Tilburg University

Compensator synthesis using (C,A,B)-pairs

Schumacher, J.M.

Published in:
IEEE Transactions on Automatic Control

Publication date:
1980

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
Schumacher, J. M. (1980). Compensator synthesis using (C,A,B)-pairs. *IEEE Transactions on Automatic Control*, AC-25(6), 1133-1138.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Compensator Synthesis Using (C, A, B) -Pairs

J. M. SCHUMACHER, MEMBER, IEEE

Abstract—This paper gives a general framework (in the geometric style) for the design of compensators for linear multivariable systems. Basic is the notion of a “ (C, A, B) -pair of subspaces.” This concept is newly defined here, and we give its simplest application (to the problem of disturbance decoupling by observation feedback). We then formulate a general compensator synthesis principle and show how several well-known synthesis techniques can be derived from it. We also show that “almost all” compensators can be interpreted as observer-based compensators, and discuss the relation between the compensator problem and the stable cover problem.

I. INTRODUCTION

The multivariable compensator synthesis problem is a difficult one if order minimization is desired. Luenberger showed [1], by using an observer, that a compensator can be constructed of order $n-p$ where n is the number of state variables and p is the number of independent observations. Because the transpose of a compensator matrix for the transposed (dual) system is a compensator matrix for the original system, it is possible to obtain from Luenberger's construction another construction which produces a compensator of order $n-m$ where m is the number of independent inputs. (This was reportedly [1] first noted by Brasch.) Brasch and Pearson [2] found a way to synthesize a compensator of order $\kappa_o - 1$ (where κ_o is the observability index of the system). By the dualization procedure referred to above, the construction can also be used to find a compensator of order $\kappa_c - 1$ (where κ_c is the controllability index of the system). There have also been, in a somewhat different vein, attempts towards stabilization by direct output feedback (zero-order compensation) (e.g., [3], [4]), but the results in this area are far from complete.

In this paper, we shall try to put multivariable compensator theory into a single (geometric) framework by formulating a *general compensator synthesis principle* in which all above-mentioned designs will fit. We do not introduce any new synthesis techniques here, but it is hoped that the general description will help to get a better view of the problems that have to be solved in order to obtain such new techniques. In particular, we shall discuss the relation of the compensator problem to the stable dynamic cover problem [5].

The central notion in the theory of this short paper is that of a (C, A, B) -pair of subspaces. It incorporates both the concept of (A, B) -invariant subspace and its dual (C, A) -invariant subspace. (Note that such symmetry is necessary for a general theory of compensator synthesis, which should contain, with any synthesis procedure, its “dualized” version.) We define this notion in Section II and show that it comes out naturally when studying the invariant subspaces of mappings defined by compensator matrices. Just like the simplest application of the notion of (A, B) -invariant subspace is to the disturbance decoupling problem [6],

Manuscript received August 1, 1979 and October 10, 1979; revised May 19, 1980. Paper recommended by B. Francis, Chairman of the Linear Systems Committee.

The author is with the Department of Mathematics, Vrije Universiteit, Amsterdam, The Netherlands.

the concept of a (C, A, B) -pair of subspaces leads directly to the solution of the problem of disturbance decoupling by observation feedback. This solution is given in Section III. In Section IV, we proceed to investigate the stability properties of compensator matrices in connection with (C, A, B) -pairs; as a result, we obtain the general compensator synthesis principle. Then we show, in Section V, how several synthesis techniques can be fit into the general framework.

II. (C, A, B) -PAIRS

Let us establish some notation first. $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ will always denote linear spaces of dimensions n , m , and p , respectively, and A , B , and C will be linear mappings, respectively, from $\mathcal{X}$ into $\mathcal{X}$, $\mathcal{U}$ into $\mathcal{X}$, and $\mathcal{X}$ into $\mathcal{Y}$. The phrase "system (C, A, B) " will refer to the dynamical system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t).\end{aligned}\quad (1)$$

To this system, we can add a compensator of the following form:

$$\begin{aligned}u(t) &= Ky(t) + Lw(t) \\ \dot{w}(t) &= Mw(t) + Nw(t)\end{aligned}\quad (2)$$

with $w(t) \in \mathcal{W}$. The order of such a compensator is by definition $\dim \mathcal{W}$, for which we shall use the letter k . Combining the two equations (1) and (2) gives rise to a new dynamic equation in the extended state space $\mathcal{X} \oplus \mathcal{W}$:

$$\frac{d}{dt} \begin{pmatrix} x \\ w \end{pmatrix} (t) = \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} (t). \quad (3)$$

The matrix appearing here, which we shall denote by A_c , is the compensator matrix.

It is a well-known fact (see [7]) that the following properties are equivalent for subspaces $\mathcal{V}$ of $\mathcal{X}$:

- 1) there exists $F: \mathcal{X} \rightarrow \mathcal{U}$ such that $(A+BF)\mathcal{V} \subset \mathcal{V}$
- 2) $A\mathcal{V} \subset \mathcal{V} + \text{ran } B$.

Subspaces having these properties are called (A, B) -invariant. By dualizing the equivalence above or by direct proof (see [8]), one can see that also the following properties are equivalent for subspaces $\mathcal{V}$ of $\mathcal{X}$:

- 1) there exists $G: \mathcal{Y} \rightarrow \mathcal{X}$ such that $(A+GC)\mathcal{V} \subset \mathcal{V}$
- 2) $A(\mathcal{V} \cap \ker C) \subset \mathcal{V}$.

These properties define (C, A) -invariant subspaces. Now, the notion of a (C, A, B) -pair of subspaces is defined as follows.

Definition: A (C, A, B) -pair of subspaces is a pair of subspaces of $\mathcal{X}$, one of which is contained in the other, the smallest one being (C, A) -invariant, and the largest one (A, B) -invariant.

As a notational convenience, let us agree to write the smallest member of a (C, A, B) -pair first, so that the expression " $(\mathcal{V}_1, \mathcal{V}_2)$ is a (C, A, B) -pair" means " $\mathcal{V}_1$ is (C, A) -invariant, $\mathcal{V}_2$ is (A, B) -invariant, and $\mathcal{V}_1 \subset \mathcal{V}_2$." Note that if $(\mathcal{V}_1, \mathcal{V}_2)$ is a (C, A, B) -pair, then $(\mathcal{V}_2^\perp, \mathcal{V}_1^\perp)$ is a (B', A', C') -pair, so the concept " (C, A, B) -pair of subspaces" may be called self-dual.

Let us now consider the closed-loop situation described in (2.3). With every subspace $\mathcal{V}$ of the extended state space $\mathcal{X} \oplus \mathcal{W}$, there are two subspaces of the original state space $\mathcal{X}$ that are naturally related to it:

$$\mathcal{V}_i := \left\{ x \in \mathcal{X} \mid \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{V} \right\} \quad (4)$$

$$\mathcal{V}_p := \left\{ x \in \mathcal{X} \mid \exists w \in \mathcal{W} : \begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{V} \right\}. \quad (5)$$

These subspaces may be viewed as the intersection of $\mathcal{V}$ with the $\mathcal{X}$ -plane and the projection of it into the $\mathcal{X}$ -plane, respectively. We note the following.

Proposition 2.1: Suppose $\mathcal{V}$ is an invariant subspace for the compensator matrix A_c appearing in (2.3). Then $(\mathcal{V}_i, \mathcal{V}_p)$ is a (C, A, B) -pair.

Proof: Of course, we have $\mathcal{V}_i \subset \mathcal{V}_p$. Take $x \in \mathcal{V}_i \cap \ker C$; we have to prove that $Ax \in \mathcal{V}_i$. This follows upon noting that

$$\begin{pmatrix} Ax \\ 0 \end{pmatrix} = \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{V}. \quad (6)$$

Next, take $x \in \mathcal{V}_p$; we have to prove that $Ax + Bu \in \mathcal{V}_p$ for some $u \in \mathcal{U}$. Take $w \in \mathcal{W}$ such that $\begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{V}$; then

$$\begin{pmatrix} Ax + B(KCx + Lw) \\ MCx + Nw \end{pmatrix} = \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{V} \quad (7)$$

so that $u = KCx + Lw$ suits our purposes. $\square$

To find a converse to this, we first prove a lemma. Following [7], we use the notation

$$F(\mathcal{V}) = \{ F: \mathcal{X} \rightarrow \mathcal{U} \mid (A+BF)\mathcal{V} \subset \mathcal{V} \} \quad (8)$$

and we introduce

$$G(\mathcal{V}) = \{ G: \mathcal{Y} \rightarrow \mathcal{X} \mid (A+GC)\mathcal{V} \subset \mathcal{V} \}. \quad (9)$$

Lemma 2.2: Let a (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$ be given. Then there exist $F \in F(\mathcal{V}_2)$, $G \in G(\mathcal{V}_1)$, and $K: \mathcal{Y} \rightarrow \mathcal{U}$ such that $F = KC + F_0$ and $G = BK + G_0$ with $\ker F_0 \supset \mathcal{V}_1$ and $\text{ran } G_0 \subset \mathcal{V}_2$.

Proof: Note that $A\mathcal{V}_1 \subset A\mathcal{V}_2 \subset \mathcal{V}_2 + \text{ran } B$; from this, it follows that there exists $G \in G(\mathcal{V}_1)$ such that $\text{ran } G \subset \mathcal{V}_2 + \text{ran } B$. Then we can choose $K: \mathcal{Y} \rightarrow \mathcal{U}$ and $G_0: \mathcal{Y} \rightarrow \mathcal{X}$ with $\text{ran } G_0 \subset \mathcal{V}_2$ such that $G = BK + G_0$. Now note that $(A+GC)\mathcal{V}_2 \subset \mathcal{V}_2 + \text{ran } B$ so that $\mathcal{V}_2$ is $(A+GC, B)$ -invariant. Because also $(A+GC)\mathcal{V}_1 \subset \mathcal{V}_1$, there exists F_0 such that $\ker F_0 \supset \mathcal{V}_1$ and $(A+GC+BF_0)\mathcal{V}_2 \subset \mathcal{V}_2$. It remains to check that $F = KC + F_0$ is in $F(\mathcal{V}_2)$; this follows from $A+BF = A+BKC+BF_0 = A+GC+BF_0 - G_0C$ and the fact that $\text{ran } G_0 \subset \mathcal{V}_2$. $\square$

We shall have occasion to use the mappings F and G as in the lemma more than one time, so let us introduce

$$\begin{aligned}P(\mathcal{V}_1, \mathcal{V}_2) &= \{ (F, G) \in F(\mathcal{V}_2) \times G(\mathcal{V}_1) \mid \text{there exists } K: \mathcal{Y} \rightarrow \mathcal{U} \\ &\text{such that } \ker(F-KC) \supset \mathcal{V}_1 \text{ and } \text{ran}(G-BK) \subset \mathcal{V}_2 \}. \end{aligned} \quad (10)$$

So Lemma 2.2 says that $P(\mathcal{V}_2, \mathcal{V}_2)$ is nonempty for every (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$.

Proposition 2.3: Given a (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$, there exists an extension space $\mathcal{W}$ of dimension $\dim \mathcal{W} = \dim \mathcal{V}_2 - \dim \mathcal{V}_1$ and mappings K, L, M , and N such that $\mathcal{V}_1 = \mathcal{V}_i$ and $\mathcal{V}_2 = \mathcal{V}_p$ for some A_c -invariant subspace $\mathcal{V}$ of $\mathcal{X} \oplus \mathcal{W}$ [where A_c is the compensator matrix of (2.3)].

Proof: Let $\mathcal{W}$ be a linear space of dimension $\dim \mathcal{V}_2 - \dim \mathcal{V}_1$, and let R be a mapping of $\mathcal{V}_2$ onto $\mathcal{W}$ such that $\ker R = \mathcal{V}_1$. Introduce

$$\mathcal{V} := \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \mathcal{V}_2 \right\} \subset \mathcal{X} \oplus \mathcal{W}. \quad (11)$$

Then $\mathcal{V}_i = \mathcal{V}_1$, $\mathcal{V}_p = \mathcal{V}_2$ and we need only show that $\mathcal{V}$ can be made A_c -invariant by a suitable choice of K, L, M , and N . To see this, take $(F, G) \in P(\mathcal{V}_1, \mathcal{V}_2)$. Let $K: \mathcal{Y} \rightarrow \mathcal{U}$ be such that $F = KC + F_0$ and $G = BK + G_0$ with $\ker F_0 \supset \mathcal{V}_1$ and $\text{ran } G_0 \subset \mathcal{V}_2$. Because $\ker F_0 \supset \mathcal{V}_1$, there exists L such that $LF_0 = 0$. Because $\text{ran } G_0 \subset \mathcal{V}_2$, we may set $M = -RG_0$. Finally, because both $\mathcal{V}_1$ and $\mathcal{V}_2$ are invariant subspaces for $A+GC+BF_0 = A+BF+G_0C$, we can define N by

$$NR = R(A+BF+G_0C). \quad (12)$$

Now everything fits together: for all $x \in \mathcal{V}_2$ we have

$$\begin{aligned} \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ Rx \end{pmatrix} &= \begin{pmatrix} (A+B(KC+LR))x \\ (MC+NR)x \end{pmatrix} \\ &= \begin{pmatrix} (A+BF)x \\ R(A+BF)x \end{pmatrix} \in \mathcal{V} \end{aligned} \quad (13)$$

and so $\mathcal{V}$ is A_c -invariant. $\square$

In particular, we see that a subspace $\mathcal{V}$ of $\mathcal{X}$ is $(A+BKC)$ -invariant for some K if and only if $\mathcal{V}$ is both (C, A) - and (A, B) -invariant.

III. DISTURBANCE DECOUPLING BY OBSERVATION FEEDBACK

For the disturbance decoupling problem, we consider the following system dynamics:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Eq(t) \\ y(t) &= Cx(t) \\ z(t) &= Dx(t).\end{aligned}\quad (14)$$

Here, $u(t)$ is the control, $q(t)$ is the disturbance, $y(t)$ is the observation, and $z(t)$ is the variable to be regulated. The problem of disturbance decoupling by observation feedback is: to find a compensator [of the form (2)] such that, in the extended system, the $q(t)$ to $z(t)$ transfer matrix is zero. The extended system has the following form:

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}(t) &= \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}(t) + \begin{pmatrix} E \\ 0 \end{pmatrix}q(t) \\ z(t) &= (D \ 0)\begin{pmatrix} x \\ w \end{pmatrix}(t).\end{aligned}\quad (15)$$

Let us write $\text{ran } E = \mathcal{E}$ and $\ker D = \mathcal{K}$; then $\text{ran} \begin{pmatrix} E \\ 0 \end{pmatrix} = \mathcal{E} \oplus \mathcal{K} \subset \mathcal{X} \oplus \mathcal{W}$ and $\ker(D \ 0) = \mathcal{K} \oplus \mathcal{W}$. Then the problem can be reformulated as follows: to find an extension space $\mathcal{W}$ and a compensator matrix A_c such that an A_c -invariant subspace $\mathcal{V}$ exists with $\mathcal{E} \oplus \mathcal{K} \subset \mathcal{V} \subset \mathcal{X} \oplus \mathcal{W}$. From the theory of Section II, the solvability conditions for this problem are quite obvious.

Theorem 3.1: The problem of disturbance decoupling by observation feedback is solvable if and only if a (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$ exists with $\mathcal{E} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{X}$. In this case, the minimal extension order which is necessary for the solution of the problem is given by

$$\min\{\dim \mathcal{V}_2 - \dim \mathcal{V}_1 \mid (\mathcal{V}_1, \mathcal{V}_2) \text{ is a } (C, A, B)\text{-pair and } \mathcal{E} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{X}\}. \quad (16)$$

Proof: The condition $\mathcal{E} \oplus \mathcal{K} \subset \mathcal{V} \subset \mathcal{X} \oplus \mathcal{W}$ is equivalent to the condition $\mathcal{E} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{X}$. $\square$

In particular, disturbance decoupling by direct output feedback is possible if and only if a subspace $\mathcal{V}$ of $\mathcal{X}$ exists such that $\mathcal{V}$ is both (A, B) - and (C, A) -invariant, and $\mathcal{E} \subset \mathcal{V} \subset \mathcal{X}$. This was proved before by Hamano and Furuta [9].

Among the (A, B) -invariant subspaces contained in $\mathcal{X}$, there is a largest one, denoted by $\mathcal{V}^*(\mathcal{X})$ [7]. One can also prove easily that among the (C, A) -invariant subspaces containing $\mathcal{E}$, there is a smallest one, which we shall denote by $\mathcal{V}_*(\mathcal{E})$. So we have the following corollary.

Corollary 3.2: The problem of disturbance decoupling by observation feedback is solvable if and only if

$$\mathcal{V}_*(\mathcal{E}) \subset \mathcal{V}^*(\mathcal{X}). \quad (17)$$

This result was also obtained, for discrete-time systems, by Akashi and Imai [10], who used a different method.

Let us close this section with a few remarks. First, consider the special case in which $\ker C \supset \ker D$. Since the (C, A) -invariant subspaces of $\ker C$ are the same as the A -invariant subspaces of $\ker C$, condition (17) now becomes

$$\langle A | \mathcal{E} \rangle \subset \mathcal{K}, \quad (18)$$

which means that the original system is already disturbance decoupled. We arrive at the following conclusion. A system that is not disturbance decoupled cannot be made so by observation feedback in case the observation is a function of the variable to be regulated.

The problem of disturbance decoupling by observation feedback is generically unsolvable. We obtain generic solvability, however, if $DE=0$, $\dim \text{ran } B + \dim \ker D > n$ and $\dim \text{ran } E + \dim \ker C < n$. Simple algorithms are available for computing $\mathcal{V}_*(\mathcal{E})$ [11], [12] and $\mathcal{V}^*(\mathcal{X})$ [7], so the condition (17) can be checked by these. Lemma 2.2 and Proposition 2.3 are also constructive. So our solution is completely constructive. The problem of effectively computing the minimal extension order (16), however, has not been solved. We have not considered stability requirements; for this, see [13].

IV. A GENERAL COMPENSATOR SYNTHESIS PRINCIPLE

Let us first introduce some notation we shall need. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are invariant subspaces for an operator A on some linear space $\mathcal{X}$, and

$\mathcal{V}_1 \subset \mathcal{V}_2$, we shall write $A: \mathcal{V}_2/\mathcal{V}_1$ for the operator induced on the factor space $\mathcal{V}_2/\mathcal{V}_1$ by the restriction of A to $\mathcal{V}_2$. The spectrum of an operator A will be written $\sigma(A)$ so we can write, for instance,

$$\sigma(A) = \sigma(A: \mathcal{X}/\mathcal{V}) \cup \sigma(A: \mathcal{V}/\mathcal{V}) \quad (19)$$

if $\mathcal{V}$ is an A -invariant subspace of $\mathcal{X}$.

We shall also need the following lemma.

Lemma 4.1: Let $\mathcal{V}$ be a subspace of an extended state space $\mathcal{X} \oplus \mathcal{W}$, and define $\mathcal{V}_i$ and $\mathcal{V}_p$ as in (4) and (5). Then the following relations hold:

$$\dim \mathcal{W} + \dim \mathcal{V}_i > \dim \mathcal{V} > \dim \mathcal{V}_p. \quad (20)$$

The proof can easily be obtained by considering the natural projections of $\mathcal{V}$ into $\mathcal{W}$ and $\mathcal{X}$, respectively, and is left to the reader.

We now prove the following, which is more or less an extension of Propositions 2.1.

Proposition 4.2: Let A_c be a compensator matrix on an extended state space. Let $\mathcal{V}$ be an A_c -invariant subspace with $\dim \mathcal{W} + \dim \mathcal{V}_i = \dim \mathcal{V}_p$. Then there exists $(F, G) \in P(\mathcal{V}_i, \mathcal{V}_p)$ such that

$$\sigma(A_c: \mathcal{V}/\mathcal{V}) = \sigma(A + BF: \mathcal{V}_p/\mathcal{V}) \quad (21)$$

and

$$\sigma(A_c: (\mathcal{X} \oplus \mathcal{W})/\mathcal{V}) = \sigma(A + GC: \mathcal{X}/\mathcal{V}_i). \quad (22)$$

Proof: Because $\dim \mathcal{V} = \dim \mathcal{V}_p$, there exists a mapping $R: \mathcal{V}_p \rightarrow \mathcal{V}$ such that $\mathcal{V} = \{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \mathcal{V}_p \}$. Because $\dim \mathcal{W} + \dim \mathcal{V}_i = \dim \mathcal{V}$, R is surjective and so there exists a mapping $S: \mathcal{W} \rightarrow \mathcal{V}_p$ such that $RS = I$. We define a mapping P from $\mathcal{V}$ to $\mathcal{V}_p$ and a mapping Q from $\mathcal{X}/\mathcal{V}_i$ to $(\mathcal{X} \oplus \mathcal{W})/\mathcal{V}$ as follows.

$$P: \begin{pmatrix} x \\ Rx \end{pmatrix} \mapsto x \quad (x \in \mathcal{V}_p) \quad (23)$$

$$Q: x + \mathcal{V}_i \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} + \mathcal{V} \quad (x \in \mathcal{X}). \quad (24)$$

(Q is well defined because $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{V}$ if $x \in \mathcal{V}_i$.) One can now check by direct computation that the following diagrams commute.

$$\begin{array}{ccc} & A_c: \mathcal{V}/\mathcal{V} & \\ P \downarrow & \xrightarrow{\quad} & \downarrow P \\ & \mathcal{V}_p & \\ & A + B(KC + LR): \mathcal{V}_p/\mathcal{V} & \end{array} \quad (25)$$

$$\begin{array}{ccc} & A_c: (\mathcal{X} \oplus \mathcal{W})/\mathcal{V} & \\ Q \uparrow & \xrightarrow{\quad} & \uparrow Q \\ & \mathcal{X}/\mathcal{V}_i & \\ & A + (BK - SM)C: \mathcal{X}/\mathcal{V}_i & \end{array} \quad (26)$$

(To see that (4.8) commutes, note that

$$\begin{pmatrix} (A + BKC)x \\ MCx \end{pmatrix} - \begin{pmatrix} (A + (BK - SM)C)x \\ 0 \end{pmatrix} = \begin{pmatrix} SMCx \\ RSMCx \end{pmatrix} \in \mathcal{V}.)$$

The mappings P and Q are easily seen to be isomorphisms, and so we conclude that the pair (F, G) with $F = KC + LR$ and $G = BK - SM$ satisfies all requirements. $\square$

We also have a converse to this.

Proposition 4.3: Let a (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$ and a pair $(F, G) \in P(\mathcal{V}_1, \mathcal{V}_2)$ be given. Then there exists an extension space $\mathcal{W}$ with $\dim \mathcal{W} + \dim \mathcal{V}_1 = \dim \mathcal{V}_2$ and a compensator matrix A_c on $\mathcal{X} \oplus \mathcal{W}$ such that $\mathcal{V}_1 = \mathcal{V}_i$ and $\mathcal{V}_2 = \mathcal{V}_p$ for some A_c -invariant subspace $\mathcal{V}$ of $\mathcal{X} \oplus \mathcal{W}$, and the relations (21) and (22) hold.

Proof: Just follow the construction in the proof of Proposition 2.3. This gives an extension space $\mathcal{W}$ and mappings K, L, M , and N such that an A_c -invariant subspace of $\mathcal{X} \oplus \mathcal{W}$ exists with $\mathcal{V}_1 = \mathcal{V}_i$ and $\mathcal{V}_2 = \mathcal{V}_p$. Moreover, we have the relations $F = KC + F_0$, $G = BK + G_0$, $F_0 = LR$ and

$M = -RG_0$ where R is such that $\mathcal{V} = \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \mathcal{V}_2 \right\}$. Thus, the following diagrams commute (with P and Q as defined by (23) and (24), respectively).

$$\begin{array}{ccc} & A_c: \mathcal{V}/\theta & \\ \mathcal{V} & \xrightarrow{\quad} & \mathcal{V} \\ P \downarrow & & \downarrow P \\ \mathcal{V}_2 & \xrightarrow{A+BF: \mathcal{V}_2/\theta} & \mathcal{V}_p \end{array} \quad (27)$$

$$\begin{array}{ccc} & A_c: (\mathcal{X} \oplus \mathcal{W})/\mathcal{V} & \\ (\mathcal{X} \oplus \mathcal{W})/\mathcal{V} & \xrightarrow{\quad} & (\mathcal{X} \oplus \mathcal{W})/\mathcal{V} \\ Q \uparrow & & \uparrow Q \\ \mathcal{X}/\mathcal{V}_1 & \xrightarrow{A+GC: \mathcal{X}/\mathcal{V}_1} & \mathcal{X}/\mathcal{V}_1 \end{array} \quad (28)$$

(To see that (28) commutes, note that

$$\begin{pmatrix} (A+BK)x \\ MCx \end{pmatrix} - \begin{pmatrix} (A+GC)x \\ 0 \end{pmatrix} = - \begin{pmatrix} G_0Cx \\ RG_0Cx \end{pmatrix} \in \mathcal{V}.$$

So we can conclude that the construction described in the proof of Proposition 2.3 gives us all we want. $\square$

We arrive at the announced "general compensator synthesis principle."

Theorem 4.4: Let a system in the form (1) be given. A sufficient condition for a stabilizing compensator of order k to exist is that there is a (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$ with $\dim \mathcal{V}_2 - \dim \mathcal{V}_1 = k$ and a pair $(F, G) \in P(\mathcal{V}_1, \mathcal{V}_2)$ such that both $A+BF: \mathcal{V}_2/\theta$ and $A+GC: \mathcal{X}/\mathcal{V}_1$ are stable.

Proof: The proof is obvious from Proposition 4.3. $\square$

Theorem 4.4 only gives a sufficient condition, but one sees from Proposition 4.2 that the condition is "almost" necessary. Indeed, let A_c be a compensator matrix on an extended state space $\mathcal{X} \oplus \mathcal{W}$, and suppose that $\dim \mathcal{W} = k$ with $k < n$. (Compensators of order larger than the system order are hardly interesting, of course.) By Proposition 4.2, the condition of the theorem holds if there is an A_c -invariant subspace $\mathcal{V}$ such that $\dim \mathcal{V}_p - \dim \mathcal{V}_1 = k$. But this relation is generically true for any subspace of $\mathcal{X} \oplus \mathcal{W}$ that has dimension r with $k < r < n$ (it is a transversality relation in the sense of [7, p. 29]). So, in general, there will be many A_c -invariant subspaces that give rise to a decomposition. (At least, this holds if one allows the decomposition to be complex; the situation may be somewhat more complicated if one insists on finding real subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ and real mappings F and G . It should be noted, though, that working with complex subspaces and mappings will not lead to implementation problems as long as the mappings that will appear in the compensator $(K, L, M$ and $N)$ come out to be real.)

V. SPECIFIC TECHNIQUES

In this section, we want to show how several well-known techniques of compensator synthesis fit into the general framework discussed above. That is, we shall show that specific choices of the (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$ lead to specific compensator synthesis procedures.

A. Observer-Based Compensators

It is immediate from the definition that subspaces having zero intersection with $\ker C$ are always (C, A) -invariant. In particular, we may take $\mathcal{V}_1$ to be a complement of $\ker C$. In this situation, we have the following useful result.

Lemma 5.1: Suppose that $(\mathcal{V}_1, \mathcal{V}_2)$ is a (C, A, B) -pair and $\mathcal{V}_1 \oplus \ker C = \mathcal{X}$. Then $P(\mathcal{V}_1, \mathcal{V}_2) = F(\mathcal{V}_2) \times G(\mathcal{V}_1)$.

Proof: Choose $F \in F(\mathcal{V}_2)$ and $G \in G(\mathcal{V}_1)$; we must show that $(F, G) \in P(\mathcal{V}_1, \mathcal{V}_2)$. Assuming that C is surjective (as we always may), there exists $T: \mathcal{V}_2 \rightarrow \mathcal{X}$ such that $\text{ran } T = \mathcal{V}_1$ and $CT = I$. Now choose K equal to FT . Then $F = KC + F(I - TC)$ with $\ker F(I - TC) \supset \ker(I - TC) = \mathcal{V}_1$ and $G = BK + (A + GC)T - (A + BF)T$ with $\text{ran}((A + GC)T - (A + BF)T) \subset (A + GC)\mathcal{V}_1 + (A + BF)\mathcal{V}_1 \subset \mathcal{V}_2$. $\square$

In order to investigate the question of pole placement with respect to $\sigma(A + GC: \mathcal{X}/\mathcal{V}_1)$ if $\mathcal{V}_1$ is a complement of $\ker C$ and $G \in G(\mathcal{V}_1)$, the following characterization is helpful.

Lemma 5.2: Suppose that $\mathcal{V}_1 \oplus \ker C = \mathcal{X}$ and let V be the projection onto $\ker C$ along $\mathcal{V}_1$. Then $\sigma(A + GC: \mathcal{X}/\mathcal{V}_1) = \sigma(VA: (\ker C)/\theta)$ for all $G \in G(\mathcal{V}_1)$.

Proof: Let H be the mapping from $\ker C$ to $\mathcal{X}/\mathcal{V}_1$ defined by

$$H: x \mapsto x + \mathcal{V}_1 \quad (x \in \ker C). \quad (29)$$

Because $\mathcal{V}_1$ and $\ker C$ are complementary, H is an isomorphism. One readily checks that the following diagram commutes for all $G \in G(\mathcal{V}_1)$.

$$\begin{array}{ccc} & A + GC: \mathcal{X}/\mathcal{V}_1 & \\ \mathcal{X}/\mathcal{V}_1 & \xrightarrow{\quad} & \mathcal{X}/\mathcal{V}_1 \\ H \uparrow & & \uparrow H \\ \ker C & \xrightarrow{VA: (\ker C)/\theta} & \ker C \end{array} \quad (30)$$

(Note that $(A + GC)x - VAx = (I - V)Ax \in \mathcal{V}_1$ for all $x \in \ker C$.) This finishes the proof. $\square$

In particular, we see that $\sigma(A + GC: \mathcal{X}/\mathcal{V}_1)$ does not depend on the choice of $G \in G(\mathcal{V}_1)$. Our pole placement result is now the following.

Lemma 5.3: If the pair (C, A) is detectable, then there exists a complement $\mathcal{V}_1$ of $\ker C$ such that $A + GC: \mathcal{X}/\mathcal{V}_1$ is stable for all $G \in G(\mathcal{V}_1)$.

Proof: Let W be a projection with $\text{ran } W = \ker C$. Suppose that for some $x \in \ker C$, $x \neq 0$, we have $(I - W)Ax = 0$ and $Wx = \lambda x$ for some $\lambda \in \mathbb{C}$; then also $Ax = \lambda x$, and by the detectability of (C, A) , it follows that $\text{Re } \lambda < 0$. Considering $(I - W)A$ as a mapping from $\ker C$ to $\ker W$ and WA as a mapping from $\ker C$ to $\ker C$, we find that this pair is detectable. So there exists $Z: \ker W \rightarrow \ker C$ such that $WA + Z(I - W)A: (\ker C)/\theta$ is stable. Using Lemma 5.2, we are done by taking $V = W + Z(I - W)$, which is easily seen to be a projection onto $\ker C$. Of course, $\mathcal{V}_1 = \ker V$. $\square$

The result is essentially the same as in [7, Theorem 3.3, p. 64] (we have replaced "observable" by "detectable"), but the proof given here is different. We are now able to derive the "Luenberger compensator."

Proposition 5.4: Suppose that, in the system (1), (C, A) is detectable and (A, B) is stabilizable. Then there exists a stabilizing compensator of order $n - p$.

Proof: Let $\mathcal{V}_1$ be a complement of $\ker C$ such that $A + GC: \mathcal{X}/\mathcal{V}_1$ is stable for $G \in G(\mathcal{V}_1)$. Put $\mathcal{V}_2 = \mathcal{X}$. Take F such that $A + BF$ is stable, and take $G \in G(\mathcal{V}_1)$. Then, by Theorem 4.4, a stabilizing compensator exists of order $\dim \mathcal{V}_2 - \dim \mathcal{V}_1 = n - p$. $\square$

Two remarks seem to be in order here. First, it is clear how the result of the proposition could be improved on. Having chosen $\mathcal{V}_1$, we can take $\mathcal{V}_2$ to be any (A, B) -invariant subspace containing $\mathcal{V}_1$, such that there exists $F \in F(\mathcal{V}_2)$ with $A + BF: \mathcal{V}_2/\theta$ stable. The compensator of minimal order (with $\mathcal{V}_1$ fixed) is then given by the smallest among these subspaces. So we have to do here with a version of the "minimal stable cover problem" (introduced in [5] with the stronger requirement of arbitrary pole placement). For recent work on this difficult problem, see, for instance, [14] and [15].

Second, it is not clear from the proof we have given why the compensator constructed in Proposition 5.4 should be called an observer-based compensator. We shall now show that not only the Luenberger compensator of order $n - p$, but, in fact, every compensator based on a (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$ with $\mathcal{V}_1 \oplus \ker C = \mathcal{X}$ can be thought of as applying feedback to an estimate of the state. This might be called a "deterministic separation theorem."

Theorem 5.5: Suppose that A_c is a compensator matrix on an extended state space $\mathcal{X} \oplus \mathcal{W}$, having an invariant subspace $\mathcal{V}$ such that $\dim \mathcal{W} + \dim \mathcal{V}_1 = \dim \mathcal{V}_p$ and, moreover, $\mathcal{V}_1 \oplus \ker C = \mathcal{X}$. Then the evolution of the state of the original system $x(t)$ is described by an equation

$$\dot{\hat{x}}(t) = A\hat{x}(t) + BF\hat{x}(t) \quad (31)$$

where F is a feedback matrix and $\hat{x}(t)$ is a linear function of the added state variable $w(t)$ and the observation $y(t)$ that estimates $x(t)$, i.e., the error $e(\cdot)$ defined by

$$e(t) = x(t) - \hat{x}(t) \quad (32)$$

satisfies an autonomous differential equation with exponents

$$\sigma(A_c: (\mathcal{X} \oplus \mathcal{W})/\mathcal{V}).$$

Proof: As in the proof of Proposition 4.2, we define $R: \mathcal{V}_p \rightarrow \mathcal{W}$ by requiring $\mathcal{V} = \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \mathcal{V}_p \right\}$, and $F: \mathcal{X} \rightarrow \mathcal{U}$ by $F = KC + LR$. Also, let $S: \mathcal{W} \rightarrow \mathcal{V}_p$ be such that $RS = I$, and $T: \mathcal{U} \rightarrow \mathcal{X}$ such that $\text{ran } T = \mathcal{V}_i$ and $CT = I$. Now, define $\hat{x}(t)$ by

$$\dot{\hat{x}}(t) = (I - TC)Sw(t) + Ty(t). \quad (33)$$

First, we have to show that this equation, together with (31) and the extended state equation

$$\dot{w}(t) = Nw(t) + My(t) \quad (34)$$

reproduces the composite system (3). That is, we have to show that $F\hat{x}(t) = Ky(t) + Lw(t)$. This follows immediately from the definitions we have made: $F(I - TC)S = (KC + LR)(I - TC)S = KCS + LRS - KCTCS - LRTCS = L$ (using $RS = I$, $CT = I$, $RT = 0$) and $FT = KCT + LRT = K$. Now let us write, for short, $V = I - TC$. We see that the error $e(t)$ may be written

$$e(t) = (V - VS)\begin{pmatrix} x \\ w \end{pmatrix}(t). \quad (35)$$

The following equivalences hold: $(V - VS)\begin{pmatrix} x \\ w \end{pmatrix} = 0 \Leftrightarrow x - Sw \in \ker V = \mathcal{V}_i = \ker R \Leftrightarrow Rx = RS w = w \Leftrightarrow \begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{V}$. Thus, $\ker(V - VS) = \mathcal{V}$ and this establishes that $e(\cdot)$ satisfies an autonomous differential equation of which the exponents are precisely those in $\sigma(A_c: (\mathcal{X} \oplus \mathcal{W})/\mathcal{V})$. $\square$

One may observe an additional feature of the split-up we have made in the theorem. It is clear from the representation (35) and the fact that V is a projection onto $\ker C$ that we shall have $e(t) \in \ker C$ for all $t > 0$ regardless of the initial values $x(0)$ and $w(0)$. This means that $C\hat{x}(t) = Cx(t)$ for all t ; one might say that the estimate is "true to the observation." A similar situation occurs in observer theory (cf. [16]).

The condition of the theorem on the A_c -invariant subspace $\mathcal{V}$ is a transversality condition in case $\dim \mathcal{V} = k + p$ and $k + p < n$ (where $k = \dim \mathcal{W}$). So we can conclude that "almost all" compensators of order $< n - p$ have an A_c -invariant subspace that satisfies the condition of Theorem 5.5, and that these may consequently be interpreted as observer-based compensators (with the same proviso as at the end of Section IV). We can even go one step further. Let a compensator of order $k < n - p$ be given. In general, every set of eigenvalues of the compensator matrix having joint multiplicity $k + p$ gives rise to an invariant subspace to which the condition of Theorem 5.5 applies; this means that these eigenvalues are then interpreted as "feedback poles," whereas the remaining ones are "error poles." We conclude that (generally speaking, still) any set of $k + p$ eigenvalues can be labeled "feedback poles," and any set of $n - p$ eigenvalues can be labeled "error poles." This indicates that it is rather pointless in compensator analysis to make such a distinction between eigenvalues.

B. Dual Observer-Based Compensators

After what has been said about observer-based compensators, we shall not dwell long on their "dual" counterparts. One can prove the following facts. If $(\mathcal{V}_1, \mathcal{V}_2)$ is a (C, A, B) -pair and $\mathcal{V}_2 \oplus \text{ran } B = \mathcal{X}$, then $P(\mathcal{V}_1, \mathcal{V}_2) = F(\mathcal{V}_2) \times G(\mathcal{V}_1)$; also, in this situation, $\sigma(A + BF: \mathcal{V}_2/\mathcal{O})$ does not depend on the choice of $F \in F(\mathcal{V}_2)$, but can be assigned freely (with the exception of uncontrollable poles) by suitably choosing $\mathcal{V}_2$ (complementary to $\text{ran } B$). Consequently, one can construct a compensator by taking $\mathcal{V}_2$ to be a suitable complement of $\text{ran } B$ and $\mathcal{V}_1$ to be $\mathcal{O}$, and this compensator has order $\dim \mathcal{V}_2 - \dim \mathcal{V}_1 = n - m$.

C. The Brasch-Pearson Compensator

As has been said before, the minimal stable cover problem is a difficult one. We do have the following result, however, on the "fixed-pole" cover problem in the case where the subspace to be covered is

one-dimensional. Let us say that a subset (with multiplicity) Λ of $\mathbb{C}$ "belongs to" an (A, B) -invariant subspace $\mathcal{V}$ if there exists $F \in F(\mathcal{V})$ such $\sigma(A + BF: \mathcal{V}/\mathcal{O}) = \Lambda$.

Lemma 5.6: Suppose that (A, B) is controllable. Let $\mathcal{V}_1$ be a one-dimensional subspace of $\mathcal{X}$. For any (symmetric) subset Λ of the complex plane (of suitable order), there exists a κ_c -dimensional (A, B) -invariant subspace $\mathcal{V}_2$ with $\mathcal{V}_2 \supset \mathcal{V}_1$ to which Λ belongs.

Here, κ_c is the controllability index of the pair (A, B) , i.e., the first value of j for which $\text{rank } [B \ AB \ \dots \ A^{j-1}B] = n$. The proof of the lemma is essentially in [5] (which in itself is an improved version of the proof in [17]), although the cover problem has a different form in these papers. The proof in [15] (which even gives a slightly sharper result) relates to a formulation of the problem that is close to ours, and so we refer to this work.

It is not difficult to use the lemma for synthesizing an observer-based compensator of order $\kappa_c - 1$, even in the case where $p > 1$.

Proposition 5.7: Suppose that (C, A) is observable and (A, B) is controllable. Then there exists a compensator for the system (1) of order $\kappa_c - 1$ with arbitrarily (within the restrictions of order and number field) assignable poles.

Proof: By the observability of the pair (C, A) , we have $Cx_i \neq 0$ for all eigenvectors $x_1, \dots, x_r$ of A . The property $HCx_i \neq 0$ is, for each i , generic for functionals H on $\mathcal{U}$; then the property $HCx_i \neq 0$ for all $i \in \{1, \dots, r\}$ is also generic for functionals H on $\mathcal{U}$. Let H be a functional with this property; write $\tilde{C} = HC$. Because $\tilde{C}x_i \neq 0$ for all $i \in \{1, \dots, r\}$, the pair $(\tilde{C}, A)$ is observable. Using the "observable" version of Lemma 5.3, we can find a complement $\mathcal{V}_1$ of $\ker \tilde{C}$ such that $\sigma(A + GC: \mathcal{X}/\mathcal{V}_1)$ has an arbitrary prescribed position for $G \in G(\mathcal{V}_1)$. Because $\mathcal{V}_1$ is one-dimensional, Lemma 5.6 gives us $\mathcal{V}_2 \supset \mathcal{V}_1$ and $F \in F(\mathcal{V}_2)$ such that $\sigma(A + BF: \mathcal{V}_2/\mathcal{O})$ is as prescribed with $\dim \mathcal{V}_2 = \kappa_c$. Clearly, this allows us to synthesize an observer-based (in the sense of $\tilde{C}$) compensator of order $\kappa_c - 1$. $\square$

Of course, the procedure can be "dualized" to produce a compensator of order $\kappa_o - 1$. The compensator order we have found is the one given by Brasch and Pearson in [2].

D. Direct Output Feedback

If a K is found such that $A + BKC$ is stable, then one can say that a zero-order compensator has been constructed. Such compensators always fit into the framework of Theorem 4.4. Just take $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$ where $\mathcal{V}$ is $(A + BKC)$ -invariant and let $F = KC$ and $G = BK$. This is trivial enough. Another question is whether the theory developed above can help to determine conditions under which zero-order compensation is possible.

In this connection, the following lemma is useful.

Lemma 5.8: Let $\mathcal{R}$ be a subspace of $\mathcal{X}$ such that $r + m > n + 1$ where $r = \dim \mathcal{R}$ (and $m = \dim \text{ran } B$). Then, generically with respect to A, B and $\mathcal{R}$ jointly, $\mathcal{R}$ is an (A, B) -controllability subspace.

Proof: Under the given relation between r, m , and n , the condition

$$\dim(\mathcal{R} + \text{ran } B) = n \quad (36)$$

is a transversality condition [7, p. 29] which, therefore, holds generically (with respect to $\mathcal{R}$). This implies that $\mathcal{R}$ is generically (A, B) -invariant. We may now use [7, Theorem 5.6, p. 111] to find the largest controllability subspace in $\mathcal{R}$.

Thus, define a sequence $\mathcal{S}_k$ according to

$$\begin{aligned} \mathcal{S}_0 &= \mathcal{O} \\ \mathcal{S}_k &= \mathcal{R} \cap (A\mathcal{S}_{k-1} + \text{ran } B) \quad (k = 1, \dots, n). \end{aligned} \quad (37)$$

Let k_0 be the first index for which $\mathcal{S}_{k+1} = \mathcal{S}_k$ and write $\mathcal{S}_{k_0} := \mathcal{S}_*$. Then $\mathcal{S}_*$ is the largest controllability subspace in $\mathcal{R}$. The following relation holds for it.

$$\mathcal{S}_* = \mathcal{R} \cap (A\mathcal{S}_* + \text{ran } B). \quad (38)$$

Note that it follows from $r + m > n + 1$ that $\mathcal{S}_1 = \mathcal{R} \cap \text{ran } B \neq \mathcal{O}$, so that $\mathcal{S}_* \neq \mathcal{O}$.

Let us write $\dim \mathcal{S}_* = q$. Then generically in A , $\dim A\mathcal{S}_* = q$ also. Using this, we have generically (in A and B) that $\dim(A\mathcal{S}_* + \text{ran } B) = \min(q + m, n)$. Finally, we obtain that generically (in A , B , and $\mathcal{R}$) the following holds.

$$\dim(\mathcal{R} \cap (A\mathcal{S}_* + \text{ran } B)) = \max(0, r + \min(q + m, n) - n). \quad (39)$$

From this and (38) we get, using the fact that $q \neq 0$,

$$q = r + \min(q + m, n) - n. \quad (40)$$

Now $\min(q + m, n)$ is equal either to $q + m$ or to n . But in the first case, (40) gives $r + m = n$, which contradicts the given relation $r + m > n + 1$. So it must be that $\min(q + m, n) = n$ and then (40) gives $q = r$, implying that $\mathcal{S}_* = \mathcal{R}$ and, thereby, that $\mathcal{R}$ is a controllability subspace. $\square$

The lemma shows, for instance, that the problem of disturbance decoupling with stability (see [7, par. 5.6] also for notation) is generically solvable if the pair (A, B) is stabilizable, $DE = 0$, and $\dim \ker D + \dim \text{ran } B > n + 1$. But we can also use it in the present context.

Corollary 5.9: Suppose that (C, A) is observable, (A, B) is controllable, and $\dim \text{ran } B > \dim \ker C + 1$. Then arbitrary pole placement by direct output feedback is generically possible.

Proof: We first use the "observable" version of Lemma 5.3 to obtain a complement $\mathcal{V}_1$ of $\ker C$ such that $\sigma(A + GC: \mathcal{X}/\mathcal{V}_1)$ has an arbitrarily prescribed location for $G \in G(\mathcal{V}_1)$. We have $\dim \mathcal{V}_1 + \dim \text{ran } B > n + 1$. So, generically, $\mathcal{V}_1$ is a controllability subspace and we can take $\mathcal{V}_2 = \mathcal{V}_1$ and $F \in F(\mathcal{V}_2)$ such that $\sigma(A + BF: \mathcal{V}_2/\theta)$ also has an arbitrarily prescribed location. So we can synthesize an observer-based compensator of order 0 with arbitrarily prescribed poles. $\square$

One may note that our method of proof also provides a simple design procedure. We have given the result in terms of genericity. Of course, one may go about to give sufficient conditions for arbitrary pole placement to be possible by direct output feedback, and indeed, several such conditions are available in the literature (see, for instance, [3], [4], and [18]). However, the relations between these results are not quite clear. Perhaps the theory of the present paper could be used to clarify this and also the further results of [18], but we shall not go into these matters here.

VI. CONCLUSIONS

We have presented a general symmetric theory of compensator synthesis, which describes the problem as a pairing problem: find a (C, A, B) -pair $(\mathcal{V}_1, \mathcal{V}_2)$ and a pair $(F, G) \in P(\mathcal{V}_1, \mathcal{V}_2)$ such that both $A + BF: \mathcal{V}_2/\theta$ and $A + GC: \mathcal{X}/\mathcal{V}_1$ are stable. As this formulation allows many ways to derive the same compensator, one can specialize—with hardly any loss of room, at least theoretically speaking—to the case in which $\mathcal{V}_1$ is a complement of $\ker C$, and so obtain compensators that may be interpreted as observer-based compensators.

Any result on the stable cover problem can now immediately be applied to compensator synthesis. (If there is a restriction on the dimension of the subspace to be covered, one may use a little trick as in the proof of Proposition 5.7.) One should note, however, that in the case of the compensator problem, the subspace to be covered is not really fixed. It is seen from Lemma 5.2 that if $\sigma(A + GC: \mathcal{X}/\mathcal{V}_1)$ is stable for $G \in G(\mathcal{V}_1)$ (where $\mathcal{V}_1$ is a complement of $\ker C$), then $\sigma(A + GC: \mathcal{X}/\mathcal{V}_1)$ will also be stable for $G \in G(\tilde{\mathcal{V}}_1)$ if $\tilde{\mathcal{V}}_1$ is sufficiently close to $\mathcal{V}_1$ (in a suitable topology to be laid on the complements of $\ker C$). The fact that the underlying subspace can be moved a bit is reassuring from a numerical point of view, but can it also be put to any theoretical use?

Another strategy for synthesizing a compensator of order k would be to first select a $(k+p)$ -dimensional (A, B) -invariant subspace $\mathcal{V}_2$ such that there exists F with $A + BF: \mathcal{V}_2/\theta$ stable, and then look for a suitable complement $\mathcal{V}_1$ of $\ker C$ such that $\mathcal{V}_1 \subset \mathcal{V}_2$. Dualizing this also gives rise to a version of the stable cover problem, but now the cover has to be complementary to a given subspace and, hence, has fixed dimension.

Obviously, there is room for further research here. One could also try to treat some variants of the compensator problem within the above

framework (for instance, when only an output function $z(t) = Dz(t)$ needs to be stabilized or when measured or unmeasured disturbances are present). Moreover, it would be of interest to see whether further results in compensator theory as in [18] can be interpreted or even extended with the help of (C, A, B) -pairs. It has already been shown that the concept of (C, A, B) -pairs, when appropriately modified, is instrumental in observer theory [16]. The author is also working on an extension of the theory to the infinite-dimensional context.

REFERENCES

- [1] D. G. Luenberger, "An introduction to observers," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 596–602, 1971.
- [2] F. M. Brach and J. B. Pearson, "Pole placement using dynamic compensators," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 34–43, 1970.
- [3] S. H. Wang and E. J. Davison, "On pole assignment in linear multivariable systems using output feedback," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 516–518, 1975.
- [4] H. Kimura, "Pole assignment by gain output feedback," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 509–516, 1975.
- [5] W. M. Wonham and A. S. Morse, "Feedback invariants of linear multivariable systems," *Automatica*, vol. 8, pp. 93–100, 1972.
- [6] —, "Decoupling and pole assignment in linear multivariable systems," *SIAM J. Contr.*, vol. 12, pp. 1–18, 1970.
- [7] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach* (2nd ed). New York: Springer Verlag, 1979.
- [8] J. M. Schumacher, " (C, A) -invariant subspaces: Some facts and uses," Wiskundig Seminarium Vrije Universiteit Amsterdam, The Netherlands, Rep. 110, 1979.
- [9] F. Hamano and K. Furuta, "Localization of disturbances and output decomposition in decentralized linear multivariable systems," *Int. J. Contr.*, vol. 22, pp. 551–562, 1975.
- [10] H. Akashi and H. Imai, "Disturbance localization and output deadbeat control through an observer in discrete-time linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 621–627, 1979.
- [11] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory," *J. Optimiz. Theory Appl.*, vol. 3, pp. 306–315, 1969.
- [12] A. S. Morse, "Structural invariants of linear multivariable systems," *SIAM J. Contr.*, vol. 11, pp. 446–465, 1973.
- [13] J. C. Willems and C. Commault, "Disturbance decoupling by measurement feedback with stability or pole placement," *SIAM J. Contr. Optimiz.*, to be published.
- [14] R. W. Scott and B. D. O. Anderson, "Least order, stable solution of the exact model matching problem," *Automatica*, vol. 14, pp. 481–492, 1978.
- [15] H. R. Sirisena, "Minimal-order observers for linear functions of a state vector," *Int. J. Contr.*, vol. 29, pp. 235–254, 1979.
- [16] J. M. Schumacher, "On the minimal stable observer problem," *Int. J. Contr.*, vol. 32, pp. 17–30, 1980.
- [17] D. G. Luenberger, "Observers for multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 190–197, 1966.
- [18] H. Kimura, "On pole assignment by output feedback," *Int. J. Contr.*, vol. 28, pp. 11–22, 1978.

A Time-Stepping Procedure for $\dot{X} = A_1 X + X A_2 + D$, $X(0) = C$

STEVEN M. SERBIN AND CYNTHIA A. SERBIN

Abstract—We develop an expression for the exact solution of the matrix differential problem $\dot{X} = A_1 X + X A_2 + D$, $X(0) = C$ based on variation of parameters and use this to devise the time-stepping relation $X(t+h) = e^{A_1 h} \{X(t) + \int_0^h e^{-A_1 s} D e^{-A_2 s} ds\} e^{A_2 h}$. We modify a procedure of Van Loan to effect efficient computation of all the terms necessary to advance the solution in time according to this relation. We consider some alternatives when sparsity is a concern. A numerical example of our procedure is included.

Manuscript received December 6, 1979; revised June 5, 1980. Paper recommended by A. J. Laub, Chairman of the Computational Methods and Discrete Systems Committee. This work was supported by USARO Grant DAAG29-78-C-0024. The Union Carbide Corporation—Nuclear Division is operated for the U.S. Department of Energy under Contract W-7405-eng-26.

S. M. Serbin is with the Department of Mathematics, University of Tennessee, Knoxville, TN 37916.

C. A. Serbin is with the Department of Mathematics and Statistics Research, Computer Sciences Division, Union Carbide Corporation—Nuclear Division, Oak Ridge, TN 37830.