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Convexity in Stochastic Cooperative Situations*

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Abstract

This paper introduces a new model concerning cooperative situations in which the payoffs are modeled by random variables. We analyze these situations by means of cooperative games with random payoffs. Special attention is paid to three types of convexity, namely coalitional-merge, individual-merge and marginal convexity. The relations between these types are studied and in particular, as opposed to their deterministic counterparts for TU games, we show that these three types of convexity are not equivalent. However, all types imply that the core of the game is nonempty. Sufficient conditions on the preferences are derived such that the Shapley value, defined as the average of the marginal vectors, is an element of the core of a convex game.

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1 Introduction

In many real-life situations payoffs to agents are uncertain. For example, consider two firms who will temporarily be working together in an R&D project. The profit of this project is yet uncertain, but the firms sign a contract beforehand in which their profit shares are written down. Another situation with uncertain payoffs is the following. Consider two musicians, a pianist and a violinist. Each of them has a contract with a hotel to give small performances. Their payoffs consist of a small wage and the tips they receive during their performances. At the end of the month their contracts will end and both their employers offer them a new contract with the same conditions. Until now these musicians always performed separately, although recently they started studying some pieces for violin and piano together. This is because they found a (third) hotel that is willing to contract both of them. This contract says

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that both musicians only perform for this hotel and their individual payoffs consist of a small wage. Ten percent of all the tips they receive during their performances will be for the hotel and the remaining 90 percent will be divided among the pianist and the violinist. Before the end of the month the musicians have to decide whether to cooperate or not. In both cases their payoffs will be uncertain because they depend on the uncertain amount of tips to be received during performances.

In 'classical' cooperative game theory, payoffs to coalitions of agents are known with certainty. Therefore, situations with uncertain payoffs in which the agents cannot await the realizations of these payoffs, cannot be modeled according to this theory. In this paper we study situations with random payoffs from a game-theoretical point of view that is close to classical cooperative game theory.

In the literature one can find a few other models that can handle uncertain payoffs, for example CHARNES and GRANOT (1973) and SUIJS, BORM, DE WAEGENAERE and TIJS (1999). CHARNES and GRANOT (1973) introduced games in stochastic characteristic function form. These are games where the payoff to coalition S, $V(S)$, is allowed to be a random variable. To allocate the payoff of the grand coalition to the players, the authors suggest a two-stage procedure. In the first stage, so called prior-payoffs are promised to the agents. These prior-payoffs are determined such that there is a relatively high chance that the promises can be realized. In the second stage the realizations of the payoffs are awaited and, if necessary, the prior-payoff vector has to be adjusted to this realization. Research on this subject was continued in CHARNES and GRANOT (1976, 1977) and GRANOT (1977). Some disadvantages of their model are the assumptions that players have to be risk-neutral and that the realizations of the payoffs can be awaited.

In SUIJS and BORM (1999) a different model is studied. They consider a set A_S of actions that coalition S can take. The stochastic value of this coalition then depends on which action $a \in A_S$ is chosen and is denoted by $X_S(a)$. An allocation of $X_S(a)$ to the members of coalition S is described as the sum of two parts. The first part is a monetary transfer between the agents and the second part is an allocation of fractions of $X_S(a)$. Work on this model was started in SUIJS, BORM, DE WAEGENAERE and TIJS (1999) and an application in insurance can be found in SUIJS, DE WAEGENAERE and BORM (1998).

In this paper we introduce a model that, when compared to the previous two models, looks the most like the model of SUIJS et al. but there are some major differences. One of these is that allocations of random payoffs are defined differently. In the model of SUIJS et al. only fractions (i.e. multiples m with $0 \leq m \leq 1$) of the random payoffs are allowed while in our model allocations may be any multiples of the random payoffs, including negative ones as well as multiples larger than 1. A second difference is caused by our focus on allocations of uncertainty only and therefore we do not allow for monetary compensations to cover risk, as opposed to the SUIJS et al. model. This prevents the players to act like little insurance companies as was a typical feature of the SUIJS et al. model. This alternative view may lead to broader insights in how uncertainty can be dealt with. Further, upon considering only multiples of the random payoffs and thus not allowing arbitrary divisions, the type of the underlying probability distribution will remain the same. Finally, this type of allocation is close to that of classical TU games since there an allocation can also be written in terms of multiples of the payoff of the grand coalition.

In this paper special attention will be paid to convexity in cooperative situations with uncertain payoffs. We define three types of convexity for games corresponding to these situations. The definitions of these types are based on three equivalent formulations of convexity for TU games based on marginalistic interpretations. For NTU games the supermodular interpretation of TU convexity was extended to ordinal convexity and cardinal convexity, introduced in Vilkov (1977) and Sharkey (1981), respectively. More recently, Suijs (1999) and Hendrickx, Borm and Timmer (2002) introduce other types of convexity for NTU games that are based on the formulations of convexity for TU games involving notions of marginal contributions.

The three types of convexity to be introduced are coalitional-merge convexity, individual-merge convexity and marginal convexity. The first two are based on the marginal contributions of a coalition of agents and a single agent, respectively, while the third type, marginal convexity, is based on whether or not all the marginal vectors belong to the core of the game. We show that coalitional-merge convexity implies individual-merge convexity, which in turn implies marginal convexity. Examples show that the reverse relations need not hold. Further, each marginal convex game has a nonempty core as well as each of its subgames. Besides, we extend the definition of the Shapley value for TU games as the average of the marginal vectors to our class of cooperative games with random payoffs. We show that the Shapley value is an element of the core of a marginal convex game for certain types of preferences.

The remainder of this paper is organized as follows. Allocations of random variables and the preference relations of the agents over these allocations are defined in section 2. After this, we give a formal description of our model in section 3 and we extend several basic notions from deterministic TU games, like allocations, imputations, superadditivity, the core and marginal vectors, to our model. In section 4 we introduce and study the three types of convexity as discussed above. As opposed to deterministic TU games, we show that these types are not equivalent. However, they all imply that the core of the game is nonempty and for certain types of preferences this core contains the Shapley value.

2 Allocations and preferences in games with random payoffs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the outcome space, $\mathcal{F}$ is a σ-algebra in Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. A stochastic variable X is a measurable function that assigns to each outcome $\omega \in \Omega$ a real number $X(\omega)$. The set of all stochastic variables X with a finite expectation is denoted by $\mathcal{L}$ and $\mathcal{L}_+$ is the set of all nonnegative stochastic variables in $\mathcal{L}$. By 0 we denote the stochastic variable that takes the value zero for sure. Note that $0 \in \mathcal{L}_+$.

A deterministic cooperative game with transferable utility, or TU game, is described by a pair (N, v) where N is the set of players. A nonempty set of players is called a coalition and $v : 2^N \to \mathbb{R}$ is the characteristic function assigning to each coalition S a value $v(S)$ and $v(\emptyset) = 0$. If we introduce uncertainty into this model, such that coalitions of players do not know for sure what payoff they will receive, then the payoffs will be random variables. Denote by $R(S)$ the stochastic payoff (reward) in $\mathcal{L}_+$ to coalition S and let $\mathcal{R} = \{R(S) | S \subset N, S \neq \emptyset\}$ be the set of all these payoffs; $R(\emptyset)$ will not be defined. For a coalition S, an *allocation* (in terms) of $R(S)$ is a vector of multiples of $R(S)$. If $p \in \mathbb{R}^S$ is an allocation for S then player $i \in S$ receives $p_i R(S)$. Such an allocation is efficient if $\sum_{i \in S} p_i = 1$. For ease of notation define $\Delta^*(S) = \{p \in \mathbb{R}^S | \sum_{i \in S} p_i = 1\}$. Allocations as defined here can be used in any situation with (nonnegative) random payoffs where one is interested in the division of some payoff among the participants.

Now that we know how the payoffs of the coalitions can be allocated, it is time to see how players compare two stochastic payoffs. First we restrict ourselves to random payoffs unequal to 0 (random payoffs that do not take the value 0 for sure), or nonzero random payoffs in short, and after that we include 0 as a payoff. Let $\mathcal{A} = \{S \subset N | S \neq \emptyset,\ R(S) \neq 0\}$ be the set of all coalitions with nonzero payoffs. Because allocations are multiples of payoffs to coalitions, the set of all possible payoffs to players equals $\{pR(S) |\, p \in \mathbb{R},\ S \in \mathcal{A}\}$. By $\succsim_i$ we denote the preference relation of player $i \in N$. For some stochastic payoffs X and Y we write $X \succsim_i Y$ if the player weakly prefers receiving the stochastic payoff X to receiving Y while $X \succ_i Y$ means that player i strictly prefers X to Y. If $X \succsim_i Y$ and $Y \succsim_i X$ then we write $X \sim_i Y$, the player is indifferent between receiving X or Y. We make the following assumption about how a player compares two payoffs.

Assumption 2.1 *For each player $i \in N$ there exist functions $f^i_S : \mathbb{R} \to \mathbb{R}$, $S \in \mathcal{A}$, that are surjective, continuous and strictly monotone increasing, such that*

1. $f^i_S(t)R(S) \succsim_i f^i_T(t')R(T)$ *if and only if* $t \geq t'$,
2. $f^i_S(0) = 0$

for any coalitions S and T in $\mathcal{A}$.

Hence, preferences are defined locally since a player's preference relation is only defined on multiples of coalitional payoffs in the game. This kind of preferences is particularly suitable for our model, which will be presented in the next section. So, if player i compares the payoffs $pR(S)$ and $qR(T)$ then $pR(S) \succsim_i qR(T)$ if and only if $t = (f^i_S)^{-1}(p) \geq t' = (f^i_T)^{-1}(q)$. Similarly $pR(S) \sim_i qR(T)$ if and only if $(f^i_S)^{-1}(p) = (f^i_T)^{-1}(q)$ and $pR(S) \succ_i qR(T)$ if and only if $(f^i_S)^{-1}(p) > (f^i_T)^{-1}(q)$. One may interpret the function $(f^i_S)^{-1}$ as some kind of utility function with respect to multiples of $R(S)$ only. The condition $f^i_S(0) = 0$ is a normalization condition.

A second implication of assumption 2.1 is that $R(S) \succ_i 0$ for all $S \in \mathcal{A}$ because $R(S) = 1 \cdot R(S) \succ_i 0 \cdot R(S) = 0$ if and only if $t > t'$ where $1 = f^i_S(t)$ and $0 = f^i_S(t') \Leftrightarrow t' = 0$. This is true because f^i_S is monotone increasing. Similarly it follows that

$$\begin{cases} pR(S) \succ_i 0 & \Leftrightarrow \quad p > 0 \\ pR(S) \sim_i 0 & \Leftrightarrow \quad p = 0 \\ pR(S) \prec_i 0 & \Leftrightarrow \quad p < 0. \end{cases} \tag{1}$$

The following example presents some preference relations that satisfy assumption 2.1.

Example 2.2 The first type of preferences we discuss here concerns expected values of random variables. Suppose that the preferences of player i are such that $X \succsim_i Y$ if and only if $\mathrm{E}(X) \geq \mathrm{E}(Y)$ for any payoffs X and Y, where $\mathrm{E}(X)$ is the expectation of X. Then $f_S^i(t) = t/\mathrm{E}(R(S))$ makes sure that $\succsim_i$ satisfies assumption 2.1. Note that here all the functions f_S^i are *linear*, that is, $f_S^i(t) = f_S^i(1)t$ for all $t \in \mathbb{R}$.

This is a special kind of von Neumann-Morgenstern preferences. Player i is said to have von Neumann-Morgenstern preferences if there exists a utility function $u_i : \mathbb{R} \to \mathbb{R}$ such that $X \succsim_i Y$ if and only if $\mathrm{E}(u_i(X)) \geq \mathrm{E}(u_i(Y))$ for any X and Y. For the preferences above we have $u_i(x) = ax + b$ with $a > 0$. If $u_i(x) = \sqrt{x}$ if $x \geq 0$ and $u_i(x) = -\sqrt{-x}$ if $x < 0$ then

$$f_S^i(t) = \begin{cases} \frac{t^2}{\mathrm{E}(u_i(R(S)))^2} & , t \geq 0 \\ -\frac{t^2}{\mathrm{E}(u_i(R(S)))^2} & , t < 0 \end{cases}$$

which is a nonlinear function.

A second type of preferences involves quantiles of random variables. Let

$$u_{\beta_i}^X = \sup\{t \in \mathbb{R} | \Pr\{X \leq t\} \leq \beta_i\}$$

be the β_i-quantile of X, where $0 < \beta_i < 1$ is such that $u_{\beta_i}^{R(S)} > 0$ for all $S \in \mathcal{A}$. Define the utility function U_i by $U_i(X) = u_{\beta_i}^X$ if $\mathrm{E}(X) \geq 0$ and $U_i(X) = u_{1-\beta_i}^X$ otherwise. We say that a player has *quantile-preferences* if $X \succsim_i Y$ if and only if $U_i(X) \geq U_i(Y)$. For these preferences $f_S^i(t) = t/u_{\beta_i}^{R(S)}$. $\square$

Since the preference relation $\succsim_i$ is complete, it is reflexive, transitive and monotone increasing. Besides, $\sim_i$ is an equivalence, and because of monotonicity, each equivalence class contains exactly one multiple of any nonzero coalitional value. Hence, a third consequence of assumption 2.1 is that for all players $i \in N$ there exists a unique number $\alpha \in \mathbb{R}$ such that $pR(S) \sim_i \alpha R(T)$ where $S, T \in \mathcal{A}$ and $p \in \mathbb{R}$. To be able to keep track of which α is connected to which variables $pR(S)$, $R(T)$ and i we define the function $\alpha_i : \mathcal{A} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ to take this unique value. One might say that the function α_i embeds $pR(S)$ in $R(T)$, so, α_i is some kind of embedding function. By assumption 2.1 $\alpha_i(S, T, p) = f_T^i((f_S^i)^{-1}(p))$. The function α_i will be restricted to the case where $i \in S \subset T$ because a player does not have to consider payoffs from coalitions to which he does not belong. Further, $S \subset T$ is not a real restriction because $\alpha_i(T, S, p)$ is the inverse of $\alpha_i(S, T, p)$. Hence, the embedding function α_i gives a complete description of the preference relation of player i.

What happens if $R(T) = 0$ for some coalition T? For all nonzero payoffs $pR(S)$ it holds by (1) that either $pR(S) \succ_i 0$ or $pR(S) \prec_i 0$. Because $\alpha R(T) = 0$ for any $\alpha \in \mathbb{R}$ it follows that for all $i \in N$ there exists no $\alpha \in \mathbb{R}$ such that $pR(S) \sim_i \alpha R(T)$. But $0 \cdot R(S) \sim_i R(T)$ and $qR(T) = 0 = R(T)$. To cover these two cases, we extend the domain of α_i and define, if $R(T) = 0$, $\alpha_i(S, T, 0) = 1$ and $\alpha_i(T, T, q) = 1$. The next theorem states some nice properties of the function α_i.

Theorem 2.3 *For all $i \in N$*

1. $\alpha_i(S,S,h) = h$ *for any* $h \in \mathbb{R}$*,* $S \in \mathcal{A}$*,*

2. $\alpha_i(T,U,\alpha_i(S,T,p)) = \alpha_i(S,U,p)$ *for any* $p \in \mathbb{R}$ *and* $S,T,U \in \mathcal{A}$*,*

3. $\alpha_i(S,T,p) = p\alpha_i(S,T,1)$ *for any* $p \in \mathbb{R}$ *and* $S,T \in \mathcal{A}$ *if the functions* f_S^i *and* f_T^i *are linear.*

Proof. For the first item, let $h \in \mathbb{R}$ and $S \in \mathcal{A}$, then $hR(S) \sim_i \alpha_i(S,S,h)R(S)$ by definition of α_i. Because $\succsim_i$ is monotone increasing, we have $h = \alpha_i(S,S,h)$.
To prove the second item, let $p \in \mathbb{R}$ and $S,T,U \in \mathcal{A}$. Then by definition of α_i $pR(S) \sim_i \alpha_i(S,T,p)R(T)$ and $\alpha_i(S,T,p)R(T) \sim_i \alpha_i(T,U,\alpha_i(S,T,p))R(U)$. Since $\succsim_i$ is transitive, $pR(S) \sim_i \alpha_i(T,U,\alpha_i(S,T,p))R(U)$. Hence, $\alpha_i(T,U,\alpha_i(S,T,p)) = \alpha_i(S,U,p)$ because $\succsim_i$ is also monotone increasing.
Finally, let $p \in \mathbb{R}$ and $S,T \in \mathcal{A}$. If the functions f_S^i and f_T^i are linear then

$$\begin{aligned} \alpha_i(S,T,p) &= f_T^i((f_S^i)^{-1}(p)) = pf_T^i((f_S^i)^{-1}(1)) \\ &= p\alpha_i(S,T,1), \end{aligned}$$

which concludes the proof. $\square$

To conclude we would like to remark that any TU game (N,v) with nonnegative values can be embedded into the framework introduced in this section. The deterministic coalition values imply that $f_S^i(t) = t/v(S)$ for $v(S) \neq 0$ and consequently $\alpha_i(S,T,p) = pv(S)/v(T)$ where $v(T) \neq 0$. Notice that an allocation x of $v(N) \neq 0$ among the players, where player i receives x_i, is equivalent to player i receiving the multiple $p_i = x_i/v(N)$ of the total payoff $v(N)$.

3 The model

In this section we will describe our model in more detail. We define the corresponding games where coalitions of players receive random values. After this we extend some basic definitions in cooperative game theory to our model and illustrate these concepts with an example.
The tuple $(N,\mathcal{R},\mathcal{A},\alpha)$ gives an extensive description of a cooperative game with random payoffs where $N = \{1,\ldots,n\}$ is the player set, $\mathcal{R} = \{R(S)|S \subset N, S \neq \emptyset\}$ is the set of coalitional payoffs, $\mathcal{A}$ contains the coalitions with a nonzero payoff and $\alpha = (\alpha_i)_{i\in N}$ with α_i the previously defined function derived from the random payoffs and preference relations that describes what multiple of one stochastic variable player i finds equivalent to another stochastic variable. Often, we will denote a game by (N,α) if $\mathcal{R}$ and $\mathcal{A}$ follow from the context, and sometimes we simply write (α).
We will now extend various notions from deterministic TU games to cooperative games with random payoffs. Recall from the previous section that $p \in \mathbb{R}^S$ describes an allocation for coalition S where player i receives a multiple p_i of its random payoff $R(S)$. Such an allocation is efficient if $p \in \Delta^*(S) = \{p \in \mathbb{R}^S | \sum_{i\in S} p_i = 1\}$. An allocation p for coalition S is *individual rational* if $p_i \geq \alpha_i(\{i\},S,1)$ for all players $i \in S$. We will denote the set of all efficient individual rational allocations for coalition S by $IR(S)$.

An allocation for coalition N is called an *imputation* if it is individual rational and efficient. The *imputation set* $I(N,\alpha)$ is the set of all imputations.

$$I(N,\alpha) = \{p \in \Delta^*(N) \,|\, p_i \geq \alpha_i(\{i\}, N, 1) \text{ for all } i \in N\}$$

Note that $I(N,\alpha) = IR(N)$. Depending upon the random values of the various coalitions we can say something more about the structure of the imputation set.

Lemma 3.1 *$I(N,\alpha) \subset \{p \in \Delta^*(N) | p_i \geq 0$ for all $i \in N\}$ if $N \in \mathcal{A}$. If $N \notin \mathcal{A}$ and $\{i\} \notin \mathcal{A}$ for all $i \in N$, then $I(N,\alpha) = \Delta^*(N)$. If $N \notin \mathcal{A}$ and $\{i\} \in \mathcal{A}$ for some $i \in N$, then $I(N,\alpha) = \emptyset$.*

Proof. Assume that $N \in \mathcal{A}$. If $I(N,\alpha) = \emptyset$ then we are done. Otherwise take an imputation p. Then $p_i \geq \alpha_i(\{i\}, N, 1) \geq 0$ where the second inequality follows from $N \in \mathcal{A}$. The two remaining statements are trivial. $\square$

The game (N,α) is *superadditive* if for all coalitions $S, T \subset N$ with $S \cap T = \emptyset$, and for all allocations $p \in IR(S)$ and $q \in IR(T)$ there exists an allocation $r \in \Delta^*(S \cup T)$ for the joint coalition such that all players are weakly better off:

$$\begin{cases} r_i \geq \alpha_i(S, S\cup T, p_i) & \text{for all } i \in S, \\ r_i \geq \alpha_i(T, S\cup T, q_i) & \text{for all } i \in T. \end{cases}$$

Notice that $r \in IR(S \cup T)$. So, if a game is superadditive then the members of two disjoint coalitions can (weakly) improve upon their payoffs by cooperating. We also could have formulated superadditivity with $k \geq 2$ disjoint coalitions: for all $T_1, T_2, \ldots, T_k \subset N$, $k \geq 2$, such that $T_i \cap T_j = \emptyset$ for all $i \neq j$, and for all $p^i \in IR(T_i)$, $i = 1, 2, \ldots, k$ there exists an allocation $r \in \Delta^*(\cup_{i=1}^k T_i)$ such that

$$r_j \geq \alpha_j(T_i, \cup_{i=1}^k T_i, p_j^i) \text{ for all } j \in T_i,\ \ i = 1, 2, \ldots, k. \tag{2}$$

Again, all players are weakly better off by joining $\cup_{i=1}^k T_i$. Obviously, this alternative definition implies superadditivity. The other way around is also true, as one can easily verify. This implies the following relation between superadditive games and the sets $IR(S)$ of individual rational allocations.

Lemma 3.2 *If a game (N,α) is superadditive then $IR(S) \neq \emptyset$ for all coalitions S.*

Proof. Let the game (N,α) be superadditive and take a coalition $S \subset N$. Assume $S = \{1, \ldots, s\}$ and define $T_i = \{i\}$ for $i = 1, \ldots, s$. Then $IR(T_i) = IR(\{i\})$ and $\cup_{i=1}^s T_i = S$. By (2) there exists an allocation $r \in \Delta^*(S)$ such that $r_i \geq \alpha_i(\{i\}, S, 1)$ for all $i \in S$. Thus $r \in IR(S)$. $\square$

For all coalitions S, the set $\text{dom}(S)$ contains the allocations for coalition N that are dominated by coalition S, i.e., there exists an allocation $q \in \Delta^*(S)$ that is strictly preferred by all members of S:

$$\text{dom}(S) = \left\{p \in \mathbb{R}^S \,|\, \exists q \in \Delta^*(S) :\ \alpha_i(S, N, q_i) > p_i \text{ for all } i \in S\right\}$$

The set of allocations that are not dominated by some coalition can take many forms, depending upon the random values. Let $p_S = \{p_i\}_{i\in S}$ be the restriction of $p \in \mathbb{R}^N$ to coalition S.

Lemma 3.3 *Let S be a coalition. Then*

$$p_S \notin dom(S) \Leftrightarrow \begin{cases} p \in \mathbb{R}^N & \text{if } S \notin \mathcal{A} \text{ and } N \notin \mathcal{A}, \\ p_i \geq 0 \text{ for some } i \in S & \text{if } S \notin \mathcal{A} \text{ and } N \in \mathcal{A}. \end{cases}$$

If $S \in \mathcal{A}$ and $N \notin \mathcal{A}$ then $p_S \in dom(S)$ for all $p \in \mathbb{R}^N$. Furthermore, if $S \in \mathcal{A}$, $N \in \mathcal{A}$ and f_S^i and f_N^i are linear then the set $\{p \in \mathbb{R}^N \,|\, p_S \notin dom(S)\}$ is convex.

Proof. We only prove the last statement. The remaining parts of the lemma are trivial.
Let (N, α) be a cooperative game with random payoffs and let S be a coalition. Assume that $S \in \mathcal{A}$ and $N \in \mathcal{A}$. Then $p_S \notin \mathrm{dom}(S)$ if and only if there exists no vector $q \in \Delta^*(S)$ such that $\alpha_i(S, N, q_i) > p_i$ for all $i \in S$. By property 3 in theorem 2.3 this is equivalent to

$$\neg \exists q \in \Delta^*(S) \;:\; q_i \alpha_i(S, N, 1) > p_i \text{ for all } i \in S,$$

so,

$$\neg \exists q \in \Delta^*(S) \;:\; q_i > p_i / \alpha_i(S, N, 1) \text{ for all } i \in S.$$

Hence,

$$\sum_{i \in S} p_i / \alpha_i(S, N, 1) \geq 1.$$

Define $h \in \mathbb{R}^S$ by $h_i = 1/\alpha_i(S, N, 1)$. Then $h_i > 0$ for all $i \in S$ and $p_S \notin \mathrm{dom}(S)$ if and only if $\sum_{i \in S} h_i p_i \geq 1$. We conclude that the set $\{p \in \mathbb{R}^N \,|\, p_S \notin \mathrm{dom}(S)\}$ is convex. □

The *core* of (N, α), denoted by $C(N, \alpha)$, consists of all payoff vectors attainable for the grand coalition that are not dominated by any coalition S, that is

$$C(N, \alpha) = \{p \in \Delta^*(N) \,|\, p_S \notin \mathrm{dom}(S) \text{ for all coalitions } S\}.$$

Because the equivalence

$$p_i \notin \mathrm{dom}(\{i\}) \Leftrightarrow p_i \geq \alpha_i(\{i\}, N, 1)$$

holds for all $i \in N$, the core is a subset of the imputation set: $C(N, \alpha) \subset I(N, \alpha)$, for all games (N, α). In particular, $C(N, \alpha) = I(N, \alpha)$ for two-person games. Using the results in the lemmas 3.1 and 3.3 we can show that the core is convex if all the functions f_S^i are linear for all players i.

Theorem 3.4 *Let (N, α) be a cooperative game with random payoffs where all the functions f_S^i are linear. Then the core $C(N, \alpha)$ is a convex set.*

Proof. Let $\mathrm{undom}(S) = \{p \in \Delta^*(N) |\, p_S \notin \mathrm{dom}(S)\}$ be the set of efficient allocations for N that are not dominated by coalition S. Then

$$I(N,\alpha) = \cap_{i\in N} \ \mathrm{undom}(\{i\})$$

and this implies that

$$C(N,\alpha) = \cap_{S\subset N,\, S\neq\emptyset} \ \mathrm{undom}(S) = I(N,\alpha) \cap \left(\cap_{S\subset N,\, |S|\geq 2} \ \mathrm{undom}(S)\right).$$

Firstly, suppose that $N \notin \mathcal{A}$. If $S \in \mathcal{A}$ for some $S \subset N$ then $\mathrm{undom}(S) = \emptyset$ according to lemma 3.3 and by this $C(N,\alpha) = \emptyset$. If $S \notin \mathcal{A}$ for all $S \subset N$ then according to the same theorem $\mathrm{undom}(S) = \Delta^*(N)$ for all $S \subset N$ and so $C(N,\alpha) = \Delta^*(N)$, which is a convex set.
Secondly, if $N \in \mathcal{A}$ then $\mathrm{undom}(\{i\}) = \{p \in \Delta^*(N) |\, p_i \geq \alpha_i(\{i\}, N, 1)\}$ is a convex set for all $i \in N$ and so is $I(N,\alpha)$. If $S \notin \mathcal{A}$ for some coalition S then $\mathrm{undom}(S) = \{p \in \Delta^*(N) |\, p_i \geq 0 \text{ for some } i \in S\}$ according to lemma 3.3 and by lemma 3.1 it follows that $\mathrm{undom}(S) \supset I(N,\alpha)$. This implies that $\mathrm{undom}(S) \cap I(N,\alpha) = I(N,\alpha)$, which is a convex set. If $S \in \mathcal{A}$ then it follows from lemma 3.3 that $\mathrm{undom}(S)$ is a convex set. We conclude that also in case $N \in \mathcal{A}$ it holds that $C(N,\alpha)$ is a convex set. $\square$

A bijection σ of the players in N is a function from $\{1,2,\ldots,n\}$ to N and $\sigma(i)$ denotes which player in N is at position i. Let $\Pi(N)$ be the set of all bijections of N. Denote by $S_i^\sigma = \{\sigma(k) |\, k \leq i\}$ the set of the first i players according to bijection σ, $i \in \{1,2,\ldots,n\}$, and let $S_0^\sigma = \emptyset$. In a deterministic TU game (N,v) the marginal vector $m^\sigma(v)$ is defined by

$$m^\sigma_{\sigma(k)}(v) = v(S_k^\sigma) - v(S_{k-1}^\sigma) = v(S_k^\sigma) - \sum_{i=1}^{k-1} m^\sigma_{\sigma(i)}(v)$$

for each $k \in \{1,2,\ldots,n\}$.
In cooperative games with random payoffs *marginal vectors* can be defined in a similar way. For this we need the following assumption.

Assumption 3.5 *If $T \notin \mathcal{A}$ for some coalition T then $S \notin \mathcal{A}$ for all coalitions $S \subset T$.*

Let $y^\sigma_{\sigma(i)}(\alpha)$ be the marginal contribution of the ith player according to the bijection σ in the game (N,α). Such a contribution is expressed as a multiple of the random payoff for S_i^σ. Now, the marginal contribution of the first player according to σ, i.e., $\sigma(1)$, equals this player's random payoff, so $y^\sigma_{\sigma(1)}(\alpha) = 1$. If the second player, $\sigma(2)$, joins then the two players together form the coalition S_2^σ. The marginal contribution of $\sigma(2)$ is all that remains of the random payoff of S_2^σ after deduction of the marginal contribution of player $\sigma(1)$:

$$y^\sigma_{\sigma(2)}(\alpha) = 1 - \alpha_{\sigma(1)}(S_1^\sigma, S_2^\sigma, y^\sigma_{\sigma(1)}(\alpha)).$$

The latter expression is well-defined according to assumption 3.5. Notice that this marginal contribution may be negative, that is, a *negative multiple* of the payoff for S_2^σ. The marginal contribution of the third player is

$$y^\sigma_{\sigma(3)}(\alpha) = 1 - \sum_{k=1}^{2} \alpha_{\sigma(k)}(S_k^\sigma, S_3^\sigma, y^\sigma_{\sigma(k)}(\alpha))$$

and the marginal contribution of $\sigma(i)$ to coalition S^σ_{i-1} is

$$y^\sigma_{\sigma(i)}(\alpha) = 1 - \sum_{k=1}^{i-1} \alpha_{\sigma(k)}(S_k^\sigma, S_i^\sigma, y^\sigma_{\sigma(k)}(\alpha))$$

for all $i \in \{1, 2, \ldots, n\}$. Then the marginal vector $m^\sigma(\alpha)$ is defined by

$$m^\sigma_{\sigma(i)}(\alpha) = \alpha_{\sigma(i)}(S_i^\sigma, N, y^\sigma_{\sigma(i)}(\alpha))$$

for $i = 1, 2, \ldots, n$, and so, this marginal vector is an efficient allocation for N. Based on these marginal vectors we define the *Shapley value* $\phi(\alpha)$ as the average of the $n!$ marginal vectors,

$$\phi(\alpha) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(\alpha)$$

just like its counterpart for deterministic TU-games (cf. Shapley (1953)). To conclude this section, we give an example of a game that illustrates the concepts introduced in this section.

Example 3.6 Consider the following situation with three players, $N = \{1, 2, 3\}$. Let $R(S) = \max\{E - \sum_{i \in N \setminus S} d_i, 0\}$ where E is the random variable

$$E = \begin{cases} 200 & & 1/4 \\ 300 & \text{with probability} & 1/2 \\ 400 & & 1/4, \end{cases}$$

and $d_1 = 200$, $d_2 = 180$ and $d_3 = 100$. Notice that $R(N) = E$. The preference relations of the players are as follows. For player 1 $X \succsim_1 Y$ if and only if $\mathrm{E}(X) \geq \mathrm{E}(Y)$. Hence, $\alpha_1(S, T, p) = p\mathrm{E}(R(S))/\mathrm{E}(R(T))$. The players 2 and 3 have quantile preferences, as defined in example 2.2, with $\beta_2 = 0.75$ and $\beta_3 = 0.9$. Consequently, $\alpha_i(S, T, p) = p u_{\beta_i}^{R(S)} / u_{\beta_i}^{R(T)}$ for $i = 2, 3$. The set of individual rational allocations is

$$I(N, \alpha) = \{p \in \Delta^*(N) | \, p_1 \geq 2/15, p_2 \geq 1/4, p_3 \geq 1/20\}$$

and the core equals

$$C(N, \alpha) = \left\{ p \in I(N, \alpha) \;\middle|\; \begin{array}{l} 9p_1 + 8p_2 \geq 6, 11p_1 + 8p_3 \geq 22/5, \\ p_2 + p_3 \geq 1/2 \end{array} \right\}.$$

The six marginal vectors are given in the table below.

σ	$m^{\sigma}(\alpha)$
$(1,2,3)$	$(2/15, 3/5, 4/15)$
$(1,3,2)$	$(2/15, 1/2, 11/30)$
$(2,1,3)$	$(4/9, 1/4, 11/36)$
$(2,3,1)$	$(1/2, 1/4, 1/4)$
$(3,1,2)$	$(4/11, 129/220, 1/20)$
$(3,2,1)$	$(1/2, 9/20, 1/20)$

The first two marginal vectors only belong to the imputation set while the remaining four are contained in the core. The Shapley value $\phi(\alpha)$ is the average of these marginal values and equals $\phi(\alpha) = (1027/2970, 29/66, 29/135)$. □

4 Three types of convexity

The following three statements about a deterministic TU game (N, v) are equivalent (cf. Shapley (1971) and Ichiishi (1981)).

i. For all $U \subset N$ and for all $S \subset T \subset N \setminus U$ it holds that $v(S \cup U) - v(S) \leq v(T \cup U) - v(T)$.

ii. For all $i \in N$ and for all $S \subset T \subset N \setminus \{i\}$ it holds that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

iii. All $n!$ marginal vectors m^{σ} of (N, v) belong to the core $C(v)$.

A game (N, v) that satisfies these statements is called a convex game. Based on these statements we define three types of convexity for cooperative games with random payoffs.

Similar to Suijs and Borm (1999), statement i can be interpreted as follows. The marginal contribution of coalition U to coalition S, $v(S \cup U) - v(S)$, is smaller than the contribution of U to T, $v(T \cup U) - v(T)$. Thus, when allocations of $v(S)$, $v(T)$ and $v(S \cup U)$ are proposed and if coalition S is willing to let U join, that is, the members of S get from $v(S \cup U)$ at least as much as what they get from $v(S)$, then there exists an allocation of $v(T \cup U)$ that makes all players in $T \cup U$ better off. The players in T get at least as much from $v(T \cup U)$ as from $v(T)$ and the players in U get at least as much from $v(T \cup U)$ as from $v(S \cup U)$. If we take into account that players will only consider individual rational allocations, then we can define a first kind of convexity as follows.

A cooperative game with random payoffs is called *coalitional-merge convex* if and only if for all coalitions U, for all $S \subset T \subset N \setminus U$ such that $S \neq T$, for all $p \in IR(S)$, for all $q \in IR(T)$ and for all $r \in IR(S \cup U)$ such that $r_i \geq \alpha_i(S, S \cup U, p_i)$ for all $i \in S$, there exists an allocation $s \in \Delta^*(T \cup U)$ such that

$$\begin{cases} s_i \geq \alpha_i(T, T \cup U, q_i) & \text{for all } i \in T, \\ s_i \geq \alpha_i(S \cup U, T \cup U, r_i) & \text{for all } i \in U. \end{cases}$$

If we restrict ourselves to $U = \{i\}$ for all $i \in N$ then we arrive at a second type of convexity, which is related to statement *ii.* A cooperative game with random payoffs is called *individual-merge convex* if and only if for all $i \in N$, for all $S \subset T \subset N \setminus \{i\}$ such that $S \neq T$, for all $p \in IR(S)$, for all $q \in IR(T)$ and for all $r \in IR(S \cup \{i\})$ such that $r_j \geq \alpha_j(S, S \cup \{i\}, p_j)$ for all $j \in S$, there exists an allocation $s \in \Delta^*(T \cup \{i\})$ such that

$$\begin{cases} s_j \geq \alpha_j(T, T \cup \{i\}, q_j) & \text{for all } j \in T, \\ s_i \geq \alpha_i(S \cup \{i\}, T \cup \{i\}, r_i). \end{cases}$$

Finally, we call a cooperative game with random payoffs *marginal convex* if and only if all its marginal vectors m^σ belong to its core. This provides sufficient conditions for the Shapley value to belong to the core.

Theorem 4.1 *Let (N, α) be a marginal convex game where all the functions f_S^i are linear. Then the Shapley value belongs to the core $C(N, \alpha)$.*

Proof. According to theorem 3.4 the core $C(N, \alpha)$ is a convex set. Because all the marginal vectors belong to the core, so does their average, the Shapley value. □

If we consider other types of preferences then this result need not hold, as is shown in the next example.

Example 4.2 Consider the game (N, α) with $N = \{1, 2, 3\}$,

$$\alpha_i(S, T, p) = \begin{cases} 0, & |S| = 1, \\ p, & |S| = |T| > 1, \\ \frac{2}{5}p, & |S| = 2, \ T = N, \end{cases}$$

for $i = 1, 3$ and

$$\alpha_2(S, T, p) = \begin{cases} 0, & |S| = 1, \\ p, & |S| = |T| > 1, \\ \frac{2}{5}p, & |S| = 2, \ T = N, \ p < 0, \\ \frac{1}{2}\sqrt[6]{p}, & |S| = 2, \ T = N, \ p \geq 0. \end{cases}$$

The core of this game,

$$C(N, \alpha) = \left\{ p \in \Delta^*(N) \;\middle|\; \begin{array}{l} p_1 \geq 0, \ p_2 \geq 0, \ p_3 \geq 0, \ 5p_1 + 128(p_2)^6 \geq 2, \\ 5p_1 + 5p_3 \geq 2, \ 128(p_2)^6 + 5p_3 \geq 2 \end{array} \right\},$$

consists of two disjoint sets in $\Delta^*(N)$. It contains all the marginal vectors and therefore this game is marginal convex. Nevertheless, the Shapley value $\phi(\alpha) = (19/60, 11/30, 19/60)$ is not an element of the core since it belongs to both $\mathrm{dom}(\{1, 2\})$ and $\mathrm{dom}(\{2, 3\})$. □

From the definitions it follows immediately that a coalitional-merge convex game is superadditive. If there exists a coalition S such that $IR(S) = \emptyset$ then the game (N, α) is not superadditive by lemma 3.2 and hence it is not coalitional-merge convex. The following theorem states a similar result with respect to marginal convexity.

Theorem 4.3 *If there exists a coalition S with $IR(S) = \emptyset$ then the game (N, α) is not marginal convex.*

Proof. Recall that

$$IR(S) = \{p \in \Delta^*(S) | p_i \geq \alpha_i(\{i\}, S, 1) \text{ for all } i \in S\}.$$

Let S be a coalition with $IR(S) = \emptyset$ such that $IR(T) \neq \emptyset$ for all subsets T of S. Note that S should contain at least two players since $IR(\{i\}) = \{1\} \neq \emptyset$ for all $i \in N$. Without loss of generality assume that $S = \{1, 2, \ldots, s\}$. Let σ be a bijection of N such that $\sigma(i) = i$. Recall that the marginal vector $m^\sigma(\alpha)$ is defined by

$$m^\sigma_{\sigma(j)}(\alpha) = \alpha_{\sigma(j)}(S^\sigma_j, N, y^\sigma_{\sigma(j)}(\alpha)) \tag{3}$$

for all $j \in N$ where

$$y^\sigma_{\sigma(j)}(\alpha) = 1 - \sum_{k=1}^{j-1} \alpha_{\sigma(k)}(S^\sigma_k, S^\sigma_j, y^\sigma_{\sigma(k)}(\alpha))$$

Now, if $y^\sigma_j(\alpha) < \alpha_j(\{j\}, S^\sigma_j, 1)$ for some $j < s$ then $m^\sigma_j(\alpha) < \alpha_j(\{j\}, N, 1)$ because of (3). Thus $m^\sigma(\alpha) \notin I(N, \alpha)$. Otherwise, if $y^\sigma_j(\alpha) \geq \alpha_j(\{j\}, S^\sigma_j, 1)$ for all $j < s$ then $IR(S) = \emptyset$ implies that $y^\sigma_s(\alpha) < \alpha_s(\{s\}, S, 1)$. From equation (3) we obtain, once again, that $m^\sigma(\alpha) \notin I(N, \alpha)$. Consequently, $m^\sigma(\alpha) \notin C(N, \alpha)$ since $I(N, \alpha) \supset C(N, \alpha)$. We conclude that this game is not marginal convex. □

Notice that this theorem is similar to lemma 3.2 about superadditivity. For two-person games marginal convexity and superadditivity coincide. But for games with at least three players there seems to be no relation between the concepts. It is easy to find an example of a game that is superadditive and not marginal convex. But a game that is marginal convex and not superadditive has not yet been found. An alternative approach is to see whether the implication series convexity ⇒ total balancedness ⇒ superadditivity holds, as it does for TU games. The first implication is shown in theorem 4.5. Unfortunately the second implication does not hold. To see this, consider the 3-person game (N, α) with $N = \{1, 2, 3\}$ and

$$\alpha_i(S, T, p) = \begin{cases} 0, & |S| = 1,\ S \neq T \\ p, & S = T, \\ \frac{1}{2}p, & S = \{1, 2\},\ T = N, \\ \frac{8+i}{10}p, & S = \{1, 3\},\ T = N, \\ \frac{1}{5}p, & S = \{2, 3\},\ T = N, \end{cases}$$

for player $i \in N$. This game has a nonempty core, and so have all of its subgames, but it is not superadditive. The definition of superadditivity fails for the allocations $p = 1 \in IR(\{2\})$ and $q = (0,1) \in IR(\{1,3\})$. Thus it remains open whether a marginal convex game is superadditive or not.

Our definitions of convexity are not equivalent for cooperative games with random payoffs, while the corresponding notions are equivalent for deterministic TU games. The relations between the three definitions are as follows:

coalitional-merge convex

$\Downarrow$

individual-merge convex

$\Downarrow$

marginal convex

The latter relation, individual-merge convex games are marginal convex, is shown in the next theorem.

Theorem 4.4 *Let (N, α) be a cooperative game with random payoffs. If it is individual-merge convex then it is marginal convex.*

Proof. Let (N, α) be an individual-merge convex game and take a bijection $\sigma \in \Pi(N)$. Without loss of generality assume that $\sigma(i) = i$ for all $i \in N$. Furthermore, let $z^{\sigma,k}(\alpha)$ be an efficient allocation for coalition $\{1, \ldots, k\} = S_k^\sigma$ defined by $z_i^{\sigma,k}(\alpha) = \alpha_i(S_i^\sigma, S_k^\sigma, y_i^\sigma(\alpha))$ for $i = 1, 2, \ldots, k$ and $k = 1, 2, \ldots, n$. Notice that $z^{\sigma,n}(\alpha) = m^\sigma(\alpha)$. We show that $z^{\sigma,k}(\alpha)$ is a core-element of the subgame Γ^k with player set S_k^σ by induction on k.

If $k = 1$ then it is clear that $z^{\sigma,1}(\alpha) \in C(\Gamma^1)$. Next, assume that $z^{\sigma,k}(\alpha) \in C(\Gamma^k)$ for $k = 1, 2, \ldots, l-1$ where $l \leq n$. We have to prove that $z^{\sigma,l}(\alpha) \in C(\Gamma^l)$. Consider a coalition $S \subset S_{l-1}^\sigma$. Then it follows from $z^{\sigma,l-1}(\alpha) \in C(\Gamma^{l-1})$ and $z_j^{\sigma,l}(\alpha) = \alpha_j(S_{l-1}^\sigma, S_l^\sigma, z_j^{\sigma,l-1}(\alpha))$ for all $j \in S_{l-1}^\sigma$ that $\{z_j^{\sigma,l}(\alpha)\}_{j\in S} \notin \mathrm{dom}(S)$. Thus, coalition S has no incentives to leave the coalition S_l^σ because it cannot improve upon $\{z_j^{\sigma,l}(\alpha)\}_{j\in S}$. Next, we show that also the coalition $S \cup \{l\}$ has no incentive to leave the coalition S_l^σ if $z^{\sigma,l}(\alpha)$ is allocated. Let $p \in IR(S)$ be such that $\sum_{j\in S} \alpha_j(S, S \cup \{l\}, p_j)$ is minimized. This particular allocation is well defined since $IR(S)$ is a compact set and $\sum_{j\in S} \alpha_j(S, S \cup \{l\}, p_j)$ is a continuous function in p. Define $r_j = \alpha_j(S, S \cup \{l\}, p_j)$ then $r_l := 1 - \sum_{j\in S} r_j$ is as large as possible. So, the allocation r is the best allocation for player l when cooperating with coalition S. Since the game (N, α) is individual-merge convex there exists an allocation $s \in \Delta^*(S_{l-1}^\sigma \cup \{l\}) = \Delta^*(S_l^\sigma)$ such that

$$\begin{cases} s_j \geq \alpha_j(S_{l-1}^\sigma, S_l^\sigma, z_j^{\sigma,l-1}(\alpha)) = z_j^{\sigma,l}(\alpha), & \text{for } j \in S_{l-1}^\sigma, \\ s_l \geq \alpha_l(S \cup \{l\}, S_l^\sigma, r_l). \end{cases}$$

Thus

$$z_l^{\sigma,l}(\alpha) = 1 - \sum_{j\in S_{l-1}^\sigma} z_j^{\sigma,l}(\alpha) \geq 1 - \sum_{j\in S_{l-1}^\sigma} s_j = s_l \geq \alpha_l(S \cup \{l\}, S_l^\sigma, r_l).$$

But we stated before that r is the best allocation for player l when cooperating with coalition S. Therefore there exists no individual rational allocation for coalition $S \cup \{l\}$ that yields player l a strictly better payoff then $z_l^{\sigma,l}(\alpha)$. Hence, coalition $S \cup \{l\}$ has no incentive to part company with coalition S_l^σ if $z^{\sigma,l}(\alpha)$ is allocated. Consequently, we have that $z^{\sigma,l}(\alpha) \in C(\Gamma^l)$. Taking $l = n$ results in $m^\sigma(\alpha) = z^{\sigma,n}(\alpha) \in C(\Gamma^n) = C(N,\alpha)$. □

For deterministic convex games it is well known that each of its subgames is convex and consequently has a nonempty core. A similar result can be derived for games with random payoffs.

Theorem 4.5 *Let (N,α) be a cooperative game with random payoffs. If it is marginal convex then so are all of its subgames.*

Proof. Let T be a coalition and consider the bijection $\sigma' \in \Pi(T)$. Take $\sigma \in \Pi(N)$ such that the players in T are first and in the same order as in σ', i.e. $\sigma'(i) = \sigma(i)$ for $i = 1, \ldots, |T|$. These identical orders imply

$$m_i^\sigma(\alpha) = \alpha_i(T, N, m_i^{\sigma'}(\alpha)) \text{ for all } i \in T, \tag{4}$$

the marginal contributions are equivalent. Because the game (N,α) is marginal convex $m^\sigma(\alpha) \in C(N,\alpha)$. This means that there exists no coalition $S \subset N$ such that $m^\sigma(\alpha) \in \text{dom}(S)$. Since $T \subset N$ the previous statement also holds for all $S \subset T$. In other words, for all coalitions $S \subset T$ there exists no vector $q \in \Delta^*(S)$ such that for all players i in S

$$\alpha_i(S,N,q_i) > m_i^\sigma(\alpha) = \alpha_i(T,N,m_i^{\sigma'}(\alpha)) \Leftrightarrow \alpha_i(S,T,q_i) > m_i^{\sigma'}(\alpha)$$

where (4) is used. Therefore $m^{\sigma'}(\alpha) \notin \text{dom}(S)$ for any coalition $S \subset T$. We conclude that $m^{\sigma'}(\alpha) \in C(T,\alpha)$ □

An immediate consequence of this theorem is that any marginal convex game and all of its subgames have a nonempty core.

For two-person games all three types of convexity are equivalent. In particular it holds that marginal convex games are individual-merge convex. The following example shows that this need not hold for games with three or more players. Because coalitional-merge convex games are by definition individual-merge convex, it follows immediately from the next example that a marginal convex game also need not be coalitional-merge convex.

Example 4.6 Consider the game (N,α) with $N = \{1,2,3\}$, $\alpha_i(\{i\},T,p) = 0$ and the values $\alpha_i(S,N,p)$ for various i and S are given in the table below.

$i \setminus S$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
1	$\frac{1}{2}p$	$\frac{1}{3}p$	—
2	$\frac{3}{10}p$	—	$\frac{3}{5}p$
3	—	$\frac{1}{7}p$	$\frac{3}{7}p$

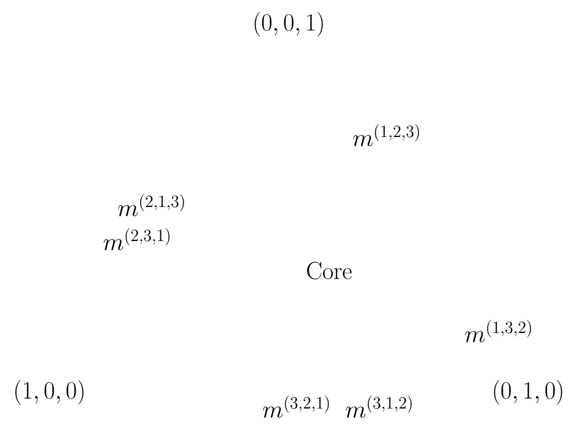

Figure 1: Marginal vectors and the core.

The core of this game equals

$$\begin{aligned} C(N,\alpha) = \{p \in \Delta^*(N) | p_i \geq 0 \text{ for all } i \in N, \\ 6p_1 + 10p_2 \geq 3,\ 6p_1 + 14p_3 \geq 2,\ 10p_2 + 14p_3 \geq 6\}. \end{aligned}$$

In figure 1 this core and all marginal vectors are shown. As can be seen, all marginal vectors belong to the core. Hence, this game is marginal convex.

This game is not individual-merge convex. To show this, take $i = 1$, $S = \{2\}$ and $T = \{2, 3\}$. Further, let $p = 1 \in IR(S)$, $q = (1, 0) \in IR(T)$ and $r = (1, 0) \in IR(S \cup \{i\}) = IR(\{1, 2\})$. If this game would be individual-merge convex then there would exist an allocation $s \in \Delta^*(N)$ such that

$$\begin{cases} s_1 \geq \alpha_1(S \cup \{i\}, N, r_1), \\ s_2 \geq \alpha_2(T, N, q_2), \\ s_3 \geq \alpha_3(T, N, q_3), \end{cases}$$

or, equivalently,

$$\begin{cases} s_1 & \geq 1/2, \\ s_2 & \geq 3/5 \\ s_3 & \geq 0. \end{cases}$$

Figure 2 shows these inequalities. They imply that $s_1 + s_2 + s_3 \geq 1/2 + 3/5 + 0 > 1$, which is in contradiction to $s \in \Delta^*(N)$. Hence, this game is not individual-merge convex. $\square$

By definition it holds that coalitional-merge convex games are individual-merge convex. One can easily see that the reverse relation will hold if the game has two players. The following theorem shows the same result for games with three players.

$s_2 \geq 0.6$

$s_1 \geq 0.5$

$s_3 \geq 0$

Figure 2: Contradictory inequalities for a non-individual-merge convex game.

Theorem 4.7 *Let (N, α) be a cooperative game with random payoffs and with three players. If the game is individual-merge convex, then it is coalitional-merge convex.*

Proof. Let (N, α) be a three-person game that is individual-merge convex and let $U \subset N$. If $U = \emptyset$ then the condition for coalitional-merge convexity is trivially satisfied. If $|U| = 1$ then $U = \{i\}$ for some $i \in N$ and we are done because of the individual-merge convexity.
If $|U| = 2$ then $|N \setminus U| = 1$. Assume that $N \setminus U = \{i\}$. The conditions $S \subset T \subset N \setminus U$ and $S \neq T$ imply that $S = \emptyset$ and $T = \{i\}$. Let $q \in IR(T)$ and $r \in IR(U)$. Then $q = 1$.
Define $S' = \emptyset$ and $T' = U$. Then $1 \in IR(\{i\})$ and $r \in IR(T')$. By individual-merge convexity there exists an allocation $s' \in \Delta^*(N)$ such that

$$\begin{cases} s'_j \geq \alpha_j(T', N, r_j) = \alpha_j(S \cup U, N, r_j) & \text{for all } j \in T' = U \\ s'_i \geq \alpha_i(S' \cup \{i\}, N, 1) = \alpha_i(T, N, 1). \end{cases}$$

This way, coalitional-merge convexity is satisfied. If $|U| = 3$ then there is nothing to check. We conclude that this game is coalitional-merge convex. $\square$

In case of four or more players, we were neither able to prove that individual-merge convex games are coalitional-merge convex nor could we find a counterexample. Hence, at this moment it remains an open problem.

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