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Schumacher, J.M.

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TECHNICAL NOTE

On a Conjecture of Basile and Marro

J. M. SCHUMACHER¹

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Abstract. An alternative characterization is given of the class of self-bounded controlled invariant subspaces that was introduced by Basile and Marro in Ref. 1. We also prove a result that was stated as a conjecture in the cited paper.

Key Words. Linear systems, controlled invariants, self-bounded controlled invariants, controllability subspaces, internal stabilizability.

1. Introduction

A recent addition to the so-called *geometric approach* to linear systems has been made (Ref. 1) by the same authors who introduced the basic concepts of this theory fourteen years ago (Ref. 2). Given a linear, finite-dimensional, time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, \quad (1)$$

and with the notation $\mathcal{F}$ for the subspace $\text{im } B$ of forcing actions, a subspace $\mathcal{S}$ is said to be an $(A, \mathcal{F})$ -controlled invariant (Ref. 2) if, for every initial point $x(0) \in \mathcal{S}$, there exists a control $u(\cdot)$ such that the corresponding state trajectory $x(\cdot)$ is completely contained in $\mathcal{S}$. When furthermore a general subspace $\mathcal{N}$ of $\mathbb{R}^n$ is given, a subspace $\mathcal{S}$ is said to be a *self-bounded controlled invariant* w.r.t. $\mathcal{N}$ (Ref. 1) if $\mathcal{S}$ is a controlled invariant contained in $\mathcal{N}$ and if all trajectories in $\mathcal{N}$ with starting point in $\mathcal{S}$ are completely contained in $\mathcal{S}$.

We shall use the following notations, consistent with Ref. 1. The class of $(A, \mathcal{F})$ -controlled invariants contained in $\mathcal{N}$ is denoted by $\text{CI}(A, \mathcal{F}, \mathcal{N})$, and the class of $(A, \mathcal{N})$ -conditioned invariants (see Ref. 2) containing $\mathcal{F}$ is

¹ Research scientist, Erasmus Universiteit, Rotterdam, Holland.

written $\text{ci}(A, \mathcal{N}, \mathcal{F})$. We also write $\text{SBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})$ for the class of all self-bounded $(A, \mathcal{F})$ -controlled invariants that are contained in $\mathcal{N}$ and that themselves contain $\mathcal{D}$. Instead of $\text{SBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{O})$, we simply write $\text{SBCI}(A, \mathcal{F}, \mathcal{N})$. Finally, if a class $\mathcal{C}$ of subspaces has a supremum (or an infimum), then we denote this supremum (or infimum) by MC (mC). Note that, in this paper, supremum and infimum are always taken with respect to the usual lattice operations on the set of all subspaces of a given vector space.

2. Alternative Characterization

The following characterization of the class of self-bounded controlled invariants contained in a given subspace $\mathcal{N}$ was given in Ref. 1. Write

$$\mathcal{J} = \text{MCI}(A, \mathcal{F}, \mathcal{N}).$$

Then, a subspace $\mathcal{S}$ belongs to $\text{SBCI}(A, \mathcal{F}, \mathcal{N})$ if and only if

$$\mathcal{J} \cap \mathcal{F} \subset \mathcal{S} \subset \mathcal{N}, \quad (2)$$

$$\mathcal{S} \text{ is controlled invariant.} \quad (3)$$

Using this, one can derive another useful characterization.

Proposition 2.1. A subspace $\mathcal{S}$ belongs to $\text{SBCI}(A, \mathcal{F}, \mathcal{N})$ if and only if

$$\mathcal{J} \cap \mathcal{F} \subset \mathcal{S} \subset \mathcal{N}, \quad (4)$$

$$(A + BH)\mathcal{S} \subset \mathcal{S}, \quad \text{for all } H \text{ such that } (A + BH)\mathcal{J} \subset \mathcal{J}. \quad (5)$$

Proof. Since a subspace is a controlled invariant if and only if it is $(A + BH)$ -invariant for some H , the “if” part is obvious. So, it remains to show that (5) follows from (2) and (3). Let H be such that

$$(A + BH)\mathcal{J} \subset \mathcal{J};$$

and let H' be such that

$$(A + BH')\mathcal{S} \subset \mathcal{S}.$$

Take $x \in \mathcal{S}$. Then,

$$(A + BH)x - (A + BH')x = B(H - H')x \in \mathcal{F} \cap \mathcal{J} \subset \mathcal{S}. \quad (6)$$

Because

$$(A + BH')x \in \mathcal{S},$$

this shows that

$$(A + BH)x \in \mathcal{S}$$

as well, which is what we wanted to prove. $\square$

From this result, it is immediately clear that the class $\text{SBCI}(A, \mathcal{F}, \mathcal{N})$ is closed under subspace addition as well as under intersection. This is also shown, in a quite laborious manner, in Ref. 1, Theorem 2.2.

3. Some Useful Identities

From the remarks above, it follows that the class $\text{SBCI}(A, \mathcal{F}, \mathcal{N})$ has an infimum. Basile and Marro show (Ref. 1, Corollary 2.1) that the following relation holds:

$$\text{mSBCI}(A, \mathcal{F}, \mathcal{N}) = \text{MCI}(A, \mathcal{F}, \mathcal{N}) \cap \text{mci}(A, \mathcal{N}, \mathcal{F}). \quad (7)$$

Through the interpretation of $\text{mSBCI}(A, \mathcal{F}, \mathcal{N})$ as the reachable set from 0 by trajectories in $\mathcal{N}$ (Ref. 1, Theorem 3.1), (7) becomes equivalent to a result proven earlier by Morse (Ref. 3). For further interpretations of $\text{mSBCI}(A, \mathcal{F}, \mathcal{N})$ and related subspaces, compare also Ref. 4. In fact, Basile and Marro prove a more general result (Theorem 2.3, Proposition 2.1); if $\mathcal{D} \subset \text{MCI}(A, \mathcal{F}, \mathcal{N})$, then

$$\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) = \text{MCI}(A, \mathcal{F}, \mathcal{N}) \cap \text{mci}(A, \mathcal{N}, \mathcal{F} + \mathcal{D}). \quad (8)$$

It is, however, easy to derive (8) from the special case (7), if one uses the following simple identities.

Proposition 3.1. Suppose that

$$\mathcal{D} \subset \text{MCI}(A, \mathcal{F}, \mathcal{N}). \quad (9)$$

Then, the following relations hold:

$$\text{MCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) = \text{MCI}(A, \mathcal{F}, \mathcal{N}), \quad (10)$$

$$\text{SBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) = \text{SBCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}). \quad (11)$$

Proof. Since

$$\text{CI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) \supset \text{CI}(A, \mathcal{F}, \mathcal{N}),$$

we have

$$\text{MCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) \supset \text{MCI}(A, \mathcal{F}, \mathcal{N}) \supset \mathcal{D}.$$

Because any $(A, \mathcal{F} + \mathcal{D})$ -controlled invariant that contains $\mathcal{D}$ is also $(A, \mathcal{F})$ -controlled invariant, it follows that

$$\text{MCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) \in \text{CI}(A, \mathcal{F}, \mathcal{N}).$$

This entails (10). Next, take

$$\mathcal{S} \in \text{SBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}).$$

Then,

$$\mathcal{S} \in \text{CI}(A, \mathcal{F}, \mathcal{N}) \subset \text{CI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}).$$

Moreover, by (9) and (10),

$$\begin{aligned} (\mathcal{F} + \mathcal{D}) \cap \text{MCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) &= (\mathcal{F} + \mathcal{D}) \cap \text{MCI}(A, \mathcal{F}, \mathcal{N}) \\ &= (\mathcal{F} \cap \text{MCI}(A, \mathcal{F}, \mathcal{N})) + \mathcal{D} \subset \mathcal{S}. \end{aligned} \quad (12)$$

So, we have

$$\mathcal{S} \in \text{SBCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}).$$

Conversely, let

$$\mathcal{S} \in \text{SBCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}).$$

Then, (12) holds again, showing this time that $\mathcal{S} \supset \mathcal{D}$, so that

$$\mathcal{S} \in \text{CI}(A, \mathcal{F}, \mathcal{N}),$$

and also that

$$\mathcal{S} \supset \mathcal{F} \cap \text{MCI}(A, \mathcal{F}, \mathcal{N}),$$

so that, in fact,

$$\mathcal{S} \in \text{SBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}).$$

If

$$\mathcal{D} \subset \text{MCI}(A, \mathcal{F}, \mathcal{N}),$$

one notes the following, using (7), (10), (11):

$$\begin{aligned} \text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) &= \text{mSBCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) \\ &= \text{MCI}(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) \cap \text{mci}(A, \mathcal{N}, \mathcal{F} + \mathcal{D}) \\ &= \text{MCI}(A, \mathcal{F}, \mathcal{N}) \cap \text{mci}(A, \mathcal{N}, \mathcal{F} + \mathcal{D}). \end{aligned} \quad (13)$$

So, in this way, it is possible to derive (8) from (7). It should be noted, though, that the proof of the crucial identity (7) in Ref. 1 can be considered

as being more straightforward than the original proof of Ref. 3. For a completely different proof, see Ref. 5, Corollary 4.10. $\square$

4. Proof of the Conjecture

Finally, let us prove a result that is given as a conjecture in Ref. 1.

Proposition 4.1. If there exists an internally stabilizable $(A, \mathcal{F})$ -controlled invariant contained in $\mathcal{N}$ and containing $\mathcal{D}$, then $\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})$ is internally stabilizable.

Proof. Let us denote the class of internally stabilizable $(A, \mathcal{F})$ -controlled invariants contained in $\mathcal{N}$ by $\text{ISCI}(A, \mathcal{F}, \mathcal{N})$. This class has a supremum (Ref. 6, p. 114; a more direct proof is given in Ref. 7, p. 26; see also Ref. 8, Lemma 3.2). So, the assumption in the statement of the proposition is, in effect,

$$\mathcal{D} \subset \text{MISCI}(A, \mathcal{F}, \mathcal{N}). \quad (14)$$

The subspace $\text{MISCI}(A, \mathcal{F}, \mathcal{N})$ is always self-bounded. This is obvious from the construction in Ref. 6, p. 114; or one can use Theorem 3.1 in Ref. 1, together with the well-known link between controllability and pole placement, to show that

$$\text{mSBCI}(A, \mathcal{F}, \mathcal{N}) \subset \text{MISCI}(A, \mathcal{F}, \mathcal{N}).$$

So, it follows from (14) that, in fact,

$$\text{MISCI}(A, \mathcal{F}, \mathcal{N}) \in \text{SBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}). \quad (15)$$

This immediately entails

$$\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) \subset \text{MISCI}(A, \mathcal{F}, \mathcal{N}). \quad (16)$$

Take H such that $\text{MISCI}(A, \mathcal{F}, \mathcal{N})$ is $(A + BH)$ -invariant and such that the restriction of $A + BH$ to this subspace is stable. It then follows from the relation (16), via the same argument that was used in the proof of Proposition 2.1, that $\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})$ is also $(A + BH)$ -invariant; and, obviously, the restriction of $A + BH$ to $\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})$ is stable. $\square$

It is not true, in general, that $\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})$ is the smallest internally stabilizable controlled invariant subspace in $\mathcal{N}$ that contains $\mathcal{D}$. In fact, such a subspace may not even exist, since the class $\text{ISCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})$ of internally stabilizable controlled invariants in $\mathcal{N}$ containing $\mathcal{D}$ is not

generally closed under intersection. For instance, when

$$\mathcal{N} = \mathbb{R}^n,$$

the whole state space, and the pair (A, B) is controllable, then

$$\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) = \mathbb{R}^n, \quad \text{for any } \mathcal{D},$$

so this subspace is not of much help in solving the important problem of finding low-dimensional internally stabilizable controlled invariants containing a given subspace $\mathcal{D}$. It is shown in Ref. 9 that this so-called *stable cover problem* is crucial in low-order compensator design.

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