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Published in:
IEEE Transactions on Automatic Control

Publication date:
1982

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
Schumacher, J. M. (1982). Regulator synthesis using (C,A,B)-pairs. *IEEE Transactions on Automatic Control*, AC-27, 1211-1221.

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Regulator Synthesis Using (C, A, B) -Pairs

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Abstract—The concept of a (C, A, B) -pair of subspaces was introduced in an earlier paper by the author to study problems of compensator design. Here, it is shown that the same concept plays a basic role in the theory of tracking and regulation. From a few fundamental properties, the main results of regulator theory are derived in a relatively simple and straightforward manner. Another new feature is that we are able to allow the presence of an unknown input, thereby unifying the fields of “tracking and regulation” and “disturbance decoupling.” All results are fully constructive, and an example is given to illustrate the theory.

I. INTRODUCTION

IT IS often true that insight in the practical handling of a computational problem depends crucially on which theoretical concepts are used to describe the basic question of solvability. Thus, the invertibility of a matrix can be described in terms of its determinant or in terms of its singular values, but from a numerical point of view the first standpoint is much less significant than the second. It is always desirable to have at least a clear conceptual framework available.

In this paper, we consider servo and regulator problems for linear multivariable systems. The basic solvability question has been studied extensively during the past decade, mostly within the state-space framework that we shall also use; see, for instance, [1]–[9]. ([9] will be our main reference.) The purpose of the present paper is to put forward a new concept which can be used to obtain the solution of the regulator problem in a direct and transparent way. Our method is to be compared with the procedure of [9], which proceeds stepwise from “restricted” problems (using state feedback with certain restrictions) to “extended” problems (using dynamic output feedback). Of course, the “extended” version is the form in which the problem has to be solved ultimately, and the main benefit of our approach is that it considers output feedback directly, without any intermediate steps. The key concept is that of a “ (C, A, B) -pair of subspaces,” which relates to dynamic

output feedback in roughly the same way as the concept “ (A, B) -invariant subspace” relates to state feedback. The notion of a (C, A, B) -pair has been used before by the author [14] to study stabilization by dynamic output feedback.

An interesting feature of our method, which acts as further corroboration of its naturalness, is that it is able to bring the disturbance decoupling aspect into the regulator problem. Up to now, the topics of “tracking and regulation” and “disturbance decoupling” have been treated separately; the solution of the latter problem involving dynamic output feedback is quite recent [10]–[13] and has, in fact, been the origin of the notion “ (C, A, B) -pair” (see [14]). We give a unified formulation from which both the tracking and regulation problems and the disturbance decoupling problems can be recovered as special cases. This general formulation has an appealing form: we consider a system which has two kinds of inputs, one that is at the controller's disposal and one that is controlled “nature,” and which also has two kinds of outputs, one of which describes the goal of the control and the other gives the available information.

For the general problems we shall pose, an algorithm will be given which checks their solvability, and we shall also give a method to construct a solution if one exists. While our treatment of these problems is thus completely constructive, it is not claimed that the presented algorithms are optimal with respect to numerical behavior. However, coming back to the point made at the beginning of the paper, it is believed that the conceptual clarification will be helpful to improve practical solution methods.

We lay down some notation and terminology in Section II where we also give the definition of a “ (C, A, B) -pair.” The next two sections give some basic results involving (C, A, B) -pairs; these are Theorem 3.1, which will be needed for the “necessity” part of our later results, and Theorem 4.1 which will be used for the constructive part. We shall pose the generalized regulator problem in three different versions, depending on the stability requirements. In Section V we consider “output stability,” and “input stability” is treated in Section VI. Complete solutions for both problems are given, together with specializations

Manuscript received December 17, 1980; revised September 2, 1981. Paper recommended by E. W. Kamen, Chairman of the Linear Systems Committee.

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earlier results. We close with an example in Section VII and conclusions in Section VIII.

II. PRELIMINARIES AND NOTATION

We shall use the following standard form for a linear multivariable system:

$$\begin{aligned}x'(t) &= Ax(t) + Bu(t) + Eq(t) \\ y(t) &= Cx(t) \\ z(t) &= Dx(t)\end{aligned}\quad (2.1)$$

where $x(t) \in X$ is the *state* of the system, $u(t) \in U$ is the *control*, $q(t)$ is the (external) *disturbance*, $y(t) \in Y$ is the *observation*, and $z(t)$ is the *variable-to-be-controlled*. A *compensator* for this system is a new linear finite-dimensional system that takes the observation $y(t)$ as its input and specifies at its output the control function $u(t)$ as follows:

$$\begin{aligned}w'(t) &= Nw(t) + My(t) \\ u(t) &= Lw(t) + Ky(t).\end{aligned}\quad (2.2)$$

The *order* of the compensator is the dimension of the state space W of $w(t)$. We can take the equations (2.1) and (2.2) together to form the following controlled system equations:

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}(t) &= \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}(t) + \begin{pmatrix} E \\ 0 \end{pmatrix} q(t) \\ z(t) &= \begin{pmatrix} D & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}(t).\end{aligned}\quad (2.3)$$

The matrix

$$A^e = \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \quad (2.4)$$

will be called the *extended system matrix*, and the space

$$X^e = X \oplus W \quad (2.5)$$

on which it acts will be called the *extended state space*. There are two natural mappings between X^e and X : the projection $P: X^e \rightarrow X$ defined by

$$P\begin{pmatrix} x \\ w \end{pmatrix} = x, \quad (2.6)$$

and the embedding $Q: X \rightarrow X^e$ defined by

$$Qx = \begin{pmatrix} x \\ 0 \end{pmatrix}. \quad (2.7)$$

For a subspace V of X , we have

$$P^{-1}V = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in X^e \mid x \in V \right\} \quad (2.8)$$

and for a subspace M of X^e we have

$$Q^{-1}M = \left\{ x \in X \mid \begin{pmatrix} x \\ 0 \end{pmatrix} \in M \right\}. \quad (2.9)$$

Less precisely, we might write $P^{-1}V = V \oplus W$, and $Q^{-1}M = M \cap X$. We shall assume that we have divided the com-

plex plane into two (self-conjugate) parts C_g and C_b which will indicate "stability" and "instability," respectively. So we shall refer to the sum of the generalized eigenspaces belonging to the eigenvalues of A in C_g as the "stable subspace" of A , and write it as $X_g(A)$. The "unstable subspace" of A is likewise defined and will be written as $X_b(A)$. Similarly, we define $X_g^e(A^e)$ and $X_b^e(A^e)$.

If $S \subset X$ is an A -invariant subspace, we shall write $A:S$ for the restriction of A to S , and $A:X/S$ for the quotient operator induced by A on the factor space X/S . Moreover, if S and T are both A -invariant subspaces and $S \subset T$, we shall write $A:T/S$ for the operator induced by the restriction of A to T on the factor space T/S (which operator can also be viewed as the restriction of $A:X/S$ to T/S).

Recall that a subspace V of X is called (A, B) -invariant [8] if there exists $F: X \rightarrow U$ such that $(A + BF)V \subset V$, or, equivalently,

$$AV \subset V + \text{im } B \quad (2.10)$$

(where $\text{im } B$ indicates the image of B). If V is (A, B) -invariant, we write

$$F(V) = \{F: X \rightarrow U \mid (A + BF)V \subset V\}. \quad (2.11)$$

Dually, a subspace T of X is called (C, A) -invariant [7] if there exists $G: Y \rightarrow X$ such that $(A + GC)T \subset T$, or, equivalently,

$$A^{-1}T \supset T \cap \ker C \quad (2.12)$$

(where $\ker C$ denotes the kernel of C). If T is (C, A) -invariant, we write

$$G(T) = \{G: Y \rightarrow X \mid (A + GC)T \subset T\}. \quad (2.13)$$

To describe stability properties related to subspaces, we introduce the following terminology. If V is A -invariant, we shall say that V is *inner stable* if $A:V$ is stable (that is, $\sigma(A:V) \subset C_g$), and *outer stable* if $A:X/V$ is stable. If V is (A, B) -invariant, we shall say that V is *inner stabilizable* if there exists $F \in F(V)$ such that $A + BF:V$ is stable, and *outer stabilizable* if there exists $F \in F(V)$ such that $A + BF:X/V$ is stable. Finally, a (C, A) -invariant subspace T will be called *inner detectable* if there exists $G \in G(T)$ such that $A + GC:T$ is stable, and *outer detectable* if there exists $G \in G(T)$ such that $A + GC:X/T$ is stable. The final notion we recall in this paragraph comes from [14]: a (C, A, B) -pair is a pair of subspaces one of which is contained in the other, the smallest one being (C, A) -invariant, and the largest one (A, B) -invariant. In the notation for such a pair, we shall always write the smaller subspace first, so that " (T, V) is a (C, A, B) -pair" will mean that T is (C, A) -invariant, V is (A, B) -invariant, and $T \subset V$.

III. NECESSARY CONDITIONS FOR CLOSED-LOOP STRUCTURAL PROPERTIES

In this section, we shall lay the foundations for the "necessity" parts of our solutions of the synthesis problems. Therefore, we always assume that W is a space of

arbitrary (but fixed) finite dimension, and that A^e is an arbitrary (but fixed) matrix of the form (2.4). Our aim is to prove the following theorem.

Theorem 3.1: The following relations hold for any A^e -invariant subspace M of $X \oplus W$.

If M is inner stable, then PM is inner stabilizable; (3.1a)

if M is outer stable, then PM is outer stabilizable; (3.1b)

if M is inner stable, then $Q^{-1}M$ is inner detectable; (3.1c)

if M is outer stable, then $Q^{-1}M$ is outer detectable. (3.1d)

That PM is (A, B) -invariant and $Q^{-1}M$ is (C, A) -invariant if M is A^e -invariant is a simple fact proved in [14]; our interest is now in the stability properties. The following lemma gives a useful characterization of inner stabilizability.

Lemma 3.2: An (A, B) -invariant subspace V of X is inner stabilizable if and only if

$$(\lambda - A)V + \text{im } B = V + \text{im } B \quad \text{for all } \lambda \in \mathbb{C}_b. \quad (3.2)$$

Proof:

"Only If": Suppose that V is inner stabilizable; then we can find F such that $A + BF:V$ is stable. For any $\lambda \in \mathbb{C}_b$, λ is not an eigenvalue of $A + BF:V$, and thus

$$(\lambda - (A + BF))V = V \quad \text{for all } \lambda \in \mathbb{C}_b. \quad (3.3)$$

The conclusion now follows upon noting that, for all $F \in F(V)$,

$$(\lambda - (A + BF))V + \text{im } B = (\lambda - A)V + \text{im } B. \quad (3.4)$$

"If": Now assume that (3.2) holds. By [9, Corollary 5.2], we can take $F \in F(V)$ such that $A + BF:R^*$ is stable, where

$$R^* = \langle A + BF | \text{im } B \cap V \rangle \quad (\text{any } F \in F(V)) \quad (3.5)$$

is the largest controllability subspace in V [9, Theorem 5.5]. With F chosen in this way, we contend that $A + BF:V$ is stable. Take any $\lambda \in \mathbb{C}_b$. From (3.2) and (3.4), we get

$$V \subset (\lambda - (A + BF))V + \text{im } B. \quad (3.6)$$

Because $(\lambda - (A + BF))V \subset V$, this is equivalent to

$$V \subset (\lambda - (A + BF))V + (\text{im } B \cap V), \quad (3.7)$$

which immediately implies

$$V \subset (\lambda - (A + BF))V + R^*. \quad (3.8)$$

Because it follows from $\lambda \notin \sigma(A + BF:R^*)$ that $(\lambda - (A + BF))R^* = R^*$, we obtain

$$\begin{aligned} V &\subset (\lambda - (A + BF))V + (\lambda - (A + BF))R^* \\ &= (\lambda - (A + BF))V \subset V. \end{aligned} \quad (3.9)$$

So our final conclusion is

$$V = (\lambda - (A + BF))V \quad (3.10)$$

which means that λ is not an eigenvalue of $A + BF:V$. ■

The dual result is the following.

Lemma 3.2': A (C, A) -invariant subspace T is outer detectable if and only if

$$(\lambda - A)^{-1}T \cap \ker C = T \cap \ker C \quad \text{for all } \lambda \in \mathbb{C}_b. \quad (3.11)$$

A characterization of outer stabilizability follows immediately from [16].

Lemma 3.3: An (A, B) -invariant subspace V is outer stabilizable if and only if the A -invariant subspace $V + \langle A | \text{im } B \rangle$ is outer stable.

Dually, we have the following.

Lemma 3.3': A (C, A) -invariant subspace is inner detectable if and only if the A -invariant subspace $T \cap \langle \ker C | A \rangle$ is inner stable. (As usual, $\langle A | \text{im } B \rangle$ denotes the smallest A -invariant subspace containing $\text{im } B$; $\langle \ker C | A \rangle$ is the largest A -invariant subspace contained in $\ker C$.) Finally, for the proof of the theorem we shall need the following pair of lemmas.

Lemma 3.4: If V is an A -invariant subspace of X containing $\text{im } B$, then $P^{-1}V$ is A^e -invariant and

$$\sigma(A: X/V) = \sigma(A^e: X^e/P^{-1}V). \quad (3.12)$$

Lemma 3.4': If V is an A -invariant subspace of X contained in $\ker C$, then QV is A^e -invariant and

$$\sigma(A: V) = \sigma(A^e: QV). \quad (3.13)$$

Proof: We shall only prove the first one. Take $\begin{pmatrix} x \\ w \end{pmatrix} \in P^{-1}V$; then

$$A^e \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} Ax + B(KCx + Lw) \\ MCx + Nw \end{pmatrix} \in P^{-1}V. \quad (3.14)$$

Thus, V is A^e -invariant. Now consider the mapping $T: X^e/P^{-1}V \rightarrow X/V$ defined by

$$T: \begin{pmatrix} x \\ w \end{pmatrix} + P^{-1}V \mapsto x + V. \quad (3.15)$$

This mapping is clearly bijective, and the proof is completed by a straightforward computation showing that the diagram

$$\begin{array}{ccc} X^e/P^{-1}V & \xrightarrow{A^e: X^e/P^{-1}V} & X^e/P^{-1}V \\ \downarrow T & & \downarrow T \\ X/V & \xrightarrow{A: X/V} & X/V \end{array} \quad (3.16)$$

commutes. ■

Proof of Theorem 3.1: To prove (3.1a), we have to show that

$$(\lambda - A)PM + \text{im } B = PM + \text{im } B \quad \text{for all } \lambda \in \mathbb{C}_b \quad (3.17)$$

if $\sigma(A^e: M) \subset \mathbb{C}_g$. It is, of course, sufficient to prove that

$$PM \subset (\lambda - A)PM + \text{im } B \quad \text{for all } \lambda \in \mathbb{C}_b. \quad (3.18)$$

So, take $\lambda \in \mathbb{C}_b$ and $x \in PM$. Let $w \in W$ be such that $\begin{pmatrix} x \\ w \end{pmatrix} \in V$. Because $\lambda \notin \sigma(A^e: M)$, there exists $\begin{pmatrix} x_1 \\ w_1 \end{pmatrix} \in V$ such that $(\lambda - A^e)\begin{pmatrix} x_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} x \\ w \end{pmatrix}$. In particular, this gives

$$x = (\lambda - (A + BKC))x_1 - BLw_1 \in (\lambda - A)PM + \text{im } B \quad (3.19)$$

which completes the proof of (3.1a).

For our second assertion, we have to show that $PM + \langle A|\text{im } B \rangle$ is outer stable (with respect to A) if M is outer stable (with respect to A^e). This follows easily from Lemma 3.4:

$$\begin{aligned} \sigma(A: X/(PM + \langle A|\text{im } B \rangle)) \\ &= \sigma(A^e: X^e/P^{-1}(PM + \langle A|\text{im } B \rangle)) \\ &\subset \sigma(A^e: X^e/M) \subset \mathbb{C}_g. \end{aligned} \quad (3.20)$$

Finally, (3.1c) is dual to (3.1b) and (3.1d) is dual to (3.1a). ■

The result (3.1a) has also been proved in [13], but the proof appearing there is rather complicated due to the fact that it is based on the characterization of stabilizability subspaces in [9], which requires the computation of an $F \in F(V)$.

IV. A CONSTRUCTION THEOREM

The regulator problem can be viewed as a generalized pole placement problem because it requires not only that certain eigenvalues are relocated, but also that the corresponding eigenspaces satisfy certain properties. The following theorem can be considered as a result on "eigenspace assignment": it gives a construction of a closed-loop system with eigenspaces of a prescribed form and having prescribed stability properties. Basically, the theorem tells us what subspaces and mappings we need to have in the original state space X , in order to obtain a closed-loop system matrix of a certain structure. The result is perhaps more difficult in its statement than in its proof; it will be seen more clearly how the theorem is used when we come to the applications in Sections V and VI.

Theorem 4.1: Let two (C, A, B) -pairs (T, V) and (T_c, V_c) be given, with $AT \subset V$. Let also be given $F \in F(V) \cap F(V_c)$ with $\ker F \supset T_c$, and $G \in G(T) \cap G(T_c)$ with $\text{im } G \subset V_c$. Let R be the canonical mapping from V_c to $W: = V_c/T_c$, and let R^+ be any right inverse to R (which exists because R is surjective). Consider the extended system matrix

$$A^e := \begin{pmatrix} A & BFR^+ \\ -RGC & R(A + BF + GC)R^+ \end{pmatrix}. \quad (4.1)$$

The subspaces M and M_c of X^e defined by

$$M := \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| x \in T \right\} + \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in V \cap V_c \right\} \quad (4.2)$$

$$M_c := \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in V_c \right\} \quad (4.3)$$

are A^e -invariant, and the following relations hold between the involved eigenvalues:

$$\sigma(A^e: X^e/M + M_c) = \sigma(A + GC: X/T + T_c) \quad (4.4a)$$

$$\begin{aligned} \sigma(A^e: M + M_c/M_c) &= \sigma(A^e: M/M \cap M_c) \\ &= \sigma(A + GC: T + T_c/T_c) \\ &= \sigma(A + GC: T/T \cap T_c) \end{aligned} \quad (4.4b)$$

$$\begin{aligned} \sigma(A^e: M + M_c/M) &= \sigma(A^e: M_c/M \cap M_c) \\ &= \sigma(A + BF: V + V_c/V) \\ &= \sigma(A + BF: V_c/V \cap V_c) \end{aligned} \quad (4.4c)$$

$$\sigma(A^e: M \cap M_c) = \sigma(A + BF: V \cap V_c). \quad (4.4d)$$

Proof: To show that M is A^e -invariant, first consider $\begin{pmatrix} x \\ 0 \end{pmatrix}$ with $x \in T$. We have

$$\begin{aligned} A^e \begin{pmatrix} x \\ 0 \end{pmatrix} &= \begin{pmatrix} Ax \\ -RGCx \end{pmatrix} \\ &= \begin{pmatrix} (A + GC)x \\ 0 \end{pmatrix} - \begin{pmatrix} GCx \\ RGCx \end{pmatrix} \in M \end{aligned} \quad (4.5)$$

because $(A + GC)x \in T$ and $GCx = (A + GC)x - Ax \in V$ (since $Ax \in AT \subset V$). Now consider $\begin{pmatrix} x \\ Rx \end{pmatrix}$ with $x \in V \cap V_c$. Applying A^e , we get

$$A^e \begin{pmatrix} x \\ Rx \end{pmatrix} = \begin{pmatrix} Ax + BFR^+Rx \\ R(A + BF + GC)R^+Rx - RGCx \end{pmatrix}. \quad (4.6)$$

Write $x = R^+Rx + x_1$; then $Rx_1 = 0$ so that $x_1 \in T_c \subset \ker F$. Thus we have $FR^+Rx = Fx$. Moreover, $R(A + GC)R^+Rx = R(A + GC)x - R(A + GC)x_1 = R(A + GC)x$ because $(A + GC)x_1 \in T_c = \ker R$. We find that

$$\begin{aligned} R(A + BF + GC)R^+Rx - RGCx \\ = R(A + BF + GC)x - RGCx = R(A + BF)x, \end{aligned}$$

so finally

$$A^e \begin{pmatrix} x \\ Rx \end{pmatrix} = \begin{pmatrix} (A + BF)x \\ R(A + BF)x \end{pmatrix} \in M. \quad (4.7)$$

By very similar reasoning, it is shown that M_c is A^e -invariant.

The rest of the proof is an exercise in finding natural isomorphisms between subspaces, and in verifying that the similarities that are obtained in this way provide precisely the intertwining relations needed to prove (4.4). For instance, one easily checks that the following diagram commutes:

$$\begin{array}{ccc}
 M/M \cap M_c & \xrightarrow{A^e: M/M \cap M_c} & M/M \cap M_c \\
 \uparrow T & & \uparrow T: x + (T \cap T_c) \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} + (M \cap M_c) \\
 T/T \cap T_c & \xrightarrow{A + GC: T/T \cap T_c} & T/T \cap T_c
 \end{array} \quad (4.8)$$

which establishes one of the equalities in (4.4b). The other relations are proved in a similar fashion. ■

The spectral relations (4.4) can be conveniently summarized in the following figure:

In the special case $T_c \subset T \subset V \subset V_c$, the situation simplifies to

inner stabilizable subspaces is closed under taking sums, and from Lemma 3.2' that the set of outer detectable

We shall need some further lemmas. The first one relates to the restriction $AT \subset V$ appearing in Theorem 4.1.

Lemma 4.2: Let (T, V) be a (C, A, B) -pair. Then there exists $K: Y \rightarrow U$ such that $(A + BKC)T \subset V$.

Proof: Let $\{x_1, \dots, x_j, x_{j+1}, \dots, x_k\}$ be a basis for T such that $\{x_1, \dots, x_j\}$ is a basis for $T \cap \ker C$. Because $T \subset V$ and V is (A, B) -invariant, there exist $u_i \in U$ and $v_i \in V$ such that

$$Ax_i = v_i + Bu_i \quad (i = 1, \dots, k). \quad (4.10')$$

Noting that the vectors $Cx_{j+1}, \dots, Cx_k$ are linearly independent, we can take $K: Y \rightarrow U$ such that

$$KCx_i = -u_i \quad (i = j+1, \dots, k). \quad (4.11)$$

Then we have $(A + BKC)x_i = v_i \in V$ for $i = j+1, \dots, k$; but also $(A + BKC)x_i = Ax_i \in T \subset V$ for $i = 1, \dots, j$ because $A(T \cap \ker C) \subset T$. ■

So replacing A by $A + BKC$ if necessary, the assumption $AT \subset V$ can always be made without restricting the general-

subspaces is closed under intersection. Consequently, there exist a largest inner stabilizable subspace and a smallest outer detectable subspace in X , and it is easily seen that they are given respectively by

$$X_{\text{stab}} = X_g(A) + \langle A | \text{im } B \rangle \quad (4.12)$$

$$X_{\text{det}} = X_b(A) \cap \langle \ker C | A \rangle. \quad (4.13)$$

(For convenience, we suppress in the notation the dependence on the matrices A , B , and C .) We can now formulate the final lemmas that will be needed.

Lemma 4.3: If the (A, B) -invariant subspace V is inner stabilizable, then there exists $F \in F(V)$ such that $A + BF: X_{\text{stab}}$ is stable.

Lemma 4.3': If the (C, A) -invariant subspace T is outer detectable, then there exists $G \in G(T)$ such that $A + GC: X/X_{\text{det}}$ is stable.

The easy proof of these lemmas, which can be based for instance on the result of [15], is left to the reader.

V. REGULATOR SYNTHESIS WITH OUTPUT STABILITY

We consider the following system equations:

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t) + Eq(t) \\ y(t) &= Cx(t) \\ z(t) &= Dx(t) \end{aligned} \quad (5.1)$$

where the interpretation of the variables has been given in Section II. The state $x(t)$ can in general be thought of as obtained by aggregation of the state of an "endosystem" ("plant") and the state of an "exosystem," which represents modeled disturbances (as opposed to the unmodeled "external" disturbance $q(t)$) and tracking signals which are to be followed. Because the exosystem is not controlled and typically unstable, the pair (A, B) will typically not be stabilizable; in other words, $X_{\text{stab}} \neq X$.

Adding a compensator of the form (2.2) to the system, we obtain the following extended system equations:

$$\frac{d}{dt} \begin{pmatrix} x \\ w \end{pmatrix} (t) = \begin{pmatrix} A + BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} (t) + \begin{pmatrix} E \\ 0 \end{pmatrix} q(t) \quad (5.2a)$$

$$z(t) = \begin{pmatrix} D & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} (t). \quad (5.2b)$$

Our first problem will be to construct a compensator (that is, to find W, K, L, M , and N) such that the variable-to-be-controlled $z(t)$ will approach zero as $t \rightarrow \infty$, regardless of the behavior of $q(t)$. We shall call this "output stability." The requirement can be formulated in terms of the extended system matrix A^e as follows:

$$X_b^e(A^e) \subset \ker \begin{pmatrix} D & 0 \end{pmatrix} \quad (5.3a)$$

$$-A^e \text{im} \begin{pmatrix} E \\ 0 \end{pmatrix} \subset \ker \begin{pmatrix} D & 0 \end{pmatrix}. \quad (5.3b)$$

An equivalent formulation is the following:

$$\begin{aligned} &\text{There exists an outer-stable } A^e\text{-invariant} \\ &\text{subspace in between } \text{im} \begin{pmatrix} E \\ 0 \end{pmatrix} \text{ and } \ker \begin{pmatrix} D & 0 \end{pmatrix}. \end{aligned} \quad (5.4)$$

Of course "V is in between R and S" means that $R \subset V \subset S$. We shall also say that a (C, A, B) -pair (T, V) is "in between" R and S if $R \subset T \subset V \subset S$.

Theorem 5.1: There exists a compensator for the system (5.1) achieving output stability if and only if there exists a (C, A, B) -pair (T, V) in between $\text{im } E$ and $\ker D$ such that

$$T \text{ is outer detectable;} \quad (5.5a)$$

$$V \text{ is outer stabilizable.} \quad (5.5b)$$

Proof:

"Only If": Suppose that A^e is an extended system matrix satisfying (5.4), and let M with $\text{im} \begin{pmatrix} E \\ 0 \end{pmatrix} \subset M \subset \ker \begin{pmatrix} D & 0 \end{pmatrix}$ be A^e -invariant and outer stable. Because $\text{im} \begin{pmatrix} E \\ 0 \end{pmatrix} = Q(\text{im } E)$ and $\ker \begin{pmatrix} D & 0 \end{pmatrix} = P^{-1}(\ker D)$, we have

$$\text{im } E \subset Q^{-1}M \subset PM \subset \ker D \quad (5.6)$$

so that $(Q^{-1}M, PM)$ is a (C, A, B) -pair in between $\text{im } E$ and $\ker D$. Moreover, $Q^{-1}M$ is outer detectable and PM is outer stabilizable, both by Theorem 3.1.

"If": Let (T, V) be a (C, A, B) -pair in between $\text{im } E$ and $\ker D$, and let (5.5) be satisfied. By Lemma 4.2, we may assume, without restricting the generality, that $AT \subset V$. Take $F \in F(V)$ such that $A + BF: X/V$ is stable, and $G \in G(T)$ such that $A + GC: X/T$ is stable. Now apply the construction of Theorem 4.1 with the pair (T, V) and the mappings F and G , setting $T_c = O$ and $V_c = X$. We obtain the extended system matrix A^e given in (4.1); if we consider W as a copy of X , we have $R = I$ and $R^+ = I$. The A^e -invariant subspace M defined by (4.2) satisfies $Q^{-1}M = T$ and $PM = V$, so it is clearly in between $\text{im} \begin{pmatrix} E \\ 0 \end{pmatrix}$ and $\ker \begin{pmatrix} D & 0 \end{pmatrix}$; moreover, it is outer stable as we can see from (4.10). ■

Of course, one asks how the condition given in the theorem can be checked constructively (that is, by an algorithm) if the system parameters are known. Note that there is a smallest outer detectable subspace containing $\text{im } E$; we shall denote it by $T_g^*(\text{im } E)$. This subspace can be constructed, for instance, by dualizing the procedure outlined in [9, pp. 114–115]. It is seen from Lemma 3.3 that any (A, B) -invariant subspace containing an outer stabilizable subspace is itself outer stabilizable. Therefore, if there are any outer stabilizable subspaces in $\ker D$ at all, then there is a largest one among them, namely, $V^*(\ker D)$: i.e., the largest (A, B) -invariant subspace in $\ker D$, which can be constructed by the algorithm given in [9, p. 91]. So we arrive at the following constructive criterion.

Corollary 5.2: There exists a compensator for the system (5.1) achieving output stability if and only if

$$X_b(A) \subset V^*(\ker D) + \langle A | \text{im } B \rangle \quad (5.7a)$$

$$T_g^*(\text{im } E) \subset V^*(\ker D). \quad (5.7b)$$

Setting $E = 0$, we recover the criterion for the "extended regulator problem" given in [9, p. 139].

Corollary 5.3: Suppose that $E = 0$ in the system (5.1). There exists a compensator for this system achieving output stability if and only if

$$X_b(A) \subset V^*(\ker D) + \langle A | \text{im } B \rangle \quad (5.8a)$$

$$X_{\text{det}} \subset \ker D. \quad (5.8b)$$

For the proof, just note that $T_g^*(O) = X_{\text{det}}$ by definition (4.18), and that $X_{\text{det}} \subset V^*(\ker D)$ is equivalent to $X_{\text{det}} \subset \ker D$ because X_{det} is A -invariant and hence also (A, B) -invariant.

VI. REGULATOR SYNTHESIS WITH PLANT STABILITY

The system equations (2.1) are derived from a model which incorporates a "plant" and a "signal," and which could be written out more explicitly as follows:

$$x_1'(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) + E_1q(t) \quad (6.1a)$$

$$x_2'(t) = A_{22}x_2(t) + E_2q(t) \quad (6.1b)$$

$$y(t) = C_1x_1(t) + C_2x_2(t) \quad (6.1c)$$

$$z(t) = D_1x_1(t) + D_2x_2(t). \quad (6.1d)$$

Here, the variable $x_1(t)$ represents the plant and the variable $x_2(t)$ represents the signal. After closing the loop by a compensator of the form (2.2), we obtain the following system:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ w \\ x_2 \end{pmatrix} (t) = \begin{pmatrix} A_{11} + B_1KC_1 & B_1L & A_{12} + B_1KC_2 \\ MC_1 & N & MC_2 \\ 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ w \\ x_2 \end{pmatrix} (t) + \begin{pmatrix} E_1 \\ 0 \\ E_2 \end{pmatrix} q(t) \quad (6.2a)$$

$$z(t) = (D_1 \ 0 \ D_2) \begin{pmatrix} x_1 \\ w \\ x_2 \end{pmatrix} (t). \quad (6.2b)$$

It is clear that the unstable modes of A_{22} will always be present in the closed-loop system; however, it may be possible to stabilize the plant, which is represented in closed loop by the upper-left 2×2 block in the above matrix. If this 2×2 block is stable, we shall say that we have achieved *plant stability*. So this means that the operation of the system will be described by modes in $\mathbb{C}_g$ if the external signal is zero ($x_2(0) = 0$).

We now want to translate the "plant stability" condition into coordinate-free terms. Because the distinction between signal and plant (that is, between $x_1(t)$ and $x_2(t)$) is coordinate-dependent, there is an inevitable loss of information involved here. However, since the plant may be characterized as the part of the system in which the control enters, it is reasonable to assume that the plant is given by the controllable part of the system, and the requirement of plant stability can then be formulated as

$$\dim X_b^e(A^e) = \text{codim } X_{\text{stab}} \quad (6.3)$$

[since the latter quantity gives precisely the joint multiplicity of the unstable uncontrollable poles of the pair (A, B)]. The same formulation would have been obtained if we had assumed that the plant is given by the stabilizable (rather than the controllable) part of the system. On the other hand, the formulation (6.3) cannot be used if the plant contains unstable uncontrollable poles; but then plant stability is, of course, not achievable.

Our concept of "plant stability" coincides with the notion of "closed-loop stability" used in [6], but it is stronger than the concept of "internal stability" studied in [9]. Wonham's treatment also covers situations of the following nature: the plant contains parts which cannot be made stable due to instabilizability and/or undetectability, but this is for some reason not bothersome, and stabilization is

only required for the controllable and observable part of the system. We shall not pursue this type of generality here.

It is immediately clear from (6.2) that a necessary condition for plant stability to be achievable is that (A_{11}, B_1) is stabilizable and (C_1, A_{11}) is detectable. It is *not* necessary that the pair (C, A) is detectable, but, as we shall see, the undetectable poles will have to be decoupled from the variables-to-be-controlled. This reason is good enough for assuming detectability at the outset (as argued in [6, p. 492–493]), but the other alternative does not lead to complications, and so in this paper we shall not assume detectability of (C, A) .

We now proceed to the main result of this section.

Theorem 6.1: There exists a compensator for the system (5.1) such that the closed-loop system satisfies (5.3) and (6.3) if and only if there exists a (C, A, B) -pair (T, V) in between $\text{im } E$ and $\ker D$ such that

$$T \text{ is outer detectable} \quad (6.4a)$$

$$V = V_b + V_g, \quad (6.4b)$$

where V_b and V_g are (A, B) -invariant subspaces such that

$$V_g \text{ is inner stabilizable} \quad (6.4b.i)$$

$$V_b \oplus X_{\text{stab}} = X \quad (6.4b.ii)$$

$$V_b \supset X_{\text{det}}. \quad (6.4b.iii)$$

Proof:

"Only If": Suppose that A^e is an extended system matrix satisfying (5.3) and (6.3). Let M be an A^e -invariant subspace containing $X_b^e(A^e)$ and $Q(\text{im } E)$, and itself contained in $P^{-1}(\ker D)$. Then $(Q^{-1}M, PM)$ is a (C, A, B) -pair in between $\text{im } E$ and $\ker D$. Moreover, $Q^{-1}M$ is outer detectable by Theorem 3.1. Because $X_b^e(A^e) \subset M$, we can write

$$M = M_b \oplus M_g \quad (6.5)$$

where we define

$$M_g := M \cap X_g^e(A^e), \quad M_b := X_b^e(A^e). \quad (6.6)$$

It is clear that M_g is inner stable; by Theorem 3.1, it follows that PM_g is inner stabilizable. In particular,

$$PM_g \subset X_{\text{stab}}. \quad (6.7)$$

Of course, M_b is outer stable and so it follows from Theorem 3.1 and Lemma 3.3 that

$$\begin{aligned} PM_b + X_{\text{stab}} &= PM_b + \langle A|\text{im } B \rangle + X_g(A) \\ &\supset X_b(A) + X_g(A) = X. \end{aligned} \quad (6.8)$$

From the obvious relation $\dim PM_b \leq \dim M_b$ and from (6.3) we now obtain that

$$PM_b \oplus X_{\text{stab}} = X. \quad (6.9)$$

From (6.5), (6.7), and (6.9), we see immediately that

$$PM = PM_g \oplus PM_b. \quad (6.10)$$

Finally, it follows from Lemma 3.4' that $QX_{\det} \subset X_b^e(A^e)$, and consequently,

$$X_{\det} \subset PM_b. \quad (6.11)$$

It is now clear that the conditions of the theorem are satisfied by

$$\begin{aligned} T &= Q^{-1}M, \quad V = PM, \\ V_g &= PM_g \quad \text{and} \quad V_b = PM_b. \end{aligned} \quad (6.12)$$

"If": Let (T, V) be a (C, A, B) -pair in between $\text{im } E$ and $\ker D$, for which (6.4) is satisfied. By Lemma 4.2, we may assume, without restricting the generality, that $AT \subset V$. We construct a mapping $F \in F(V)$ in the following way.

By (6.4b.ii), we can specify F by giving its action on V_b and X_{stab} separately. Using Lemma 4.3, we can choose F on X_{stab} in such a way that $F \in F(V_g)$ and $A + BF: X_{\text{stab}}$ is stable. On V_b we take F such that $(A + BF)V_b \subset V_b$; because $AX_{\det} \subset X_{\det} \subset V_b$, it is possible (and, as a little thought will show, even necessary) to do this with $X_{\det} \subset \ker F$. Putting our feedback map together, we get that $A + BF: X/V_b$ is stable, and $F \in F(V)$.

By Lemma 4.3', we can take $G \in G(T)$ such that $A + GC: X/X_{\det}$ is stable. Using the pair (T, V) and the mappings F and G as constructed, we apply Theorem 4.1 with $T_c = X_{\det}$ and $V_c = X$. We can read off immediately from (4.10) and the definition (4.2) that $Q(\text{im } E) \subset M \subset P^{-1}(\ker D)$ and that $X_b^e(A^e) \subset M$. Moreover, we have

$$\begin{aligned} M \cap M_c &= \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in V \right\} \\ &= \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in V_g \right\} \oplus \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in V_b \right\}. \end{aligned} \quad (6.13)$$

Using the easily established isomorphism between $V \cap V_c$ and $M \cap M_c$ that is given by the commuting diagram below,

$$\begin{array}{ccc} M \cap M_c & \xrightarrow{A^e: M \cap M_c} & M \cap M_c \\ \uparrow T: x \mapsto \begin{pmatrix} x \\ Rx \end{pmatrix} & & \uparrow T: x \mapsto \begin{pmatrix} x \\ Rx \end{pmatrix} \\ V \cap V_c & \xrightarrow{A + BF: V \cap V_c} & V \cap V_c \end{array} \quad (6.14)$$

we get

$$X_b^e(A^e) = \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| x \in V_b \right\}. \quad (6.15)$$

Thus, it follows that

$$\dim X_b^e(A^e) = \dim V_b = \text{codim } X_{\text{stab}}. \quad (6.16)$$

This completes the proof. ■

We now want to put the condition (6.4) into a constructive form. The following result, which can be easily obtained from Theorem 6.1 using an argument similar to the one preceding Corollary 5.2 is a preliminary step.

Corollary 6.2: There exists a compensator for the system (5.1) achieving plant stability if and only if there exists a

subspace V_b of X with the following properties:

$$V_b \oplus X_{\text{stab}} = X \quad (6.20a)$$

$$AV_b \subset V_b + \text{ran } B \quad (6.20b)$$

$$V_b \subset \ker D \quad (6.20c)$$

$$V_b \supset X_{\det} \quad (6.20d)$$

$$V_b + V_g^*(\ker D) \supset T_g^*(\text{im } E). \quad (6.20e)$$

Here, $V_g^*(\ker D)$ is the largest inner stabilizable subspace in $\ker D$, which can be constructed using the method of [9, pp. 114–115]. The next step is to rewrite the equations (6.20) as a set of matrix equations. For this purpose, we fix a basis $\{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$ for X such that $\{x_1, \dots, x_k\}$ is a basis for X_{stab} . With respect to this basis and this partitioning, let the matrices of A , B , and D be given by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad (6.21a)$$

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad (6.21b)$$

$$D = (D_1 \quad D_2). \quad (6.21c)$$

Also, we introduce basis matrices for the known subspaces $X_{\det}$, $T_g^*(\text{im } E)$ and $V_g^*(\ker D)$:

$$X_{\det} = sp \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad (6.21d)$$

$$T_g^*(\text{im } E) = sp \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \quad (6.21e)$$

$$V_g^*(\ker D) = sp \begin{pmatrix} V_1 \\ 0 \end{pmatrix}. \quad (6.21f)$$

Finally, for the unknown subspace V_b we use the notation

$$V_b = sp \begin{pmatrix} T \\ S \end{pmatrix}. \quad (6.22)$$

Now we can start translating (6.20) into matrix terms. It follows from (6.20a) that V_b should be $(n - k)$ -dimensional, so that the matrix S in (6.22) can be taken square. Then (6.20a) says that the $n \times n$ -matrix

$$\begin{pmatrix} I_k & T \\ 0 & S \end{pmatrix} \quad (6.23)$$

should be invertible. This will be the case if and only if S is invertible. But then we have

$$sp \begin{pmatrix} T \\ S \end{pmatrix} = sp \begin{pmatrix} T \\ S \end{pmatrix} S^{-1} = sp \begin{pmatrix} TS^{-1} \\ I \end{pmatrix} \quad (6.24)$$

so that it is no restriction of the generality to assume that $S = I$, and we shall do this from now on.

The requirement (6.20b) can now be reformulated as: there exist matrices Q and R such that

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} T \\ I \end{pmatrix} = \begin{pmatrix} T \\ I \end{pmatrix} Q + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} R. \quad (6.25)$$

It readily follows that we must have $Q = A_{22}$, and then (6.25) boils down to: there exists a matrix R such that

$$A_{11}T + A_{12} = TA_{22} + B_1R. \quad (6.26)$$

To translate (6.20c) is very simple:

$$(D_1 \ D_2) \begin{pmatrix} T \\ I \end{pmatrix} = D_1T + D_2 = 0. \quad (6.27)$$

Further, (6.20d) says that there exists a matrix Q such that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} T \\ I \end{pmatrix} Q. \quad (6.28)$$

This is clearly equivalent to

$$X_1 = TX_2. \quad (6.29)$$

Finally, (6.20e) is equivalent to the existence of matrices Q and R such that

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ 0 \end{pmatrix} Q + \begin{pmatrix} T \\ I \end{pmatrix} R. \quad (6.30)$$

We get $R = W_2$, and so (6.30) is the same as

$$W_1 = V_1Q + TW_2. \quad (6.31)$$

Summarizing, the conclusion is as follows.

Corollary 6.3: Using the notation (6.21), there exists a compensator for the system (5.1) achieving plant stability if and only if there exist matrices T , Q , and R such that

$$A_{11}T - TA_{22} - B_1R = -A_{12} \quad (6.32a)$$

$$TX_2 = X_1 \quad (6.32b)$$

$$D_1T = -D_2 \quad (6.32c)$$

$$TW_2 + V_1Q = W_1. \quad (6.32d)$$

Because the mapping $(T, Q, R) \mapsto (A_{11}T - TA_{22} - B_1R, TX_2, D_1T, TW_2 + V_1Q)$ is a linear mapping between linear spaces of finite (although, in general, large) dimension, the condition in the theorem can be read in the form "a given vector is in the range of a given linear mapping." Therefore, the condition is (at least in principle) constructive. To conclude this paragraph, let us briefly consider some special cases. If the pair (A, B) is stabilizable so that $X_{\text{stab}} = X$, "plant stability" means that $X_g(A^e) = O$; in this case, we shall use the term "complete stability." Of course, stabilizability of the pair (A, B) is also a necessary condition for a stable A^e to exist (Lemma 3.4). So we have the following.

Corollary 6.4: There exists a compensator for the system (5.1) achieving complete stability if and only if (A, B) is stabilizable, (C, A) is detectable, and there exists a (C, A, B) -pair (T, V) in between $\text{im } E$ and $\ker D$ such that

$$T \text{ is outer detectable} \quad (6.33a)$$

$$V \text{ is inner stabilizable.} \quad (6.33b)$$

This is (slightly restated) the result appearing in [12] and [13]. While this specialization emphasizes the "disturbance

localization" property, we can also completely remove this aspect by setting E equal to zero. Then the result is as follows.

Corollary 6.5: Suppose that $E = 0$ in the system (5.1). There exists a compensator for this system achieving plant stability if and only if there exists an (A, B) -invariant subspace $V \subset \ker D$ such that

$$V \oplus X_{\text{stab}} = X \quad (6.34a)$$

$$V \supset X_{\text{det}}. \quad (6.34b)$$

This corollary follows immediately from Corollary 6.2 because the condition (6.20e) is implied by (6.20d) if $\text{im } E = O$. If we assume that the pair (C, A) is detectable, our criterion of "plant stability" coincides with the criterion of "internal stability" [9]; and if we further assume that $X_g(A) \subset \langle A | \text{im } B \rangle$ ("the exogenous signals are completely unstable"), Corollary 6.5 specializes to [9, Theorem 8.1].

VII. AN EXAMPLE

Suppose we have the following system equations:

$$\begin{aligned} x_1'(t) &= -x_1(t) + 2x_2(t) - 4x_3(t) + u(t) \\ x_2'(t) &= -x_1(t) - x_2(t) + x_3(t) \\ x_3'(t) &= -x_2(t) - x_3(t) + q(t) \\ x_4'(t) &= 0 \\ z(t) &= x_1(t) - x_4(t) \\ y(t) &= (x_3(t)x_4(t))^T. \end{aligned} \quad (7.1)$$

The variable $x_4(t)$ represents a constant reference variable which $x_1(t)$ is required to track. The plant is described by the first three state variables, which are influenced by the control $u(t)$ (entering through x_1) and by an unknown disturbance function $q(t)$ (entering through x_3). One can check that the plant is open-loop unstable. Let us try to solve the regulator problem with plant stability for this system. That is, we want to add a compensator of the form (2.2) to the system (7.1), such that the variable $x_1(t)$ will track the constant reference value $x_4(t)$, irrespective of the behavior of $q(t)$, and such that the whole system (except for the reference variable) is stabilized so that a bounded input through $q(t)$ will make the system behave boundedly. One easily computes the following:

$$X_{\text{det}} = O \quad (7.2a)$$

$$V_g^*(\ker D) = \text{sp} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (7.2b)$$

$$T_g^*(\text{im } E) = \text{sp} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (7.2c)$$

So according to Corollary 6.3, the problem is solvable if and only if there exist a 3×1 -matrix T , a 1×1 -matrix R , and a 2×1 -matrix Q such that

$$\begin{pmatrix} -1 & 2 & -4 \\ -1 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (7.3a)$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (7.3b)$$

$$(1 \ 0 \ 0) \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 1 \quad (7.3c)$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} 0 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (7.3d)$$

It is easily verified that this set of linear equations has a unique solution, given by $t_1 = 1$, $t_2 = -\frac{1}{2}$, $t_3 = \frac{1}{2}$, $r = -4$, $q_1 = 0$, $q_2 = 1$.

We proceed to construct the compensator. Since we have found $T = (1 \ -\frac{1}{2} \ \frac{1}{2})^T$, we must use

$$V_b = sp \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}. \quad (7.4)$$

We apply Theorem 6.1 with $T = T_g^*(\text{im } E)$, $V_g = V_g^*(\ker D)$, and V_b as above. Following the "If" part of the proof of this theorem, we see that we first have to find a matrix K such that $(A + BKC)T \subset V$. Such a matrix can be constructed as in the proof of Lemma 4.2; for instance, we may use

$$K = (4 \ 0). \quad (7.5)$$

This gives

$$A + BKC = \begin{pmatrix} -1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.6)$$

For brevity, we shall denote this matrix by A_K in the following. Now we have to find $F \in F(V_g) \cap F(V_b)$ such that $A_K + BF: X_{\text{stab}}$ is stable. Such an F is given by

$$F = (0 \ -2 \ 0 \ 1) \quad (7.7)$$

where the second and third entry are determined by the requirement $F \in F(V_g)$, the first entry is selected such that $A_K + BF: X_{\text{stab}}$ is stable, and finally the fourth entry is determined by the requirement $F \in F(V_b)$. As our final step, we have to find $G \in G(T)$ such that $A_K + GC$ is stable. Such a G is given by, for instance,

$$G = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.8)$$

We can now construct the required compensator as in the proof of Theorem 4.1. The result is as follows:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} (t) &= \begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} (t) \\ &\quad - \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} (t) \end{aligned} \quad (7.9a)$$

$$u(t) = -2w_2(t) + w_4(t) + 4y_1(t). \quad (7.9b)$$

This completes the solution of the problem. The closed-loop system equations are given by

$$\begin{aligned} x_1'(t) &= -x_1(t) + 2x_2(t) - 2w_2(t) + w_4(t) \\ x_2'(t) &= -x_1(t) - x_2(t) + x_3(t) \\ x_3'(t) &= -x_2(t) - x_3(t) + q(t) \\ x_4'(t) &= 0 \\ w_1'(t) &= -w_1(t) + w_4(t) \\ w_2'(t) &= -w_1(t) - w_2(t) + x_3(t) \\ w_3'(t) &= -w_2(t) - w_3(t) \\ w_4'(t) &= -w_4(t) + x_4(t). \end{aligned} \quad (7.10)$$

One can now verify, for instance, that the tracking job is done as it should be by deriving the following equations from the above:

$$\begin{aligned} \frac{d}{dt} (x_1 - x_4)(t) &= -(x_1 - x_4)(t) + 2(x_2 - w_2)(t) \\ &\quad - (x_4 - w_4)(t) \\ \frac{d}{dt} (x_2 - w_2)(t) &= -(x_2 - w_2)(t) - (x_1 - w_1)(t) \\ \frac{d}{dt} (x_1 - w_1)(t) &= 2(x_2 - w_2)(t) - (x_1 - w_1)(t) \\ \frac{d}{dt} (x_4 - w_4)(t) &= -(x_4 - w_4)(t). \end{aligned} \quad (7.11)$$

One sees that $z(t)$ depends on the variables of a stable subsystem in which the disturbance function $q(t)$ does not appear.

VIII. CONCLUSIONS

We have shown that the concept of a (C, A, B) -pair of subspaces is the key that opens the door to a direct treatment of regulation problems, thereby avoiding the "state-feedback-plus-observer" approach which gives rise to considerable complications. With hardly any extra effort, we have been able to include the disturbance decoupling aspect in the regulation context. Generally speaking, (C, A, B) -pairs are central in the description of the candidate closed-loop structures associated with a given input-output system, and this is the basic reason for their importance in solving design problems.

The main emphasis of this paper has been on the clarification of the fundamental structure of the regulation problem. We have used the state-space approach as a convenient tool for doing this, although it may be true that a reformulation of the results in frequency domain terms will help to bridge the gap to robustness and numerical problems. Although the concepts of (A, B) - and (C, A) -invariant subspaces have been studied rather extensively now from the transfer matrix standpoint (see, for instance [16]–[18]), it should be noted here that a proper way to describe (C, A, B) -pairs in such terms does not seem to have been found yet.

A discussion of order reduction for regulating compensators in the style of this paper is given in [19]. In [20], a different formulation of the regulator problem is given, which leads to a general theorem from which the solutions to the problems with “output” and “plant” stability can be derived as special cases. The same work also gives a treatment of regulator problems for several classes of infinite-dimensional systems (parabolic, delayed).

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