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On the structure of strongly controllable systems[†]

J. M. SCHUMACHER[‡]

An important problem in the structure theory of linear systems is the construction of a canonical form for strongly controllable systems (i.e. surjective systems without finite zeros). The problem has been solved in principle (Morse 1973, Kronecker 1890), but without a clear identification of the system-theoretic meaning of the construction steps. This paper proposes the concept of a 'train basis' in order to give this interpretation. The second main idea of the paper is the 'system associated with a subspace', through which concept it is possible to translate problems concerning (A, B) -invariance etc. of subspaces in terms of input-output systems. In this way, we make contact with results of Willems (1980) on almost-invariance.

1. Introduction

The structure at infinity of a linear system is well known to play an important role in the study of invertibility and of asymptotic properties related to the root loci. As to the description of this structure in state space terms, a crucial step has been made by Morse (1973) in the study of invariants of linear systems under the action of a certain transformation group. Morse (1976, pp. 67-68) described a set of invariants consisting of four lists; one of these lists gives the zeros at infinity of the system transfer matrix. A more explicit proof was provided later (Commault and Dion 1982).

The transformation group used by Morse (1976) is essentially the same as the group that is used in defining equivalence for singular matrix pencils, and in this context a complete set of invariants has already been given by Kronecker, almost a century ago (Gantmacher 1957, Thorp 1973). However, stating and solving the problem in system-theoretic terms still remains a desirable goal. An interesting attempt to clarify the system-theoretic meaning as well as the technical content of Morse's (1973) paper has been made by Molinari (1978). Unfortunately, Molinari's method of proof contains considerable gaps at a crucial point (Commault and Dion 1981, Molinari 1981). The present paper represents another attempt to straighten things out. Our principal aim is to obtain a conceptual picture that displays the main issues involved. For an algorithmic and computation oriented approach, see Van der Weiden and Bosgra (1979).

It appears that the difficult part of the problem of obtaining a canonical form is in the 'irreducible' (terminology of Dion and Commault 1982) part of the given system, which represents the infinite zeros. This is, indeed, the point where Molinari's proof breaks down. So it would be reasonable to concentrate on this part of the system, i.e. to construct a canonical form for a system which is *a priori* assumed to be irreducible. It is then a relatively easy exercise

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(Morse 1973, Molinari 1978, Van der Weiden and Bosgra 1979) to fit this special procedure into a canonical basis construction for a general linear system. However, it turns out that it is technically more convenient to start with a somewhat weaker assumption, and so we shall consider strongly controllable systems, i.e. surjective systems without finite invariant zeros. Specifically, our purpose will be to derive a canonical form for the state space description of such systems when we allow the following transformations: state feedback, output injection, and change of basis in input space, output space and state space. The derivation will involve a minimum of matrix manipulation, and the key step consists of proving that, for every strongly controllable system, there exists a so called 'train basis' of the state space. Unlike canonical bases that are related exclusively to the state input pair of mappings (see, for instance, Kailath 1980, § 6.4), the train basis is defined in terms of all the system mappings.

It is known that there is a close connection between the results of Morse (1973) and those of Willems (1980, 1981, 1982) on almost invariant subspaces (see Commault and Dion 1981). More generally, we shall show in this paper that it is possible to obtain a direct connection between the study of subspaces (in relation to a given state input pair of mappings) and the study of systems (with inputs and outputs). This connection is established via the idea of a 'system associated with a subspace'. It turns out that almost controllability subspaces (Willems 1980) correspond to strongly controllable systems, whereas sliding subspaces (Willems 1981) are related to irreducible systems. Hence we obtain the geometric characterization of almost controllability subspaces (Willems 1980) using the train basis, and we are able to extract some more detailed information about sliding subspaces.

There is quite an extensive literature on the class of strongly controllable systems, or rather on the dual class of strongly observable systems. The problem of strong observability (which is: to reconstruct the state from the observations, when unknown inputs are present) seems to have been posed for the first time by Basile and Marro (1969 b). They gave a necessary and sufficient condition for the solvability of this problem in terms of the concept of controlled invariance, that had just been developed (Basile and Marro 1969 a). The fact that strong observability is equivalent to the transfer matrix being injective and having no finite zeros then follows from results of Morse and Wonham (1971), Morse (1973), Moore and Silverman (1974), Hosoe (1975) and Anderson (1975, 1976). Nikolskii (1970) proved that strong observability is equivalent to 'feedback irreducibility' (which means that no state feedback can cause unobservability), and this is easily seen to be equivalent to the solution given by Basile and Marro (1969 b). The concept of feedback irreducibility has turned up in several further studies of the structure of linear systems (for instance Heymann 1972, Morse 1976, Fuhrmann and Willems 1979). Strong observability was studied together with the dual concept of strong controllability by Molinari (1976). A module-theoretic approach was applied by Hammer and Heymann (1980) to strongly observable systems. Hautus (1979) gave a rank condition for strong observability. Dion and Commault (1982) define a system to be irreducible if it is both strongly controllable and strongly observable, and they connect this to the existence of a polynomial inverse for the transfer matrix of the system.

The organization of the present paper is as follows. In § 2, we define a 'train basis', and shows that a system is strongly controllable if and only if there is a train basis for it. Once this special basis is available, it is easy to obtain Morse's canonical form; this is shown in § 3. In § 4, we define the 'system associated with a subspace', and we translate the results on strongly controllable and irreducible systems into results on almost controllability subspaces and on sliding subspaces. The conclusions are briefly summarized in § 5.

2. The train basis

Throughout this paper, we shall consider a given finite dimensional, time-invariant, linear system Σ

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), & x(t) \in \mathcal{X}, & u(t) \in \mathcal{U} \\ y(t) &= Cx(t) + Du(t), & y(t) \in \mathcal{Y} \end{aligned} \right\} \quad (1)$$

We shall always assume that $\begin{pmatrix} B \\ D \end{pmatrix}$ is injective and $(C \ D)$ is surjective. Two sequences of subspaces are defined as follows

$$\left. \begin{aligned} \mathcal{T}^0 &= \{0\} \\ \mathcal{T}^{k+1} &= \{x \in \mathcal{X} \mid \exists u \in \mathcal{U}, w \in \mathcal{T}^k \\ &\quad Aw + Bu = x, \quad Cw + Du = 0\} \end{aligned} \right\} \quad (2)$$

such that

$$\left. \begin{aligned} \mathcal{V}^0 &= \mathcal{X} \\ \mathcal{V}^{k+1} &= \{x \in \mathcal{X} \mid \exists u \in \mathcal{U}, w \in \mathcal{V}^k \\ &\quad Ax + Bu = w, \quad Cx + Du = 0\} \end{aligned} \right\} \quad (3)$$

such that

It is easily shown that the sequence $\{\mathcal{T}^k\}$ is non-decreasing and that the sequence $\{\mathcal{V}^k\}$ is non-increasing. So both sequences have limits, which we shall denote by $\mathcal{T}^*$ and $\mathcal{V}^*$, respectively. System (1) is said to be strongly controllable if $\mathcal{T}^* = \mathcal{X}$; it is called strongly observable if $\mathcal{V}^* = \{0\}$.

To motivate this terminology, we note the following. Let an impulsive input with Laplace transform

$$\hat{u}(s) = \sum_{i=-k}^0 u_i s^{-i} \quad (4)$$

be applied to system (1) when it is in the initial state $x_{in} = x(0-)$. Then the state and output responses are given by

$$\hat{x}(s) = \sum_{i=-k+1}^{\infty} x_i s^{-i}, \quad \hat{y}(s) = \sum_{i=-k}^{\infty} y_i s^{-i} \quad (5)$$

where the vector sequences $\{x_i\}$ and $\{y_i\}$ are defined recursively by the equations

$$\left. \begin{aligned} x_{-k+1} &= Bu_{-k} & y_{-k} &= Du_{k-} \\ x_{-k+2} &= Ax_{-k+1} + Bu_{-k+1} & y_{-k+1} &= Cx_{-k+1} + Du_{-k+1} \\ &\vdots & &\vdots \\ x_0 &= Ax_{-1} + Bu_{-1} & y_{-1} &= Cx_{-1} + Du_{-1} \\ x_1 &= Ax_0 + Bu_0 + x_{in} & y_0 &= Cx_0 + Du_0 \\ &\vdots & &\vdots \end{aligned} \right\} \quad (6)$$

One verifies immediately that the condition of strong controllability, $\mathcal{F}^* = \mathcal{X}$, means exactly that there exists, for every initial condition $x(0-) = x_{in}$, an impulsive control (4) such that $x(0+) = x_1 = 0$, while $y_{-k} = y_{-k+1} = \dots = y_0 = 0$ (i.e. the output is a regular function). This result is taken from Hautus and Silverman (1983), and we refer to this work for a more precise derivation and connections with invertibility.

In the above, the 'impulse train' which takes the initial state to the origin in zero time is represented by the sequence $(x_{-k+1}, x_{-k+2}, \dots, x_0)$. We shall be particularly interested in sequences of this type; for short, we shall call them 'trains'. The formal definition runs as follows. Given a system Σ as specified by (1), a 'train' for this system is a mapping $\mathbf{x}$ from the non-positive integers $\mathbb{Z}_-$ into $\mathcal{X}$ that satisfies the following points:

- (i) there exists a non-negative integer $l(\mathbf{x})$ such that $\mathbf{x}(i) = 0$ for all $i \leq -l(\mathbf{x})$ and $\mathbf{x}(i) \neq 0$ for $i = -l(\mathbf{x}) + 1$;
- (ii) for all $i \leq -1$, there exists $\mathbf{u}(i)$ such that

$$\left. \begin{aligned} \mathbf{x}(i+1) &= A\mathbf{x}(i) + B\mathbf{u}(i) \\ 0 &= C\mathbf{x}(i) + D\mathbf{u}(i) \end{aligned} \right\} \quad (7)$$

The number $l(\mathbf{x})$ of condition (i) is called the 'length' of the train. If $\mathbf{x}(i) \neq 0$ for all $i \in \{-l(\mathbf{x}) + 1, \dots, 0\}$, then we say that the train is 'connected'; otherwise, the train is 'disconnected'.

The set of all trains for a given system has an obvious linear structure. We can also define a shift operator in the usual way

$$(\sigma\mathbf{x})(i) = \mathbf{x}(i-1), \quad i \in \mathbb{Z}_- \quad (8)$$

Note that the set of trains is closed under this operation. Non-negative powers of the shift will be used as well.

Given a set of trains $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$, we denote by $\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ the set of all linear combinations of elements of the form $\mathbf{x}_j(i)$, $1 \leq j \leq k$, $i \in \mathbb{Z}_-$. A set of trains is said to constitute a 'train basis' if its span is the state space $\mathcal{X}$, and if the sum of the lengths of the trains in the set equals $\dim \mathcal{X}$. Of course, the actual basis is then formed by the non-zero values assumed by the mappings $\mathbf{x}_j$. It is easily seen that if a system has a train basis, then it must be strongly controllable. The main result of this section is that the converse is also true.

Theorem 1

Suppose that the system Σ as specified by eqns. (1) is strongly controllable. Then there exists a train basis for Σ .

Before we proceed to a proof of this fact, it is convenient to have a number of manipulations available that can be applied to a set of trains without affecting its span. The following two lemmas are concerned with this.

Lemma 1

Let a set $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ of trains be given. Let $\tilde{\mathbf{x}}_1$ be of the form

$$\tilde{\mathbf{x}}_1 = \sum_{j=1}^k \sum_{r=0}^{M(j)} \alpha_{jr} \sigma^r \mathbf{x}_j, \quad \alpha_{10} \neq 0 \quad (9)$$

Then $\text{span}\{\tilde{\mathbf{x}}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} = \text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$

Proof

We have

$$\tilde{\mathbf{x}}_1(i) = \alpha_{10} \mathbf{x}_1(i) + \sum_{r=1}^{M(1)} \alpha_{1r} \mathbf{x}_1(i-r) + \sum_{j=2}^k \sum_{r=0}^{M(j)} \alpha_{jr} \sigma^r \mathbf{x}_j \quad (10)$$

So we can view the replacement of $\mathbf{x}_1$ by $\tilde{\mathbf{x}}_1$ as the result of a process, in which first $\mathbf{x}_1(0)$ has been replaced by $\tilde{\mathbf{x}}_1(0)$ as defined by (10), then $\mathbf{x}_1(-1)$ has been replaced, etc. In each of these steps, the subspace spanned by the totality of available vectors does not change. Consequently, at the end of the process this subspace must still be the same, which is what we wanted to prove. $\square$

Lemma 2

Let $\mathbf{x}$ be a train, and suppose that $\mathbf{x}(I) = 0$. Then $\hat{\mathbf{x}}$, defined by

$$\left. \begin{aligned} \hat{\mathbf{x}}(i) &= 0, & i \leq I \\ \hat{\mathbf{x}}(i) &= \mathbf{x}(i), & I < i \leq 0 \end{aligned} \right\} \quad (11)$$

is also a train. Moreover, we have

$$\text{span}\{\mathbf{x}\} = \text{span}\{\hat{\mathbf{x}}, \sigma^{1-I}\mathbf{x}\} \quad (12)$$

Proof

The proof of this lemma is immediate from the definitions.

The point of the second lemma is that one can replace a disconnected train by two (or more) connected trains, thereby reducing the sum of the lengths of the trains that one needs to span a certain subspace. This will be used in the proof of the theorem, which follows next.

Proof of Theorem 2.1

The proof is in two steps. In the first, we construct a set of trains whose span is $\mathcal{X}$. To do this, start with an arbitrary non-zero vector, and call it $\mathbf{x}_1(0)$. Let $k \in \mathbb{N}$ be such that $\mathcal{T}^k = \mathcal{T}^* = \mathcal{X}$. By the definition of the sequence

$\{\mathcal{T}^k\}$ (see eqn. (2)), there exist vectors $x_1(-1) \in \mathcal{T}^{k-1}$ and $u_1(-1) \in \mathcal{U}$ such that

$$\left. \begin{aligned} Ax_1(-1) + Bu_1(-1) &= x_1(0) \\ Cx_1(-1) + Du_1(-1) &= 0 \end{aligned} \right\} \quad (13)$$

Obviously, we can repeat the process, so that after a finite number of steps we obtain $x_1(j) \in \mathcal{T}^1 = \{Bu \mid Du = 0\}$. Then the prescription

$$\left. \begin{aligned} x_1(i) &= 0, & i < j \\ x_1(i) &= x_1(i), & j \leq i \leq 0 \end{aligned} \right\} \quad (14)$$

defines a train. If $\text{span}\{x_1\} = \mathcal{X}$, this concludes the first step of the proof. If not, we can find a vector $x_2(0) \in \mathcal{X}$ that is not contained in $\text{span}\{x_1\}$, and we can repeat the procedure to obtain a new train x_2 . If $\text{span}\{x_1, x_2\}$ is still not equal to $\mathcal{X}$, we pick a third independent vector $x_3(0)$, and so on. After a finite number of repetitions of the procedure, we must obtain a situation in which we have a set of trains $\{x_1, \dots, x_k\}$ that spans $\mathcal{X}$.

Of course, the sum of the lengths of these trains may very well exceed the dimension of $\mathcal{X}$, and in that case we do not have a train basis. The second step of the proof consists of a reduction procedure which will remedy this situation, without destroying the train structure.

First of all, note that we may assume, by virtue of Lemma 2, that all trains in the set $\{x_1, \dots, x_k\}$ are connected. Write

$$r(j) = 1 - l(x_j) \quad (15)$$

Then the set $\{x_j(i) \mid 1 \leq j \leq k, r(j) \leq i \leq 0\}$ contains exactly all non-zero elements from the given set of trains. If these elements are independent, then we already have a train basis, and the work is finished. If they are not, then there exist scalars $\alpha(j; i)$ ($1 \leq j \leq k, r(j) \leq i \leq 0$), not all equal to zero, such that

$$\sum_{j=1}^k \sum_{i=r(j)}^0 \alpha(j; i) x_j(i) = 0 \quad (16)$$

Define

$$I = \max \{i \mid \alpha(j; i) \neq 0 \text{ for some } j\} \quad (17)$$

and let J be such that $\alpha(J; I) \neq 0$. Next, define

$$\tilde{x}_J = \sum_{j=1}^k \sum_{i=r(j)}^I \alpha(j; i) \sigma^{I-i} x_j \quad (18)$$

It is clear from eqn. (16) that $\tilde{x}_J(I) = 0$. So we can construct a new train $\hat{x}_J$ from $\tilde{x}_J$, as in Lemma 2. Now, we can write down the following sequence of equalities

$$\left. \begin{aligned} \mathcal{X} &= \text{span}\{x_1, \dots, x_J, \dots, x_k\} \\ &= \text{span}\{x_1, \dots, \tilde{x}_J, \dots, x_k\} \\ &= \text{span}\{x_1, \dots, \hat{x}_J, \sigma^{1-I} \tilde{x}_J, \dots, x_k\} \\ &= \text{span}\{x_1, \dots, \hat{x}_J, \sigma^{1-I} x_J, \dots, x_k\} \end{aligned} \right\} \quad (19)$$

We have used Lemmas 1 and 2 in succession and the fact that $\sigma^{1-l}\tilde{\mathbf{x}}_j$ can be written as a linear combination of (shifted versions of) the trains $\sigma^{1-l}\mathbf{x}_j$ ($1 \leq j \leq k$), which is an obvious consequence of the fact that $\tilde{\mathbf{x}}_j$ can be written as a linear combination of (shifted versions of) the trains $\mathbf{x}_j$. In effect, we have replaced $\mathbf{x}_j$ by the two trains $\tilde{\mathbf{x}}_j$ and $\sigma^{1-l}\mathbf{x}_j$, and we have obtained a new set of trains which still spans $\mathcal{X}$, whereas the sum of the lengths of the trains has been reduced by one.

If the new set of trains happens to be a train basis, then we are done. In the other instance, the above procedure can be repeated so that the sum of the lengths of the trains is reduced again. It is clear that after a finite number of steps, this process must come to an end, and the desired train basis will have been obtained. $\square$

The reduction procedure that was employed in the proof above was chosen because it is relatively simple to describe, but it is by no means the only possible procedure. We shall come back to this point in the next section. This section will now be concluded with a few simple properties of train bases.

Proposition 1

Let the system Σ of eqns. (1) be strongly controllable, and denote by Σ_F the system

$$\left. \begin{aligned} \dot{x}(t) &= (A + BF)x(t) + Bu(t) \\ y(t) &= (C + DF)x(t) + Du(t) \end{aligned} \right\} \quad (20)$$

where F is a linear mapping from $\mathcal{X}$ into $\mathcal{U}$. Suppose that $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a train basis for Σ . Then $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is also a train basis for Σ_F .

Proof

We know that, for every $j \in \{1, \dots, k\}$ and $i \in \mathbb{Z}_-$, there exists an $\mathbf{u}_j(i)$ such that

$$\left. \begin{aligned} \mathbf{x}_j(i+1) &= A\mathbf{x}_j(i) + B\mathbf{u}_j(i) \\ 0 &= C\mathbf{x}_j(i) + D\mathbf{u}_j(i) \end{aligned} \right\} \quad (21)$$

and we have to prove that, for all these j and i , there exists an $\tilde{\mathbf{u}}_j(i)$ such that

$$\left. \begin{aligned} \mathbf{x}_j(i+1) &= (A + BF)\mathbf{x}_j(i) + B\tilde{\mathbf{u}}_j(i) \\ 0 &= (C + DF)\mathbf{x}_j(i) + D\tilde{\mathbf{u}}_j(i) \end{aligned} \right\} \quad (22)$$

Obviously, it suffices to take $\tilde{\mathbf{u}}_j(i) = \mathbf{u}_j(i) - F\mathbf{x}_j(i)$. $\square$

Proposition 2

Let the system Σ of eqn. (1) be strongly controllable, and denote by Σ^G the system

$$\left. \begin{aligned} \dot{x}(t) &= (A + GC)x(t) + (B + GD)u(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \right\} \quad (23)$$

where G is a linear mapping from $\mathcal{Y}$ into $\mathcal{X}$. Suppose that $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a train basis for Σ . Then $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is also a train basis for Σ^G .

Proof

We know that, for every $j \in \{1, \dots, k\}$ and $i \in \mathbb{Z}_-$, there exists an $\mathbf{u}_j(i)$ such that

$$\left. \begin{aligned} \mathbf{x}_j(i+1) &= A\mathbf{x}_j(i) + B\mathbf{u}_j(i) \\ 0 &= C\mathbf{x}_j(i) + D\mathbf{u}_j(i) \end{aligned} \right\} \quad (24)$$

But then we also have

$$\left. \begin{aligned} \mathbf{x}_j(i+1) &= (A + GC)\mathbf{x}_j(i) + (B + GD)\mathbf{u}_j(i) \\ 0 &= C\mathbf{x}_j(i) + D\mathbf{u}_j(i) \end{aligned} \right\} \quad (25)$$

which shows that $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is also a train basis for Σ^G . $\square$

Lemma 3

Let the system Σ of eqn. (1) be strongly controllable and let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a train basis for Σ , with $\mathbf{x}_j \neq \mathbf{0}$ for all j . Write $m = \dim \mathcal{U} = \text{rank} \begin{pmatrix} B \\ D \end{pmatrix}$, $p = \dim \mathcal{Y} = \text{rank} \begin{pmatrix} C & D \end{pmatrix}$, and $r = \text{rank } D$. Then the following inequalities hold

$$p \leq k + r \leq m \quad (26)$$

Proof

By the definition of a train basis, the elements $\mathbf{x}_j(-l(\mathbf{x}_j) + 1)$ ($j = 1, \dots, k$) are independent and belong to the subspace $\mathcal{T}^1 = \{Bu \mid Du = 0\}$.

Therefore, we have

$$k \leq \dim \mathcal{T}^1 = m - r \quad (27)$$

Also, we know that all elements $\mathbf{x}_j(i)$ with $i \leq -1$ belong to the subspace $\mathcal{V}^1 = \{x \mid \exists u \text{ s.t. } Cx + Du = 0\}$. It follows that $\mathcal{X}$ can be written as the sum of $\mathcal{V}^1$ and the subspace spanned by the independent elements $\mathbf{x}_j(0)$ ($j = 1, \dots, k$), so that

$$k \geq \text{codim } \mathcal{V}^1 = p - r \quad (28)$$

This completes the proof. $\square$

3. The Morse canonical form for irreducible systems

A system Σ of the form (1) is said to be irreducible (Dion and Commault 1982) if it is both strongly controllable and strongly observable. The terminology stems from the fact that the McMillan degree of the transfer function of such a system cannot be reduced by applying a transformation $\Sigma \rightarrow \Sigma_{\mathcal{P}}^G$ as defined in Propositions 1 and 2. For a system that is already given to be strongly controllable, we have the following characterizations of irreducibility. Recall that a system Σ is said to be (left/right) invertible if its transfer function is (left/right) invertible as a matrix over the field of rational functions.

Lemma 4

Let the system Σ of (1) be an m -input- p -output strongly controllable system. Then the following are equivalent :

- (i) Σ is irreducible ;
- (ii) Σ is left invertible ;
- (iii) Σ is invertible ; and
- (iv) $m=p$.

Proof

It is well known that a system Σ is left invertible if and only if $\mathcal{V}^* \cap \mathcal{T}^* = \{0\}$, and that it is right invertible if and only if $\mathcal{V}^* + \mathcal{T}^* = \mathcal{X}$ (Morse and Wonham 1971, Morse 1973, Hautus and Silverman 1983). So if $\mathcal{T}^* = \mathcal{X}$, Σ is left invertible if and only if $\mathcal{V}^* = \{0\}$, i.e. Σ is irreducible. Since a strongly controllable system is always right invertible, it is clear that (ii) is, for such a system, equivalent to (iii). If Σ is right invertible, then its transfer matrix must obviously be square, so that $m=p$. On the other hand, a square right invertible matrix must be invertible *tout court*, so that the implication from (iv) to (iii) holds as well.

Corollary 1

The number of trains in any train basis for an irreducible system is equal to $p-r$, where p is the number of outputs and r is the rank of the transfer matrix at infinity.

Proof

The proof of this corollary follows immediately from Lemmas 3 and 4.

By change of basis in the state space and in the input and output spaces, and by transformations of the form $\Sigma \rightarrow \Sigma_F^G$, one can bring the system matrices of an irreducible system into a canonical form (Morse 1975). We now proceed to derive this form by employing the concept of the train basis. $\square$

Theorem 2

Let the system Σ , as given by eqn. (2), be irreducible. Let $\{\mathbf{x}_1, \dots, \mathbf{x}_{p-r}\}$ be any train basis for Σ , and let l_i be the length of $\mathbf{x}_i$. Then there exist linear mappings $F: \mathcal{X} \rightarrow \mathcal{U}$ and $G: \mathcal{Y} \rightarrow \mathcal{X}$, and bases in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$, such that the system matrices of the transformed system Σ_F^G have the following form, with respect to the selected bases

$$A + BF + GC + GDF = \text{block diag } (\tilde{A}_i), \quad i = 1, \dots, p-r \quad (29)$$

$$\tilde{A}_i = \begin{pmatrix} 0 & & & & 1 \\ & \ddots & & & \\ & & \bigcirc & & \\ & & & \ddots & \\ & & & & 1 \\ & & \bigcirc & & & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \in \mathbb{R}^{l_i \times l_i}$$

$$B + GD = (\tilde{B} \quad 0_{n \times r}), \quad \tilde{B} = \text{block diag } (\tilde{B}_i), \quad i = 1, \dots, p-r \quad (30)$$

$$\tilde{B}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{l_i \times 1}$$

$$C + DF = \begin{pmatrix} \tilde{C} \\ 0_{r \times n} \end{pmatrix}, \quad \tilde{C} = \text{block diag } (\tilde{C}_i), \quad i = 1, \dots, p-r \quad (31)$$

$$\tilde{C}_i = (1 \ 0 \ \dots \ 0) \in \mathbb{R}^{1 \times l_i}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} \quad (32)$$

Proof

First, we show that it is sufficient to solve the problem for $D=0$ (which is actually the case that was considered by Morse). Decompose $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ with $\mathcal{U}_1 = \ker D$ and $\mathcal{Y}_2 = \text{im } D$. With respect to this decomposition, we can write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & D_{22} \end{pmatrix} \quad (33)$$

where D_{22} is invertible. After application of

$$F_0 = \begin{pmatrix} F_1 \\ -D_{22}^{-1} C_2 \end{pmatrix}, \quad G_0 = (G_1 \ -B_2 D_{22}^{-1}) \quad (34)$$

the corresponding matrix for the transformed system is

$$\begin{pmatrix} A_1 + B_1 F_1 + G_1 C_1 & B_1 & 0 \\ C_1 & 0 & 0 \\ 0 & 0 & D_{22} \end{pmatrix} \quad (35)$$

where we have written

$$A_1 = A - B_2 D_{22}^{-1} C_2 \quad (36)$$

Since the matrix of D_{22} can, of course, be made equal to the identity matrix by a suitable selection of bases in $\mathcal{U}_2$ and $\mathcal{Y}_2$, it remains to choose suitable bases in $\mathcal{U}_1$, $\mathcal{Y}_1$ and $\mathcal{X}$, and suitable mappings F_1 and G_1 , such that the canonical form is obtained. It is easily verified that the system $\hat{\Sigma}$ specified by the four-tuple $(A_1, B_1, C_1, 0)$ is irreducible (in fact, one has $\mathcal{T}^i(\Sigma) = \mathcal{T}^i(\hat{\Sigma})$ and $\mathcal{V}^i(\Sigma) = \mathcal{V}^i(\hat{\Sigma})$ for all i), and so we may just as well assume that $D=0$ in the original system Σ .

We start by selecting a basis for $\mathcal{X}$. Not surprisingly, we pick the non-zero elements of the given train basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ (note : we have $r=0$, now) which we number as follows, writing $l_j = l(\mathbf{x}_j)$

$$\left. \begin{aligned} x_1 &= \mathbf{x}_1(0), \quad x_2 = \mathbf{x}_1(-1), \dots, x_{l_1} = \mathbf{x}_1(-l_1+1) \\ x_{l_1+1} &= \mathbf{x}_2(0), \dots, x_n = \mathbf{x}_p(-l_p+1) \end{aligned} \right\} \quad (37)$$

Because of the irreducibility, the elements $\{\mathbf{x}_j(-l_j+1) | j=1, \dots, p\}$ form a basis for $\text{im } B$ (cf. the proof of Lemma 3). Likewise, the elements $\{\mathbf{x}_j(0) | j=1, \dots, p\}$ span a direct complement of $\ker C$. So we can form bases of $\mathcal{U}$ and $\mathcal{Y}$ by the following prescriptions

$$Bu_j = \mathbf{x}_j(-l_j+1), \quad j=1, \dots, p \quad (38)$$

$$y_j = C\mathbf{x}_j(0), \quad j=1, \dots, p \quad (39)$$

Moreover, because of the train basis structure, we can find $F : \mathcal{X} \rightarrow \mathcal{X}$ such that

$$(A + BF)\mathbf{x}_j(i) = \mathbf{x}_j(i+1), \quad j=1, \dots, p; \quad i = -l_j+1, \dots, -1 \quad (40)$$

And finally, we can define $G : \mathcal{Y} \rightarrow \mathcal{X}$ by

$$Gy_j = -(A + BF)\mathbf{x}_j(0), \quad j=1, \dots, p \quad (41)$$

With these definitions, it is easily verified that the canonical form is obtained. $\square$

One sees that the existence of the canonical form can easily be derived from the existence of a train basis. Other canonical forms can be obtained by numbering the train basis elements in a different way than was done in eqn. (37). There are several corollaries that now follow immediately.

Corollary 2

Let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a train basis for an irreducible system Σ . Then we have, for all $i=0, 1, 2, \dots$

$$\mathcal{T}^i(\Sigma) = \text{span} \{\mathbf{x}_j(l) | j=1, \dots, k, \quad l \leq -l(\mathbf{x}_j) + i\} \quad (42)$$

$$\mathcal{V}^i(\Sigma) = \text{span} \{\mathbf{x}_j(l) | j=1, \dots, k, \quad l \leq -i\} \quad (43)$$

Proof

If we change Σ to $\hat{\Sigma}$ and then $\hat{\Sigma}$ to $\hat{\Sigma}_F^G$ as in the proof of Theorem 2, we will still have $\mathcal{T}^i(\hat{\Sigma}_F^G) = \mathcal{T}^i(\Sigma)$ and $\mathcal{V}^i(\hat{\Sigma}_F^G) = \mathcal{V}^i(\Sigma)$. So we may just as well assume that $D=0$ and that the matrices A , B and C are in the canonical form (with $r=0$) with respect to the given train basis. The relations (42) and (43) then follow by inspection.

Corollary 3

Let Σ be an irreducible system. Write

$$\tau_i = \dim(\mathcal{T}^i(\Sigma)), \quad \nu_i = \dim(\mathcal{V}^i(\Sigma)) \quad (44)$$

Let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be any train basis for Σ , and write

$$\eta_i = \{j | l(\mathbf{x}_j) \geq i\} \quad (45)$$

Then the following relations between the numbers τ_i , ν_i and η_i hold for all $i = 0, 1, 2, \dots$

$$\tau_i + \nu_i = n \quad (46)$$

$$\eta_i = \tau_{i+1} - \tau_i = \nu_{i+1} - \nu_i \quad (47)$$

Proof

The proof of this corollary follows from counting, using eqns. (42) and (43).

Corollary 4

Let Σ be an irreducible system. The lengths of the trains are the same (up to permutation) for any train basis for Σ , and they coincide with the orders of the zeros at infinity of the transfer matrix of Σ . Moreover, they form a complete set of invariants on the set of four-tuples (A, B, C, D) representing irreducible systems with p inputs and outputs and with rank r at infinity, under the action of the following group

$$(A, B, C, D) \rightarrow (T(A + BF + GC + GDF)T^{-1}, \quad T(B + GD)S, \\ R(C + DF)T^{-1}, \quad RDS) \quad (48)$$

where

$$T \in GL(n, \mathbb{R}), \quad R \in GL(p, \mathbb{R}), \quad S \in GL(p, \mathbb{R}), \quad F \in \mathbb{R}^{p \times n}, \quad G \in \mathbb{R}^{n \times p}$$

Proof

The first statement is immediate from Corollary 3. The relation with the infinite zeros is given by Commault and Dion (1982) (cf. Morse 1973, 1976). Finally, the fact that the train lengths form a complete set of invariants is clear from Corollary 3 and Theorem 2.

Remark

Irreducible systems are strongly controllable systems for which $\mathcal{V}^* = \{0\}$. As a contrast, we may also consider strongly controllable systems for which $\mathcal{V}^* = \mathcal{X}$. For the purpose of constructing a train basis, we may replace such a system by a strictly proper version, by the construction that was used in the first step of the proof of the Theorem 2. Indeed, it is readily verified that a set of functions $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ from $\mathbb{Z}_-$ to $\mathcal{X}$ is a train basis for the transformed system if and only if it is a train basis for the original system. For the strictly proper version, we have $D = 0$; but then it follows from $\mathcal{V}^* = \mathcal{X}$ that $C = 0$ also. Under these circumstances, the concept of strong controllability coincides with that of ordinary controllability, and a train is just a function $\mathbf{x} : \mathbb{Z}_- \rightarrow \mathcal{X}$ which satisfies $\mathbf{x}(i+1) - A\mathbf{x}(i) \in \text{im } B$ for all $i < 0$. It can be seen from simple examples that the number of trains in a train basis is not fixed; in fact, it can be anything from one to m , the number of inputs. However, a canonical form can still be obtained by adding the requirements that the number of trains should be m and that we should have $A\mathbf{x}_j(0) \in \text{im } B$ for all j . Indeed, in this way one exactly obtains the Brunovsky canonical form for a controllable pair (A, B) (see for instance Wonham 1979, pp. 116–122).

For the general situation in which $\mathcal{V}^*$ is not necessarily equal to either $\{0\}$ or $\mathcal{X}$, we can use the above remarks to give an alternative version of the reduction procedure that was used in the proof of Theorem 1. In this version, one first selects a train basis for $\mathcal{V}^*$. This can be done by the procedure of the proof of Theorem 1, or by one of the procedures that are available to obtain the Brunovsky canonical form. Next, one forms additional trains just as in the proof of Theorem 1, and one starts the reduction process. However, instead of replacing $\mathbf{x}_J$ by the two trains $\hat{\mathbf{x}}_J$ and $\sigma^{1-I}\mathbf{x}_J$ (where we follow the notation of the proof of Theorem 1; note that $\mathbf{x}_J$ can always be chosen to be one of the additionally selected trains), it suffices to replace $\mathbf{x}_J$ just by $\hat{\mathbf{x}}_J$. The reason for this is that, as is immediate from the definitions, it follows from $Ax + Bu \in \mathcal{V}^*$ and $Cx + Du = 0$ that $x \in \mathcal{V}^*$, so that the subspace spanned by the train $\sigma^{1-I}\hat{\mathbf{x}}_J$ is contained in $\mathcal{V}^*$, since $\hat{x}_J(I) = 0$. As a consequence, the total length of the trains may be reduced by more than one in each of the steps of the reduction process. So in general, it may be expected that the modified procedure will lead to the desired train basis in fewer steps. The price one pays, of course, is that $\mathcal{V}^*$ has to be computed first.

4. Systems and subspaces

The purpose of this section is to show how results on the structure of systems can be made applicable to the study of various classes of subspaces, that are defined with respect to a given state input pair (A, B) . The connection is made via the idea of a 'system associated with a subspace', which we shall now describe.

Consider the following controlled state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U} \quad (49)$$

Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then the system associated with $\mathcal{K}$ (and with the pair (A, B)), which we shall denote by $\Sigma_{\mathcal{K}}$, is the system specified by the spaces $\tilde{\mathcal{X}}, \tilde{\mathcal{U}}, \tilde{\mathcal{Y}}$ and the mappings $\tilde{A} : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}, \tilde{B} : \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{X}}, \tilde{C} : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}, \tilde{D} : \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{Y}}$, which are defined in the following way. Set

$$\tilde{\mathcal{X}} = \mathcal{K} + A\mathcal{K} + \text{im } B \quad (50)$$

and let $\tilde{A}$ be any mapping such that $Ax = \tilde{A}x$ for $x \in \mathcal{K}$. Take $\tilde{\mathcal{U}} = \mathcal{U}$, and let $\tilde{B}$ be equal to the mapping B , when understood as a mapping into $\tilde{\mathcal{X}}$. Select an output space $\tilde{\mathcal{Y}}$ and an output mapping $\tilde{C}$ in such a way that $\tilde{C}$ is surjective and $\ker \tilde{C} = \mathcal{K}$. Finally, set $\tilde{D} = 0$.

In matrix terms, the construction can be described as follows. When the matrices of A and B are written down with respect to a basis that is adapted to the chain $\{0\} \subset \mathcal{K} \subset \mathcal{K} + A\mathcal{K} + \text{im } B \subset \mathcal{X}$, we obtain

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix} \quad (51)$$

Then we can take

$$\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \tilde{C} = (0 \quad I) \quad (52)$$

From the construction, it is clear that the properties of the system $\Sigma_{\mathcal{X}}$ will reflect the 'holdability' characteristics of the subspace $\mathcal{K}$ with respect to the trajectories defined by eqn. (49). To quote an extreme case, it is easy to see that every trajectory with $x(0)=0$ will satisfy $x(t) \in \mathcal{K}$ for all $t \geq 0$ if and only if the transfer function of $\Sigma_{\mathcal{X}}$ is zero. But the next result will show that one can capture also much finer properties through the associated system. For definitions of the various classes of subspaces we refer to Wonham (1979), Willems (1980, 1981), Schumacher (1982).

Theorem 3

Consider eqn. (49), and let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Let $\Sigma_{\mathcal{X}}$ be the system associated with $\mathcal{K}$, as described above. In each of the following five cases, the statements under (i) and (ii) are equivalent.

- (A) (i) $\mathcal{K}$ is almost (A, B) -invariant.
(ii) $\Sigma_{\mathcal{X}}$ is right invertible.
- (B) (i) $\mathcal{K}$ is an almost controllability subspace.
(ii) $\Sigma_{\mathcal{X}}$ is strongly controllable,
- (C) (i) $\mathcal{K}$ is (A, B) -invariant.
(ii) $\tilde{C}\tilde{B}$ is surjective.
- (D) (i) $\mathcal{K}$ is a sliding subspace.
(ii) $\Sigma_{\mathcal{X}}$ is irreducible.
- (E) (i) $\mathcal{K}$ is a coasting subspace.
(ii) $\tilde{C}\tilde{B}$ is invertible.

Proof

Consider the following algorithms (Wonham 1979, Willems 1981, Schumacher 1982)

$$\left. \begin{aligned} \mathcal{V}^1(\mathcal{K}) &= \mathcal{K} \\ \mathcal{V}^{j+1}(\mathcal{K}) &= \mathcal{K} \cap A^{-1}(\mathcal{V}^j(\mathcal{K}) + \text{im } B) \end{aligned} \right\} \quad (53)$$

$$\left. \begin{aligned} \mathcal{R}_b^1(\mathcal{K}) &= \text{im } B \\ \mathcal{R}_b^{j+1}(\mathcal{K}) &= \text{im } B + A(\mathcal{R}_b^j(\mathcal{K}) \cap \mathcal{K}) \end{aligned} \right\} \quad (54)$$

It is easily verified that $\mathcal{V}^j(\mathcal{K}) = \mathcal{V}^j(\Sigma_{\mathcal{X}})$ and $\mathcal{R}_b^j(\mathcal{K}) = \mathcal{T}^j(\Sigma_{\mathcal{X}})$ for all $j \in \mathbb{N}$, so that, in particular, the limits are equal

$$\mathcal{V}^*(\mathcal{K}) = \mathcal{V}^*(\Sigma_{\mathcal{X}}), \quad \mathcal{R}_b^*(\mathcal{K}) = \mathcal{T}^*(\Sigma_{\mathcal{X}}) \quad (55)$$

Case (A)

First note that we always have $\mathcal{V}^*(\mathcal{K}) \subset \mathcal{K}$, and, moreover

$$\mathcal{R}_b^*(\mathcal{K}) = \text{im } B + A(\mathcal{R}_b^*(\mathcal{K}) \cap \mathcal{K}) \quad (56)$$

As a consequence, the inclusion

$$\mathcal{V}^*(\mathcal{K}) + \mathcal{R}_b^*(\mathcal{K}) \subset \mathcal{K} + A\mathcal{K} + \text{im } B \quad (57)$$

holds for all subspaces $\mathcal{K}$. Almost (A, B) -invariance of $\mathcal{K}$ is characterized by (Willems 1980, 1981, 1982)

$$\mathcal{K} = \mathcal{V}^*(\mathcal{K}) + (\mathcal{R}_b^*(\mathcal{K}) \cap \mathcal{K}) \quad (58)$$

If eqn. (58) holds, then obviously $\mathcal{K} \subset \mathcal{V}^*(\mathcal{K}) + \mathcal{R}_b^*(\mathcal{K})$. Furthermore

$$\left. \begin{aligned} A\mathcal{K} &= A(\mathcal{V}^*(\mathcal{K})) + A(\mathcal{R}_b^*(\mathcal{K}) \cap \mathcal{K}) \\ &\subset \mathcal{V}^*(\mathcal{K}) + \text{im } B + A(\mathcal{R}_b^*(\mathcal{K}) \cap \mathcal{K}) \\ &= \mathcal{V}^*(\mathcal{K}) + \mathcal{R}_b^*(\mathcal{K}) \end{aligned} \right\} \quad (59)$$

Since we always have $\text{im } B \subset \mathcal{R}_b^*(\mathcal{K})$, eqn. (58) is seen to imply

$$\mathcal{V}^*(\mathcal{K}) + \mathcal{R}_b^*(\mathcal{K}) = \mathcal{K} + A\mathcal{K} + \text{im } B \quad (60)$$

In terms of $\Sigma_{\mathcal{X}}$, this says that

$$\mathcal{V}^*(\Sigma_{\mathcal{X}}) + \mathcal{T}^*(\Sigma_{\mathcal{X}}) = \tilde{\mathcal{X}} \quad (61)$$

which is equivalent to saying that $\Sigma_{\mathcal{X}}$ is right invertible (Morse and Wonham 1971, Hautus and Silverman 1983).

Conversely, suppose that $\Sigma_{\mathcal{X}}$ is right invertible, so that eqn. (61) and hence (60) holds. Intersecting both sides of this equation with $\mathcal{K}$, we obtain (58) immediately.

Case (B)

The subspace $\mathcal{K}$ is an almost controllability subspace if and only if

$$\mathcal{K} \subset \mathcal{R}_b^*(\mathcal{K}) \quad (62)$$

(Willems 1980, 1981, 1982). If (62) holds, then

$$\mathcal{R}_b^*(\mathcal{K}) = \text{im } B + A(\mathcal{R}_b^*(\mathcal{K}) \cap \mathcal{K}) = \text{im } B + A\mathcal{K} \quad (63)$$

Adding $\mathcal{K}$ to both sides of this equation, we get

$$\mathcal{R}_b^*(\mathcal{K}) = \mathcal{K} + A\mathcal{K} + \text{im } B \quad (64)$$

By eqns. (50) and (55), this means that $\Sigma_{\mathcal{X}}$ is strongly controllable. Conversely, if $\Sigma_{\mathcal{X}}$ is strongly controllable, then (64) holds, and (62) follows immediately.

Case (C)

The subspace $\mathcal{K}$ is (A, B) -invariant if and only if

$$A\mathcal{K} \subset \mathcal{K} + \text{im } B \quad (65)$$

(Wonham 1979). This is the same as

$$\mathcal{K} + \text{im } B = \mathcal{K} + A\mathcal{K} + \text{im } B \quad (66)$$

In terms of Σ_x , this says that $\ker \tilde{C} + \text{im } \tilde{B} = \tilde{\mathcal{X}}$, which is equivalent to the statement that $\tilde{C}\tilde{B}$ is surjective.

Case (D)

A sliding subspace is an almost controllability subspace $\mathcal{K}$ for which $\mathcal{V}^*(\mathcal{K}) = \{0\}$ (Willems 1980). Hence, the characterization (ii) is clear from Case (B) and eqn. (50).

Case (E)

A coasting subspace is an (A, B) -invariant subspace $\mathcal{K}$ for which $\mathcal{K} \cap \mathcal{R}_b^*(\mathcal{K}) = \{0\}$ (Willems 1980). Since it is always true that $\mathcal{R}_b^*(\mathcal{K}) \subset \text{im } B$, it follows from $\mathcal{K}$ being coasting that $\tilde{\mathcal{X}} = \text{im } \tilde{B} \oplus \ker \tilde{C}$ (direct sum), which means that $\tilde{C}\tilde{B}$ is invertible. Conversely, if this condition holds, then it is easily verified that $\mathcal{R}_b^*(\mathcal{K})$ is in fact equal to $\text{im } B$, so that $\mathcal{R}_b^*(\mathcal{K}) \cap \mathcal{K} = \{0\}$. At the same time, the fact that $\mathcal{K}$ is (A, B) -invariant is guaranteed by Case (C). So the proof is complete. $\square$

Remark 1

The class of controllability subspaces (Wonham 1979) is simply the intersection of the classes of almost controllability subspaces and of (A, B) -invariant subspaces (Jaffe and Karcanias 1981), so that this category can also be described in terms of the associated system.

Remark 2

It follows from the above that a subspace is almost (A, B) -invariant if and only if the transfer function of the associated system is right invertible (cf. Morse and Wonham 1971, Hautus and Silverman 1983). Condition (ii) in Case (C) is easily seen to be equivalent to the existence of a right inverse of the form $R(s) = R_1s + R_0 + R_{-1}s^{-1} + \dots$ for the transfer function of Σ_x . Condition (ii) of Case (E) is equivalent to the existence of a (left and right) inverse of that same form. Conditions in (B) and (D) are not so easily expressed in terms of the existence of a (right) inverse of a certain type, although we do have the following result.

Proposition 3

Let $\Sigma(A, B, C, D)$ be a minimal realization of the transfer function $W(s)$. Then Σ is strongly controllable if and only if $W(s)$ has a polynomial right inverse.

Proof

The transfer function $W(s)$ has a polynomial right inverse if and only if its Smith–McMillan form appears as follows

$$U(s)W(s)V(s) = \begin{pmatrix} 1 & & & \\ \Psi_1(s) & 0 & & \\ & \ddots & & 0 \\ 0 & & 1 & \\ & & \Psi_p(s) & \end{pmatrix} \quad (67)$$

where $U(s)$ and $V(s)$ are unimodular, and the $\Psi_i(s)$ are polynomials. The condition that $W(s)$ is right invertible is equivalent to the condition $\mathcal{V}^* + \mathcal{T}^* = \mathcal{X}$; the condition that the numerator polynomials in the Smith-McMillan form are all equal to unity is, under the minimality assumption, equivalent to the condition $\mathcal{V}^* \subset \mathcal{T}^*$ (cf. Anderson (1975), who gives a proof in the case $D \neq 0$). Both conditions together are therefore equivalent to $\mathcal{T}^* = \mathcal{X}$, which is the definition of strong controllability.

The minimality assumption is essential in the above proposition. It is quite clear that if a realization of a transfer function with a polynomial right inverse is not controllable, then it will certainly not be strongly controllable. But even non-observability can spoil the relation, as is shown by the following example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = (1 \ 0 \ 0), \quad D = 0 \quad (68)$$

This system is controllable, and its transfer function is $(1/(s-1) \ 0)$, which has a polynomial inverse. Still, one easily verifies that $\mathcal{T}^* = \text{im } B \neq \mathcal{X}$. The reason is that eqn. (68) is not observable. (At this point, we note that the result for irreducible systems that corresponds to Proposition 3 is stated incorrectly by Dion and Commault (1982), since the minimality assumption is omitted.) Thus, the fact that the transfer function of $\Sigma_{\mathcal{X}}$ has a polynomial right inverse does not necessarily imply that $\mathcal{X}$ is an almost controllability subspace. The reverse implication, however, is correct; this follows from a result of Bengtsson (1974, Theorem 1).

From the theorem above, it transpires that our discussion of strongly controllable systems will apply to almost controllability subspaces, whereas the specific remarks on irreducible systems can be made applicable to sliding subspaces. In particular, we have the following corollary.

Corollary 5

Consider eqn. (49) and let $\mathcal{X}$ be a subspace of $\mathcal{X}$. Then $\mathcal{X}$ is an almost controllability subspace if and only if there exists a chain of subspaces $\mathcal{B}_1 \supset \mathcal{B}_2 \supset \dots \supset \mathcal{B}_n$, with $\mathcal{B}_1 \subset \text{im } B$, and a feedback mapping F such that

$$\mathcal{X} = \mathcal{B}_1 + (A + BF)\mathcal{B}_2 + \dots + (A + BF)^{n-1}\mathcal{B}_n \quad (69)$$

Moreover, the subspaces $\mathcal{B}_i$ are uniquely determined if $\mathcal{X}$ is a sliding subspace, in which case they are given by

$$\mathcal{B}_i = \text{im } B \cap \mathcal{V}^i(\mathcal{X}) \quad (70)$$

Proof

Suppose that $\mathcal{X}$ is an almost controllability subspace. Then the associated system $\Sigma_{\mathcal{X}}$ is strongly controllable, and so we can form a train basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for it. Define

$$\mathcal{B}_l = \text{span } \{\mathbf{x}_j(-l(\mathbf{x}_j) + 1) \mid j \text{ s.t. } l(\mathbf{x}_j) \geq l + 1\} \quad (71)$$

There exists a feedback mapping F such that

$$(A + BF)\mathbf{x}_j(i) = \mathbf{x}_j(i+1), \quad j=1, \dots, k, \quad i = -l(\mathbf{x}_j) + 1, \dots, -1 \quad (72)$$

With these choices of $\mathcal{B}_l$ ($l=1, \dots, n$) and F , it is easily seen that the relation (69) holds, and that the chain property $\mathcal{B}_1 \supset \dots \supset \mathcal{B}_n$ and the property $\mathcal{B}_1 \supset \text{im } B$ are satisfied.

Conversely, suppose that $\mathcal{K}$ is given by (4.21) for some chain $\mathcal{B}_1 \supset \mathcal{B}_2 \supset \dots \supset \mathcal{B}_n$ in $\text{im } B$ and some feedback mapping F . By the chain property, the subspace $\mathcal{K}(s)$, defined for $s \in \mathbb{C}$ by

$$\mathcal{K}(s) = \mathcal{B}_1 + (sI - (A + BF))\mathcal{B}_2 + \dots + (sI - (A + BF))^{n-1}\mathcal{B}_n \quad (73)$$

does not depend on s . In particular, we have $\mathcal{K} = \mathcal{K}(0) = \mathcal{K}(s)$ for all $s \in \mathbb{C}$. It follows that

$$\mathcal{K} \subset (sI - (A + BF))\mathcal{K} + \text{im } B = (sI - A)\mathcal{K} + \text{im } B \quad (74)$$

for all $s \in \mathbb{C}$. By Schumacher (1982), Theorem 7.1 (ii), this means that $\mathcal{K}$ is an almost controllability subspace.

Now, let $\mathcal{K}$ be a sliding subspace, and suppose that $\mathcal{K}$ is represented in the form (69). We can select a basis $\{b_1, \dots, b_m\}$ for the subspace $\text{im } B$ in such a way that

$$\mathcal{B}_i = \text{span} \{b_1, \dots, b_{r(i)}\} \quad (75)$$

where the $r(i)$ are numbers such that $r(1) \geq r(2) \geq \dots \geq r(n)$, and where we set $r(i) = 0$ if $\mathcal{B}_i = \{0\}$. We also set $r(0) = m (= \dim \text{im } B)$. The numbers $l(j)$ are defined by

$$l(j) = \# \{i \in \mathbb{Z}_+ | r(i) \geq j\} \quad (76)$$

for $j = 1, \dots, m$. For the same j and for $i \in \mathbb{Z}_-$, the vectors $\mathbf{x}_j(i)$ are defined by

$$\left. \begin{aligned} \mathbf{x}_j(i) &= 0 && \text{for } i \leq -l(j) \\ \mathbf{x}_j(i) &= (A + BF)^{i+l(j)-1}b_j && \text{for } -l(j) + 1 \leq i \leq 0 \end{aligned} \right\} \quad (77)$$

It will follow that the set $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ of mappings from $\mathbb{Z}_-$ to $\tilde{\mathcal{X}} = \mathcal{K} + A\mathcal{K} + \text{im } B$ is a train basis, if we can show that the vectors $\{\mathbf{x}_j(i) | j=1, \dots, m; i = -l(j) + 1, \dots, 0\}$ constitute a basis for $\tilde{\mathcal{X}}$. To see that this is indeed true, first note that the set spanned by these vectors is $(A + BF)\mathcal{K} + \text{im } B = A\mathcal{K} + \text{im } B = \mathcal{K} + A\mathcal{K} + \text{im } B$, since the inclusion $\mathcal{K} \subset A\mathcal{K} + \text{im } B$ is true for every almost controllability subspace (eqn. (54)). To prove the linear independency, suppose that there exist scalars $\alpha(j; i)$ such that

$$\sum_{j=1}^m \sum_{i=-l(j)+1}^0 \alpha(j; i) \mathbf{x}_j(i) = 0 \quad (78)$$

This can be rewritten as

$$\begin{aligned} (A + BF) \sum_{j=1}^{r(1)} \sum_{i=-l(j)+2}^0 \alpha(j; i) \mathbf{x}_j(i-1) \\ = - \sum_{j=1}^m \alpha(j; -l(j)+1) \mathbf{x}_j(-l(j)+1) \\ = - \sum_{j=1}^m \alpha(j; -l(j)+1) b_j \end{aligned} \quad (79)$$

We can now use the fact that

$$\mathcal{K} \cap A^{-1}[\text{im } B] = \{0\} \quad (80)$$

which is true for every subspace $\mathcal{K}$ for which $\mathcal{V}^*(\mathcal{K}) = \{0\}$ (simply apply the second line of (53), with j large enough so that $\mathcal{V}^j(\mathcal{K}) = \mathcal{V}^{j+1}(\mathcal{K}) = \mathcal{V}^*(\mathcal{K}) = \{0\}$). Noting that the left-hand side of eqn. (79) is of the form $(A + BF)x$ with $x \in \mathcal{K}$, and that the right hand side is in $\text{im } B$, we obtain

$$\sum_{j=1}^{r(1)} \sum_{i=-l(j)+2}^0 \alpha(j; i) \mathbf{x}_j(i-1) = 0 \quad (81)$$

$$\sum_{j=1}^m \alpha(j; -l(j)+1) b_j = 0 \quad (82)$$

Since the b_j s are linearly independent, it follows from eqn. (82) that $\alpha(j; -l(j)+1) = 0$ for $j = 1, \dots, m$. Equation (81) can be rewritten as

$$\begin{aligned} (A + BF) \sum_{j=1}^{r(2)} \sum_{i=-l(j)+3} \alpha(j; i) \mathbf{x}_j(i-2) \\ = - \sum_{j=1}^{r(1)} \alpha(j; -l(j)+2) \mathbf{x}_j(-l(j)+1) \\ = - \sum_{j=1}^{r(1)} \alpha(j; -l(j)+2) b_j \end{aligned} \quad (83)$$

Repeating the argument above, we can conclude that $\alpha(j; -l(j)+2) = 0$ for $j = 1, \dots, r(1)$, and we obtain another equation of the form of (78) and (81). Going on in this way, we finally find that $\alpha(j; i) = 0$ for all j and i . So this proves that the set $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ is indeed a train basis for the system associated with $\mathcal{K}$. We now apply Corollary 2, to the effect that

$$\begin{aligned} \text{im } B \cap \mathcal{V}^i(\mathcal{K}) &= \text{im } B \cap \text{span} \{\mathbf{x}_j(l) \mid j = 1, \dots, m; \quad l \leq -i\} \\ &= \text{span} \{\mathbf{x}_j(-l(j)+1) \mid j \text{ s.t. } -l(j)+1 \leq -i\} \end{aligned} \quad (84)$$

By eqn. (76), the condition $-l(j)+1 \leq -i$ is equivalent to

$$\#\{k \in \mathbb{Z}_+ \mid r(k) \geq j\} \geq i+1 \quad (85)$$

Because $r(0) \geq r(1) \geq \dots \geq r(n)$, (85) means simply

$$j \leq r(i) \quad (85)$$

So we find

$$\text{im } B \cap \mathcal{V}^i(\mathcal{K}) = \text{span} \{b_j \mid j = 1, \dots, r(i)\} = \mathcal{B}_i \quad (87)$$

This completes the proof. $\square$

The first part of the above corollary is a result due to Willems (1980, Theorem 7 (iii)), which gives a somewhat involved direct proof. From our proof, it would seem that here again the 'train basis' is the natural concept to use.

As another example of an application of a result on strongly controllable systems to almost controllability subspaces, consider the rank criterion for strong controllability given by Hautus (1979): A system $\Sigma(A, B, C, D)$ is strongly controllable if and only if

$$\text{rank} \begin{pmatrix} sI - A & B \\ -C & D \end{pmatrix} = n + p \quad \text{for all } s \in \mathbb{C} \quad (88)$$

With the notation of eqns. (51) and (52), this means that $\mathcal{K}$ is an almost controllability subspace if and only if

$$\text{rank} \begin{pmatrix} sI - A_{11} & -A_{12} & B_1 \\ -A_{21} & sI - A_{22} & B_2 \\ 0 & -I_p & 0 \end{pmatrix} = k + 2p \quad \text{for all } s \in \mathbb{C} \quad (89)$$

where $k = \dim \mathcal{K}$ and $p = \dim (\mathcal{K} + A\mathcal{K} + \text{im } B) - \dim \mathcal{K}$. Condition (89) is clearly equivalent to

$$\text{rank} \begin{pmatrix} sI - A_{11} & B_1 \\ -A_{21} & B_2 \end{pmatrix} = k + p \quad \text{for all } s \in \mathbb{C} \quad (90)$$

This criterion has also been derived, by different methods, in an earlier paper by the author (Schumacher 1982, Corollary 7.3).

5. Conclusions

In this paper, the concept of a train basis has been proposed as a means of giving a natural account of the structure at infinity of a linear system. The train basis is a canonical basis for the state space, defined in terms of the input, state, and output mappings of the system. Its definition is motivated from the condition of strong controllability. The train basis is constructed in a way that is similar to the way one constructs a basis from a spanning set of vectors via elementary column operations. There is some notational complexity involved in writing down the general situation, but the basic principles are simple. Once the train basis is available, the desired canonical form can be derived in a straightforward way. From this form, one can obtain further results by inspection.

A second point of the paper has been to connect the study of (A, B) -invariance and various related concepts to the structure theory of input-output systems. It has been shown that a fairly complete categorization of the classes of subspaces which have appeared in the literature can be given in terms of an associated system, but a complete test in terms of the associated transfer function has not been obtained in all cases. In particular, results on strongly controllable systems have been applied to the class of almost controllability subspaces introduced by Willems (1980), leading us to a new derivation of the 'geometric' characterization of this class, and to more precise results for the subclass of sliding subspaces.

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