begin{pmatrix} B_1 & V_1 \\ B_2 & 0 \end{pmatrix}$, where V_1 is any matrix such that $\text{Ran} \begin{pmatrix} V_1 \\ 0 \end{pmatrix} = \mathcal{V}$; note that $\text{Ran} \begin{pmatrix} B_1 & V_1 \\ B_2 & 0 \end{pmatrix} = \mathfrak{B} + \mathcal{V}$. $\square$

Now, let us introduce the following notation. If $\mathcal{V}$ and $\mathfrak{W}$ are invariant subspaces for some linear mapping A , and $\mathcal{V} \subset \mathfrak{W}$, then we shall write $\sigma(A|\mathfrak{W}/\mathcal{V})$ (the *spectrum of A between $\mathfrak{W}$ and $\mathcal{V}$*) for the spectrum of the mapping that is induced on the quotient space $\mathfrak{W}/\mathcal{V}$ by the restriction of A to $\mathfrak{W}$.

In matrix terms, this means the following. With respect to a basis adapted to the chain $\mathcal{V} \subset \mathfrak{W} \subset \mathfrak{X}$ the matrix of A looks like

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}. \quad (2)$$

Then $\sigma(A|\mathfrak{W}/\mathcal{V})$ is just $\sigma(A_{22})$. The rule

$$\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22}) \cup \sigma(A_{33}) \quad (3)$$

can now also be written as

$$\sigma(A) = \sigma(A|\mathcal{V}/\mathcal{V}) \cup \sigma(A|\mathfrak{W}/\mathcal{V}) \cup \sigma(A|\mathfrak{X}/\mathfrak{W}). \quad (4)$$

Of course, this rule can be extended to longer chains of invariant subspaces; in the theorem following, we shall have occasion to apply it to a five-term chain.

Theorem: Let $\mathcal{V}$ be an (A, B) -invariant subspace. Then $\mathfrak{R} \equiv \langle A + BF|\mathfrak{B} \cap \mathcal{V} \rangle$ and $\mathfrak{S} \equiv \langle A + BF|\mathfrak{B} + \mathcal{V} \rangle$ do not depend on the particular choice of $F \in F(\mathcal{V})$, so that $\mathcal{V} \subset \mathfrak{R} \subset \mathfrak{S} \subset \mathfrak{X}$ is a chain of invariant subspaces for all such F . The freedom we have in placing the poles of $A + BF$ when F is restricted to $F(\mathcal{V})$ can be described as follows:

$$\left. \begin{array}{l} \sigma(A + BF|\mathfrak{X}/\mathfrak{S}) \text{ is fixed} \\ \sigma(A + BF|\mathfrak{S}/\mathcal{V}) \text{ is free} \\ \sigma(A + BF|\mathcal{V}/\mathfrak{R}) \text{ is fixed} \\ \sigma(A + BF|\mathfrak{R}/\mathcal{V}) \text{ is free.} \end{array} \right\} \quad (5)$$

Moreover, $\mathfrak{S} = \mathcal{V} + \langle A|\mathfrak{B} \rangle$ and $\sigma(A + BF|\mathfrak{X}/\mathfrak{S}) = \sigma(A|\mathfrak{X}/\mathfrak{S})$.

Proof: Take any $F \in F(\mathcal{V})$, and write $\mathfrak{R} = \langle A + BF|\mathfrak{B} \cap \mathcal{V} \rangle$ and $\mathfrak{S} = \langle A + BF|\mathfrak{B} + \mathcal{V} \rangle$. Clearly, we have the inclusions $\mathcal{V} \subset \mathfrak{R} \subset \mathfrak{S} \subset \mathfrak{X}$, and with respect to a basis adapted to this chain of $(A + BF)$ -invariant subspaces, we obtain

$$A + BF \equiv Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ 0 & Z_{22} & Z_{23} & Z_{24} \\ 0 & 0 & Z_{33} & Z_{34} \\ 0 & 0 & 0 & Z_{44} \end{bmatrix}. \quad (6)$$

The matrix of B with respect to the same basis looks like

$$B = \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{bmatrix}. \quad (7)$$

We now see immediately that $\langle A + BF|\mathfrak{B} \cap \mathcal{V} \rangle \subset \mathfrak{R}$ for all $\tilde{F} \in F(\mathcal{V})$ and that $\langle A + BF|\mathfrak{B} + \mathcal{V} \rangle \subset \mathfrak{S}$ for all $\tilde{F}$. Because our initial F was chosen arbitrarily from $F(\mathcal{V})$, this implies that $\mathfrak{R}$ and $\mathfrak{S}$ do not depend on $F \in F(\mathcal{V})$. With respect to $\mathfrak{S}$ we can even say more: $\langle A + BF|\mathfrak{B} + \mathcal{V} \rangle$ does not depend on F and so $\mathfrak{S} = \langle A|\mathfrak{B} + \mathcal{V} \rangle$.

A Complement on Pole Placement

J. M. SCHUMACHER

Abstract—A complement is given on Wonham's description of the spectral assignability properties of $A + BF$ when F is varied under the restriction that a given subspace $\mathcal{V}$ be $(A + BF)$ -invariant. We apply the theory to give a simple solution to the output stabilization problem.

I. INTRODUCTION

In this correspondence, we shall follow the notational conventions of [1]. So, let $\mathcal{V}$ be an (A, B) -invariant subspace, and let us write $F(\mathcal{V}) = \{F: \mathfrak{X} \rightarrow \mathfrak{U} | (A + BF)\mathcal{V} \subset \mathcal{V}\}$. A natural question is, what freedom do we have in placing the poles of $A + BF$ when F is restricted to $F(\mathcal{V})$. The spectrum of the restriction of $A + BF$ to $\mathcal{V}$ is described in [1, Corollary 5.2], but this only gives a partial answer. To obtain a full description, we shall determine the remaining part of the spectrum below. The usefulness of the thus provided description will be illustrated in a simple solution to the output stabilization problem.

II. A SPECTRAL LATTICE DIAGRAM

For all clarity, let us state the basic fact we shall use in a separate lemma.

Lemma: Suppose $\mathcal{V}$ is an A -invariant subspace, so that, in a basis adapted to the chain $\mathcal{V} \subset \mathfrak{W} \subset \mathfrak{X}$, we have

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \quad (1)$$

Manuscript received October 12, 1979.

The author is with the Department of Mathematics, Vrije Universiteit, Amsterdam, The Netherlands.

For $\sigma(A + BF|\mathcal{R}/\mathcal{O})$, we refer to [1]. Applying the lemma to $A + BF|\mathcal{S}$ and $\mathcal{V}$ shows that (Z_{33}, B_3) is controllable. Moreover, it is clear that Z_{22} and Z_{44} are the same for any $F \in F(\mathcal{V})$. This proves (5).

We have already shown that $\mathcal{S} = \langle A|\mathcal{B} + \mathcal{V} \rangle$, and it is an easy exercise to verify that $\langle A|\mathcal{B} + \mathcal{V} \rangle = \langle A|\mathcal{B} \rangle + \mathcal{V}$.

Taking quotients with respect to $\mathcal{S}$, we have $\overline{A + BF} = \overline{A}$ because $\mathcal{B} \subset \mathcal{S}$; and consequently, $\sigma(A + BF|\mathcal{K}/\mathcal{S}) = \sigma(A|\mathcal{K}/\mathcal{S})$. $\square$

We can get the standard spectral assignability result for unrestricted F from the Theorem, by taking $\mathcal{V}$ equal to any subspace that is $(A + BF)$ -invariant for all F (such as $\mathcal{K}$, $\mathcal{O}$, or $\langle A|\mathcal{B} \rangle$).

Our findings can be conveniently summarized in a "spectral lattice diagram" for $\sigma(A + BF)$, $F \in F(\mathcal{V})$:

III. OUTPUT STABILIZATION

The output stabilization problem (OSP) is the following [1, Section 4.4]: given, in addition to the pair (A, B) , an output map $D: \mathcal{X} \rightarrow \mathcal{Y}$, together with a symmetric partition $\mathcal{C} = \mathcal{C}_g \cup \mathcal{C}_b$, find $F: \mathcal{X} \rightarrow \mathcal{U}$ such that

$$\mathcal{X}_b(A + BF) \subset \ker D. \quad (9)$$

The result on solvability of OSP appears in [1], but its proof might seem involved. However, it follows naturally from the above theory.

Corollary [1, Theorem 4.4]: OSP is solvable if and only if

$$\mathcal{X}_b(A) \subset \langle A|\mathcal{B} \rangle + \mathcal{V}^* \quad (10)$$

where $\mathcal{V}^*$ is the largest (A, B) -invariant subspace of $\ker D$.

Proof: By [1, Lemma 4.5], OSP is solvable if and only if there exists an (A, B) -invariant subspace $\mathcal{V} \subset \ker D$ and $F \in F(\mathcal{V})$ such that

$$\sigma(A + BF|\mathcal{K}/\mathcal{V}) \subset \mathcal{C}_g. \quad (11)$$

By the theorem, such an F exists (for a given $\mathcal{V}$) if and only if

$$\sigma(A|\mathcal{K}/(\langle A|\mathcal{B} \rangle + \mathcal{V})) \subset \mathcal{C}_g \quad (12)$$

or, equivalently (using [1, Lemma 4.5]),

$$\mathcal{X}_b(A) \subset \langle A|\mathcal{B} \rangle + \mathcal{V}. \quad (13)$$

Clearly, (13) holds for some (A, B) -invariant subspace $\mathcal{V} \subset \ker D$ if and only if it holds for $\mathcal{V}^*$. $\square$

IV. FINAL REMARK

It would be interesting to investigate whether the subspaces $\mathcal{S}$, $\mathcal{V}$, and $\mathcal{R}$ can be traced solely from their properties as elements of the lattice

$$\{\mathcal{V} | (A + BF)\mathcal{V} \subset \mathcal{V} \text{ for all } F \in F(\mathcal{V})\}. \quad (14)$$

If the answer is yes, a more general study might be undertaken into the connections between the structural properties of the lattice

$$\{\mathcal{V} | (A + BF)\mathcal{V} \subset \mathcal{V} \text{ for all } F \in \mathcal{F}_0\} \quad (15)$$

on one hand (where $\mathcal{F}_0$ is any subset of the class of all linear maps from $\mathcal{X}$ to $\mathcal{U}$), and the spectral assignability properties of $A + BF$ for $F \in \mathcal{F}_0$ on the other. For instance, taking $\mathcal{F}_0 = \{F | \ker F \supset \ker C\}$ would amount to studying the problem of stabilization by direct output feedback.

REFERENCES

- [1] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*. New York: Springer-Verlag, 1979.