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Output deadbeat control using state feedback of discrete-time multivariable systems

JACOB C. ENGWERDA†

In this note the problem of output deadbeat regulation with internal stability using state feedback is solved in its full generality. That is, the only assumption made is that the system is described by a linear time-invariant recurrence equation. By making an appropriate state-space decomposition it is shown that the results obtained by Kimura *et al.* (1981) can be generalized in a straightforward way. An advantage of this approach is that it facilitates an easy proof of the basic solvability condition, gives a good insight into the basics of the problem, and provides a synthesis procedure to construct a minimal-time deadbeat controller. The approach is used to parametrize a set of weight matrices which turn the corresponding minimum variance controller into a minimal-time deadbeat one.

1. Introduction

Deadbeat control problems have been studied for a long time. In the sampled-data control theory the deadbeat regulation has been regarded as a simple and effective control policy. In particular in the design of tracking systems, for systems where control cost are of minor importance, fast settling time properties for the system are desired. Possible applications may be found in macroeconomic control problems and robot control systems. As an example, consider the stabilization of a simple multiplier–accelerator model as developed by Hicks (1950) and Samuelson (1939). The Hicks–Samuelson approach is based on

$$C(k) = C_0 + cY(k-1)$$

$$I(k) = I_0 + v(Y(k-1) - Y(k-2))$$

according to which, the real consumption C depends on the last period's real income Y , whereas real investment I adjusts fully to lagged changes in income (Turnovsky 1979, pp. 319, 354). Assuming that the product market equilibrium holds in each period:

$$Y(k) = C(k) + I(k) + G(k)$$

where G is real government expenditure, the question is to choose $G(\cdot)$ such that income is stabilized about the target level $\bar{Y}$ as fast as possible, subject to the dynamics specified above. Now introduce the variables $\Delta y(k) = Y(k) - \bar{Y}$ and $\Delta g(k) = G(k) - \bar{G}$, where $\bar{G} := (1-c)\bar{Y} - C_0 - I_0$ defines the equilibrium level of government expenditure needed to achieve $\bar{Y}$ on average. Then the system can be transformed into

$$\Delta y(k) = c\Delta y(k-1) + v(\Delta y(k-1) - \Delta y(k-2)) + \Delta g(k)$$

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or equivalently

$$x(k) = \begin{bmatrix} c + v & -v \\ 1 & 0 \end{bmatrix} x(k-1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k-1)$$

$$y(k) = (1 \quad 0)x(k)$$

where $x(k) := (\Delta y(k) \quad \Delta y(k-1))^T$ and $u(k-1) := \Delta g(k)$.

The question considered in this paper is how to design a state feedback controller $u(k) = Fx(k)$ such that the output $y(k)$ of such a system is forced to zero and kept at zero for any initial condition of the system in a minimal number of time steps; and, moreover, that the internal state of the system is stabilized. The last requirement is made, of course, for robustness reasons.

In an early contribution Bergen and Ragazzini (1954) designed a z-transform regulator which makes the error in the response to a step input zero at the sampling instants after a finite number of time steps. Kalman (1960) considered the design of a linear state feedback that transfers the state of a single-input sampled-data system from its initial state to the origin in a minimum number of time steps. Using Brunovsky's state space canonical form, Porter and Bradshaw (1975) showed how Kalman's result to design a minimal state deadbeat controller can be generalized for a multi-input controllable system. The problem of zeroing the output of a single-input single-output sampled-data system in a minimum number of time steps, using a linear state feedback, was first formulated and solved by Kuçera (1972). Leden (1977) extended Kuçera's result for multivariable systems, whereas Akashi and Imai (1978) considered a variant of this problem. They considered the design of a controller which settles the output in zero as fast as possible under the additional requirement that the applied control converges to zero if time goes to infinity. A more general formulation of this problem that we will describe below, was considered by Kimura and Tanaka (1981). Eising (1984) considered the deadbeat control problem from a numerical point of view. He presented numerically reliable algorithms, allowing for a trade-off between the size of the feedback gains and the order of nilpotency of the regulated system. The generalization of the deadbeat problem for a generalized state-space system was first considered by Beelen and van Dooren (1988). They suggested a method to design semistate deadbeat control for regular generalized state-space systems. Emami-Naeini (1992) treated the general (singular) case and addressed both the semistate and output deadbeat control problems using an optimal control formulation, as did Marrari *et al.* (1989).

Marrari *et al.* discussed in their paper the problem of designing a state feedback which in minimum time drives the outputs of the system to zero. In their paper two algorithms are presented which partially solve the problem in the case when the system is completely reachable. The problem is only partially solved in the sense that their algorithms force the output to zero, within a time which is in general smaller than the time that a state feedback controller needs to force all states to zero. Their claim that this time period is minimal seems, however, premature. A serious proof is lacking. Since the design of an appropriate weighted minimum variance controller seems to be rather involved it seems reasonable first to answer the question of under which conditions a minimal-time state feedback deadbeat output regulator, with no additional restrictions on control structure, exists.

Kimura and Tanaka (1981) solved the problem of output deadbeat regulation with internal stability using state feedback, for a linear discrete-time system

described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(k)$$

$$z(k) = D_1 x_1(k) + D_2 x_2(k)$$

under the assumptions that B_1 is monic, D_1 is epic, (D_2, A_2) is observable, (A_1, B_1) is controllable, (D_1, A_1) is observable and the spectrum of A_2 , i.e. $\sigma(A_2)$, contains no good modes. It is obvious that by introducing the auxiliary state variable $x^T(k) := (x_1^T(k) \ x_2^T(k))$, this system is a special type of the general discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k)$$

Though many systems can be described by the model considered by Kimura and Tanaka, it is from a theoretical point of view more satisfactory to have solvability conditions for the output deadbeat regulation problem with respect to the general discrete-time system.

Therefore, this paper considers the output deadbeat regulation with internal stability using state feedback in its full generality. First both necessary and sufficient conditions for the existence of a state feedback controller which drives the output to zero in minimum time are presented, and then an algorithm to construct such a minimum-time deadbeat controller is given.

Apart from the fact that this result is more general than that of Kimura and Tanaka (1981), an additional advantage of considering the state-space representation in its full generality is that now optimally a state-space decomposition technique can be exploited in constructing a set of minimal-time state feedback deadbeat controllers. Subsequently it is shown that this set of controllers can also be obtained by making appropriate choices for the weight matrix in the minimum variance controller.

2. Preliminaries

In this section the problem is formulated and some notation and well-known concepts are introduced. The problem is as follows.

Given the finite-dimensional, linear, time-invariant, discrete-time system described by the difference equation

$$\left. \begin{aligned} x(k+1) &= \mathbf{A}x(k) + \mathbf{B}u(k), \quad x(0) = x \\ y(k) &= \mathbf{C}x(k) \end{aligned} \right\} \quad (1)$$

where $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^m$ and $y(\cdot) \in \mathbb{R}^q$, the problem is to find a linear state feedback $u(k) = \mathbf{F}x(k)$ such that the outputs are forced to zero and kept at zero for any initial state x in a minimum number of time steps μ , i.e.

$$y(k) = \mathbf{C}x(k) = \mathbf{C}(\mathbf{A} + \mathbf{B}\mathbf{F})^k x = 0, \quad \forall x, k \geq \mu$$

with μ as small as possible. Moreover, it is required that $\sigma(\mathbf{A} + \mathbf{B}\mathbf{F}) \subset \mathbb{C}_g$, where $\mathbb{C}_g$ is some prespecified part of the complex plane containing zero. The smallest possible integer for which this can be accomplished is referred to as the minimal settling time, and the corresponding map is a minimal-time controller. Note that this integer may depend on the choice of $\mathbb{C}_g$.

Note that whenever a controller exists that makes the output deadbeat, a minimal-time controller exists.

To solve this problem, a state-space decomposition of system (1) is made. This decomposition uses the notions of the reachability and a particular stabilizability subspace. These concepts are formalized in Definition 1. That they are well defined can be seen in Hautus and Heymann (1980).

Definition 1: The *reachability subspace* denoted by $\langle \mathbf{A} | \text{Im } \mathbf{B} \rangle$ is equal to

$$\text{Im } \mathbf{B} + \mathbf{A} \text{Im } \mathbf{B} + \cdots + \mathbf{A}^{n-1} \text{Im } \mathbf{B}$$

where $\text{Im } \mathbf{B}$ is the image of the matrix $\mathbf{B}$. V is called a *stabilizability subspace* (with respect to $\mathbb{C}_g$) if, for any $\mathbf{x}(0) \in V$, there exists a Bohl function $\mathbf{u}(\cdot)$ such that $\mathbf{x}(0, k, \mathbf{x}(0), \mathbf{u}) \in V$, for all $k \geq 0$ and $\mathbf{x}(0, \cdot, \mathbf{x}(0), \mathbf{u})$ is stable (with respect to $\mathbb{C}_g$). $V_g^*(\ker \mathbf{C})$ denotes the largest stabilizability subspace (with respect to $\mathbb{C}_g$) that is contained in the kernel ($\ker$) of $\mathbf{C}$. A subspace V is called *M-invariant* if $MV \subset V$. A subspace V is called *controlled invariant* if $\mathbf{A}V \subset V + \text{Im } \mathbf{B}$. An eigenvalue λ is called *controllable* if $\text{rank } [\mathbf{A} - \lambda \mathbf{I} | \mathbf{B}] = n$. An eigenvalue λ is called *observable* if

$$\text{rank} \begin{bmatrix} \mathbf{A} - \lambda \mathbf{I} \\ \mathbf{C} \end{bmatrix} = n \quad \square$$

The following two lemmas, with respect to the reachability subspace and the largest stabilizability subspace, are well known (Wonham 1979).

Lemma 1: $\langle \mathbf{A} | \text{Im } \mathbf{B} \rangle$ is $\mathbf{A}$ -invariant. $V_g^*(\ker \mathbf{C})$ is controlled invariant.

Lemma 2: A subspace V is a stabilizability subspace with respect to $\mathbb{C}_g$ if and only if (iff) there exists a map F that satisfies

$$(\mathbf{A} + \mathbf{B}F)V \subset V \quad \text{and} \quad \sigma(\mathbf{A} + \mathbf{B}F | V) \subset \mathbb{C}_g$$

For the derivation of results it will appear to be convenient to first apply a feedforward $\mathbf{u} = K\mathbf{x} + \mathbf{u}'$ to (1), where K makes $V_g^*(\ker \mathbf{C})$ $\mathbf{A} + \mathbf{B}K$ -invariant and the spectrum of $\mathbf{A} + \mathbf{B}K$ with respect to $V_g^*(\ker \mathbf{C})$ is contained in $\mathbb{C}_g$. So without loss of generality it can be assumed that the spectrum of $\mathbf{A} + \mathbf{B}K$ with respect to $V_g^*(\ker \mathbf{C}) \cap \langle \mathbf{A} | \text{Im } \mathbf{B} \rangle$ is contained in $\mathbb{C}_g$. So, instead of the system (1), look at

$$\left. \begin{aligned} \mathbf{x}'(k+1) &= (\mathbf{A} + \mathbf{B}K)\mathbf{x}'(k) + \mathbf{B}\mathbf{u}'(k), & \mathbf{x}'(0) &= \mathbf{x} \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}'(k) \end{aligned} \right\} \quad (2)$$

Now consider the following state space decomposition:

$$X_1 := \langle \mathbf{A} | \text{Im } \mathbf{B} \rangle \cap V_g^*(\ker \mathbf{C})$$

$$X_1 \oplus X_2 := V_g^*(\ker \mathbf{C})$$

$$X_1 \oplus X_3 := \langle \mathbf{A} | \text{Im } \mathbf{B} \rangle$$

$$X_1 \oplus X_2 \oplus X_3 \oplus X_4 := \mathbb{R}^n$$

Application of the invariance properties from Lemmas 1 and 2 then yields the next state-space decomposition.

Theorem 1: Let $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4$. With a basis adapted to this decomposition

system, (2) is described by the recurrence equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = (0 \quad 0 \quad C_3 \quad C_4) \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

where

$$\left\{ \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} \right\}$$

is reachable, and $\sigma(A_{11}) \subset \mathbb{C}_g$.

Using this basic result the next section shows how the general problem can be solved in a straightforward way.

3. Problem solution

In this section the central problem of this paper is solved. First a necessary and sufficient condition for the existence of a minimal-time controller is given. This condition generalizes the result obtained by Kimura and Tanaka (1981, Theorem 1), where the same problem was considered for the system mentioned in the introduction.

Theorem 2: *There exists a minimal-time deadbeat-output state-feedback controller iff*

$$\sigma[\mathbf{A} | \mathbb{R}^n / (\langle \mathbf{A} | \text{Im } \mathbf{B} \rangle + V_g^*(\ker \mathbf{C}))] = \{0\}$$

Proof

Only if. Consider the state-space decomposition introduced in Theorem 1. From this decomposition it is clear that

$$\sigma[\mathbf{A} | \mathbb{R}^n / (\langle \mathbf{A} | \text{Im } \mathbf{B} \rangle + V_g^*(\ker \mathbf{C}))] = \sigma(A_{44})$$

(note that this is independent of the chosen preliminary feedback in system (2)). Now let λ be an eigenvalue of A_{44} . Then λ is also an eigenvalue of $\mathbf{A}$. Since the eigenvalue λ cannot be removed by feedback, it must belong to $\mathbb{C}_g$. Now let $\mathbf{x}$ be an eigenvector corresponding with $\lambda \neq 0$. Then $\mathbf{C}\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{C}\mathbf{x} = 0$. Consequently $\mathbf{x} \in V_g^*(\ker \mathbf{C})$, which contradicts the assumption that $\lambda \in \sigma(A_{44})$. So the stated condition is a necessary one.

If. To prove that the condition is sufficient too, we take $F = (0 \quad 0 \quad F_3 \quad 0)$, where F_3 is such that $A_{33} + B_3F_3$ is nilpotent (this is possible since (A_{33}, B_3) is reachable). Using this state feedback controller simple calculation shows that the system becomes deadbeat-output, which completes the proof. $\square$

Note that Theorem 2 merely states an existence result. In general the resulting time index from which the output is always zero will not be minimal for the chosen state-feedback controller.

To obtain a characterization for the minimal settling time, an algorithm based on the analysis performed by Kimura and Tanaka (1981) will be given. To that end consider the following sequence of subspaces: $T_0, T_1, \dots$

$$\left. \begin{aligned} T_0 &:= V_g^*(\ker \mathbf{C}) \\ T_i &:= \mathbf{A}^{-1}(T_{i-1} + \text{Im } \mathbf{B}), \quad i \geq 1 \end{aligned} \right\} \quad (3)$$

The interpretation of subspace T_i is that it contains all initial states which can be brought in T_0 at time i , by applying some input sequence $u(0), u(1), \dots, u(i-1)$.

Using the following property, an estimate can easily be obtained for the minimal settling time.

Lemma 3: *Assume that there exists a minimal-time deadbeat-output state-feedback controller K and that the minimal settling time is k . Then $T_k = \mathbb{R}^n$.*

Proof: It will be shown by induction on i that

$$\text{Im } (\mathbf{A} + \mathbf{BK})^{k-i} \subset T_i, \quad i = 0, 1, \dots, k \quad (4)$$

If $i = 0$, it must be proved that $\text{Im } (\mathbf{A} + \mathbf{BK})^k \subset V_g^*(\ker \mathbf{C})$. This inclusion, however, is trivially satisfied, since, by assumption, $\mathbf{C}(\mathbf{A} + \mathbf{BK})^k = 0$ and $\sigma(\mathbf{A} + \mathbf{BK}) \subset \mathbb{C}_g$. So, assume now that the inclusion holds for $i = \tau$, i.e. $\text{Im } (\mathbf{A} + \mathbf{BK})^{k-\tau} \subset T_\tau$. From this inclusion it is immediately clear that

$$(\mathbf{A} + \mathbf{BK}) \text{Im } (\mathbf{A} + \mathbf{BK})^{k-\tau-1} \subset T_\tau$$

which implies that $\text{Im } (\mathbf{A} + \mathbf{BK})^{k-\tau-1} \subset \mathbf{A}^{-1}(T_\tau + \text{Im } \mathbf{B})$. Since by definition

$$\mathbf{A}^{-1}(T_\tau + \text{Im } \mathbf{B})$$

is equal to $T_{\tau+1}$, the induction argument is complete with this. The stated result now follows by taking $i = k$ in (4). $\square$

Corollary 1: *The minimal settling time is not smaller than*

$$k_s := \min_i \{i \mid T_i = \mathbb{R}^n\}$$

In fact the minimal settling time is always equal to this number k_s . This is the content of the next algorithm.

Algorithm 1: Let the subspace $\tilde{T}_i$, $i = 0, \dots, k$, be defined as follows:

$$\begin{aligned} \tilde{T}_0 &:= T_0 \\ T_{i-1} \oplus \tilde{T}_i &:= T_i, \quad i = 1, \dots, k_s \end{aligned}$$

Next define maps G_i that satisfy

$$G_0 := 0 \quad \text{and} \quad (\mathbf{A} + \mathbf{BG}_i)\tilde{T}_i \subset T_{i-1}, \quad i = 1, \dots, k_s \quad (5)$$

Then any map G defined by

$$G|_{\tilde{T}_i} = G_i|_{\tilde{T}_i}, \quad i = 0, \dots, k_s$$

solves the minimal time deadbeat-output state-feedback control problem.

Proof: The proof is along the lines of Theorem 2 in Kimura and Tanaka (1981). First note that, since $V_g^*(\ker \mathbf{C})$ is controlled invariant, it is easily shown that the inclusion $T_{i-1} \subset T_i$, $i = 1, \dots, k_s$ holds. So the definition of $\tilde{T}_i$ makes sense. Moreover, since $\mathbf{A}\tilde{T}_i \subset T_{i-1} + \text{Im } \mathbf{B}$ it is clear that a map G_i always exists that satisfies (5).

So what is left to be shown is that $(\mathbf{A} + \mathbf{BG})^{k_s} \subset V_g^*(\ker \mathbf{C})$. To that end first note that

$$(\mathbf{A} + \mathbf{BG})T_i = (\mathbf{A} + \mathbf{BG})(\tilde{T}_0 + \cdots + \tilde{T}_i) \subset \sum_{j=1}^i (\mathbf{A} + \mathbf{BG}_j)\tilde{T}_j \subset \sum_{j=1}^{i-1} T_j \subset T_{i-1}$$

Consequently

$$(\mathbf{A} + \mathbf{BG})^{k_s}T_{k_s} \subset (\mathbf{A} + \mathbf{BG})^{k_s-1}T_{k_s-1} \subset \cdots \subset V_g^*(\ker \mathbf{C})$$

As $T_{k_s} = \mathbb{R}^n$, this completes the proof. $\square$

Combination of the results of Lemma 3 and Algorithm 1 then yields the next theorem.

Theorem 3: *The minimal settling time equals*

$$\min_i \{i \mid T_i = \mathbb{R}^n\}$$

4. The relationship with least-squares minimization

This section considers the connection between minimum variance (MV) control and the deadbeat-output state-feedback control problem.

Using the results of the preceding section, it is shown that there always exists a minimal-time deadbeat-output state-feedback controller within the set of controllers resulting from the minimization of the MV cost criterion

$$\mathbf{J} := \mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k), \quad \text{with respect to } \sum : \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k+1) + \mathbf{B}\mathbf{u}(k)$$

by making an appropriate choice of the weight matrix $\mathbf{Q}$.

In fact a set of deadbeat-output state-feedback controllers is first parametrized that satisfies the requirement that every state component is controlled as fast as possible to zero (in the sense discussed in Note 1 below).

Then this parametrization is used to present a set of minimal-time deadbeat-output MV-controllers.

The following lemma is crucial in deriving the results. Since its proof is somewhat technical, it is deferred to the Appendix. In the lemma the notion of a Jordan block is used. Recall that a matrix $\mathbf{D} \in \mathbb{R}^{n \times n}$ is called a Jordan block of order n if it has the form

$$\begin{bmatrix} a & & 1 & & \\ & \ddots & & \ddots & \\ & & a & & 1 \\ & & & a & \\ & & & & a \end{bmatrix}$$

Lemma 4: *Let*

$$\mathbf{A} := \begin{bmatrix} J_1 & \mathbf{D} \\ 0_{m \times n} & J_2 \end{bmatrix}$$

where J_1 and J_2 are nilpotent Jordan blocks of order n and m , respectively; $0_{m \times n} \in \mathbb{R}^{m \times n}$

is the zero matrix, and $\mathbf{D} \in \mathbb{R}^{n \times m}$ has the form

$$\mathbf{D} := \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \\ d_{n-11} & \cdots & d_{n-1n} \\ f_1 & \cdots & f_m \end{bmatrix}$$

Denote $\max(n, m)$ by $\bar{r}$, $\min(n, m)$ by $\underline{r}$, $\max(0, m - n)$ by $\bar{s}$, and $\max(0, n - m)$ by $\underline{s}$. Then

$$\mathbf{A}^{\bar{r}} = \left[\begin{array}{c|c|c} 0_{n \times n} & 0_{n \times \bar{s}} & \bar{\mathbf{D}}_{\bar{r}} \\ \hline & & 0_{\underline{s} \times m} \\ \hline 0_{m \times n} & & 0_{m \times m} \end{array} \right]$$

where $\bar{\mathbf{D}}_{\bar{r}} \in \mathbb{R}^{(m - \bar{s}) \times (n - \underline{s})}$ equals

$$\begin{bmatrix} f_1 & f_2 + d_{n-11} & f_3 + d_{n-12} + d_{n-21} & \cdots & f_{\underline{r}} + \sum_{i=1}^{\underline{r}-1} d_{n-i\underline{r}-i} \\ 0 & & & & \vdots \\ \vdots & & & & \\ & & & f_3 + d_{n-12} + d_{n-21} & \\ & & & f_2 + d_{n-11} & \\ 0 & \cdots & 0 & f_1 & \end{bmatrix}$$

(Note that either $0_{n \times \bar{s}}$ or $0_{\underline{s} \times m}$ does not exist and $\bar{\mathbf{D}}_{\bar{r}}$ is always a square matrix.)

Corollary 2: Assume that in Lemma 4 the parameters $f_1, \dots, f_m$ can be arbitrarily chosen. Then $\mathbf{A}$ is nilpotent of minimal order $\bar{r}$ iff $f_1 = 0$, $f_2 = -d_{n-11}, \dots$,

$$f_{\bar{r}} = - \sum_{i=1}^{\underline{r}-1} d_{n-i\underline{r}-i}$$

Proof: That the order of nilpotency of $\mathbf{A}$ is not less than $\bar{r}$ is obvious. On the other hand it is clear from Lemma 4 that $\mathbf{A}^{\bar{r}} = 0$ iff the parameters $f_1, \dots, f_{\underline{r}}$ are chosen as stated above. $\square$

Remark 1: It is easily verified that if the Jordan block J_2 in Lemma 4 is replaced by a nilpotent Jordan matrix containing k Jordan blocks J_i of order p_i , $i = 1, \dots, k$, the content of this lemma remains almost the same. All that changes is that the right-upper block of matrix $\mathbf{A}^{\bar{r}}$ is replaced by $kn \times p_i$ blocks of that form.

The parametrization of all deadbeat-output controllers is obtained by transforming the transformed system discussed in Theorem 1 once again. To avoid some technicalities, assume, without loss of generality, that the matrix B_3 in Theorem 1 is injective.

Lemma 5: There exists a similarity transformation S such that (2) is described by the

recurrence equation

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \\ \bar{x}_3(k+1) \\ \bar{x}_4(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \bar{A}_{13} & \bar{A}_{14} \\ 0 & A_{22} & 0 & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{33} & \bar{A}_{34} \\ 0 & 0 & 0 & \bar{A}_{44} \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \\ \bar{x}_3(k) \\ \bar{x}_4(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \\ \bar{B}_3 \\ 0 \end{bmatrix} \bar{u}(k) \quad (6)$$

$$y(k) = (0 \quad 0 \quad \bar{C}_3 \quad \bar{C}_4) \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \\ \bar{x}_3(k) \\ \bar{x}_4(k) \end{bmatrix}$$

where

$$\left\{ \begin{bmatrix} A_{11} & \bar{A}_{13} \\ 0 & \bar{A}_{33} \end{bmatrix}, \begin{bmatrix} \bar{B}_1 \\ \bar{B}_3 \end{bmatrix} \right\}$$

is reachable, $\sigma(A_{11}) \subset \mathbb{C}_g$, and $\bar{A}_{33}$, $\bar{B}_3$ and $\bar{A}_{44}$ are as follows:

$$\bar{A}_{33} := (E_{i1}), \quad \bar{B}_3 := (\underline{e}_{1i})$$

with

$$E_{i1} := n_i \begin{bmatrix} \overbrace{0 \cdots 0}^{n_i} & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_{i1} & & & & & & a_{in} \end{bmatrix}$$

and $\underline{e}_{1i}$ the $(n_1 + \cdots + n_i)$ th standard unit vector in $\mathbb{R}^n$, $i = 1, \dots, l$ (here $\underline{n}_i := n_0 + \cdots + n_{i-1} + 1$, with $n_0 := 0$)

$$\bar{A}_{44} := \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

with $J_i \in \mathbb{R}^{p_i \times p_i}$ nilpotent Jordan blocks of order p_i . Here $n_1 \geq n_2 \geq \cdots \geq n_l \geq 1$ with

$$\sum_{i=1}^l n_i = n$$

are the controllability indices, with n_1 as the controllability index; a_{ij} denotes an arbitrary real number, and $p_1 \geq \cdots \geq p_k \geq 1$ with

$$\sum_{i=1}^k p_i = m$$

Proof: Since (A_{33}, B_3) is reachable, it follows from Luenberger (1967) that there exist non-singular transformation matrices S_1 and T such that $\bar{A}_{33} = S_1 A_{33} S_1^{-1}$ and $\bar{B}_3 = S_1 B_3 T$. Furthermore, since A_{44} is nilpotent, there exists a non-singular

transformation matrix S_2 such that $\bar{A}_{44} = S_2 A_{44} S_2^{-1}$. Applying the similarity transformation

$$\mathbf{x}(k) := \begin{bmatrix} I & & & \\ & I & & \\ & & S_1^{-1} & \\ & & & S_2^{-1} \end{bmatrix} \bar{\mathbf{x}}(k)$$

and input transformation $\mathbf{u}(k) = T\bar{\mathbf{u}}(k)$ then yields the result. $\square$

Theorem 4: Consider the transformed system (6). Let

$$\bar{A}_{34} := \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nm} \end{bmatrix}$$

For this system $\bar{\mathbf{u}}(k) = (0 \ 0 \ F_3 \ F_4)\bar{\mathbf{x}}(k)$ is a minimal-time deadbeat-output controller if

$$F_3 := \begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & & \vdots \\ -a_{l1} & \cdots & -a_{ln} \end{bmatrix} \quad \text{and} \quad F_4 := (f_{st}), \quad s = 1, \dots, l, t = 1, \dots, k$$

where $f_{st} \in \mathbb{R}^{1 \times p_s}$ is given by

$$f_{st} := \left[-d_{\underline{n}_s \underline{p}_{t-1} + 1} \cdots - \sum_{i=1}^{r_{st}-1} d_{\underline{n}_s - i \underline{p}_{t-1} + r_{st} - i} * \cdots * \right]$$

Here

$$\underline{n}_s := \sum_{i=1}^s n_i, \quad \underline{p} := \sum_{j=1}^i p_j, \quad \underline{p}_0 = 0, \quad r_{ij} := \min(n_i, p_j)$$

and $*$ denotes a parameter that can be chosen arbitrarily. The minimal settling time is equal to $\max(n_1, p_1)$.

Proof: From (3) it is clear that there exist states in X_3 which can be brought in $V_g^*(\ker \mathbf{C})$ by applying an appropriate input sequence whose minimal length is equal to the controllability index n_1 of $(\bar{A}_{33}, \bar{B}_3)$. Similarly, it is obvious that there exist states in X_4 which, whatever control sequence is used, have a minimal settling time of p_1 . In other words, the minimal settling time is at least $\max(n_1, p_1)$. Using the results of Corollary 2 (and Remark 1) it is now a matter of straightforward calculation to show that the advertised feedback controller F yields a closed-loop matrix of the form

$$\begin{bmatrix} A_{11} & A_{12} & \bar{A}_{13} + \bar{B}_1 F_3 & \bar{A}_{14} + \bar{B}_1 F_4 \\ 0 & A_{22} & 0 & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{33} + \bar{B}_3 F_3 & \bar{A}_{34} + \bar{B}_3 F_4 \\ 0 & 0 & 0 & \bar{A}_{44} \end{bmatrix} \quad (7)$$

where the matrix

$$\begin{bmatrix} \bar{A}_{33} + \bar{B}_3 F_3 & \bar{A}_{34} + \bar{B}_3 F_4 \\ 0 & \bar{A}_{44} \end{bmatrix}$$

is nilpotent of order $\max(n_1, p_1)$. So this controller makes the system deadbeat in $\max(n_1, p_1)$ time steps, and moreover, does not affect the eigenvalues of A_{11} and A_{22} . Consequently, it is an appropriate minimal-time deadbeat controller. $\square$

Note 1: Note that the construction of Corollary 2 shows that if one considers the following decomposition of the state variables

$$\begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}^T = (v_1^T v_2^T \cdots v_l^T w_1^T \cdots w_k^T)$$

where each vector $v_i, i = 1, \dots, l$, contains n_i variables and $w_i, i = 1, \dots, k$, contains p_i variables, then all variables v_i and w_i are minimal deadbeat (i.e. $v_i(t) = 0, \forall \bar{x}$ and $t \geq s_i$, with s_i as small as possible, $i = 1, \dots, l$ (and similarly for w_i)).

Next, the obtained results are linked with MV state feedback deadbeat output control by suggesting an appropriate choice of the weight matrix for the MV controller. To that end, first a preliminary result is proved.

Lemma 6: Consider (1) (with, without loss of generality, B injective) and a system obtained from (1) by applying a state transformation $\bar{x} = Sx$ and input transformation $\bar{u} = Tu$. Assume that (1) and its transformation both are controlled by a MV-regulator with weight matrix Q and $\bar{Q}$, respectively. Then both closed-loop systems are identical if $\bar{Q} = SQS^{-1}$.

Proof: It is easily verified that the closed-loop system of the MV-controlled system (1) is equal to

$$x(k+1) = (I - B(B^TQB)^{-1}B^TQ)Ax(k)$$

For the transformed closed-loop system one similarly obtains

$$\bar{x}(k+1) = (I - \bar{B}(\bar{B}^T\bar{Q}\bar{B})^{-1}\bar{B}^T\bar{Q})\bar{A}\bar{x}(k)$$

Since $\bar{B} = SBT$, $\bar{A} = SAS^{-1}$, $\bar{x} = Sx$ and $\bar{Q} = SQS^{-1}$ it is easily seen by substitution of these transformed variables into the transformed closed-loop expression that the closed-loop system displayed above is reobtained. $\square$

A consequence of this lemma is that to construct an appropriate minimal-time output-deadbeat state-feedback MV-controller it suffices to design such a controller for the transformed system (6).

Theorem 5: Choose the weight matrix in the MV-controller as follows (using the notation of Lemma 5 and Theorem 4):

$$Q := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & Q_{34} \\ 0 & 0 & Q_{34}^T & Q_{44} \end{bmatrix}$$

where Q_{44} can be chosen arbitrarily and Q_{34} has the following structure: $Q_{34} := (\tilde{Q}_{st})$.

$s = 1, \dots, l, t = 1, \dots, k$, with

$$\tilde{Q}_{st} := n_s \begin{bmatrix} & * & \dots & & * \\ & \vdots & & & \vdots \\ & * & \dots & \dots & * \\ d_{\underline{n}_s - 1 \underline{p}_{t-1} + 1} & \dots & \sum_{i=1}^{r_{st}-1} d_{\underline{n}_s - i \underline{p}_{t-1} + r_{st} - i} & * & \dots & * \end{bmatrix}$$

where $\underline{n}_s$ is used to denote

$$\sum_{i=1}^s n_i$$

and p_i to denote

$$\sum_{j=1}^i p_j, \quad i = 1, \dots, t-1$$

and $p_0 := 0$.

Then the MV-controller yields a closed-loop system that is identical to (7). Consequently this MV-controller is minimal-time output-deadbeat too, and it has the same additional transient deadbeat minimality property as mentioned in Remark 1.

Before the proof of this theorem is given, first note that the matrix Q_{34} is obtained from F_4 in Theorem 4 as follows. Consider $-F_4$. Drop the first term of every element in this matrix and consecutively shift each p_i -block one to the left and add at the end of each block an arbitrary element (so that it contains again p_i elements). Choose row number n_i of Q_{34} equal to row number i of the matrix constructed above, and all other rows of Q_{34} arbitrarily.

Proof: It is easily verified that using MV-control with $\mathbf{Q}$ as indicated above yields the closed-loop system matrix

$$\begin{bmatrix} A_{11} & A_{12} & \bar{A}_{13} - \bar{B}_1 \bar{B}_3^T \bar{A}_{33} & \bar{A}_{14} - \bar{B}_1 \bar{B}_3^T \bar{A}_{34} - \bar{B}_1 \bar{B}_3^T Q_{34} \bar{A}_{44} \\ 0 & A_{22} & 0 & \bar{A}_{24} \\ 0 & 0 & (I - \bar{B}_3 \bar{B}_3^T) \bar{A}_{33} & (I - \bar{B}_3 \bar{B}_3^T) \bar{A}_{34} - \bar{B}_3 \bar{B}_3^T Q_{34} \bar{A}_{44} \\ 0 & 0 & 0 & \bar{A}_{44} \end{bmatrix}$$

Note that the structure of this matrix coincides with that of (7). Taking a close look at both matrices, if

$$F_3 = -\bar{B}_3^T \bar{A}_{33} \quad \text{and} \quad F_4 = -\bar{B}_3^T \bar{A}_{34} - \bar{B}_3^T Q_{34} \bar{A}_{44}$$

then both matrices are in fact identical.

First it is shown now that $F_3 = -\bar{B}_3^T \bar{A}_{33}$. To that end consider an arbitrary row j of F_3 . By definition this row equals $(-a_{j1} \dots - a_{jn})$. Next consider row number j of $\bar{B}_3^T \bar{A}_{33}$. This row is equal to row number j of $\bar{B}_3^T$ multiplied by $\bar{A}_{33}$. It is easily verified that this product yields row number $n_1 + \dots + n_j$ of $\bar{A}_{33}$, i.e. $(a_{j1} \dots a_{jn})$, which completes this part of the proof.

In a similar way the second equality

$$F_4 = -\bar{B}_3^T \bar{A}_{34} - \bar{B}_3^T Q_{34} \bar{A}_{44}$$

is proved. Note that row number j of $\bar{B}_3^T \bar{A}_{34} + \bar{B}_3^T Q_{34} \bar{A}_{44}$ (see the argument above) is equal to the sum of row number $n_1 + \dots + n_j$ of the matrices $\bar{A}_{34}$ and $Q_{34} \bar{A}_{44}$, i.e.

$$(d_{j1} \dots d_{jm}) + \left[\begin{array}{c} 0 \ d_{j-11} \dots \sum_{i=1}^{r_{j1}-1} d_{j-ir_{j1}-i} * \dots * 0 \ d_{j-1p_1+1} \dots \sum_{i=1}^{r_{j2}-1} d_{j-ip_1+r_{j2}-i} * \dots * \\ p_1 \qquad \qquad \qquad p_2 \\ 0 \ d_{j-1p_{k-1}} + 1 \dots \sum_{i=1}^{r_{jk}-1} d_{j-ip_{k-1}+r_{jk}-i} * \dots * \\ p_k \end{array} \right]$$

where, for notational purposes again, $n_1 + \dots + n_j$ is denoted by j and $p_1 + \dots + p_{k-1}$ by p_{k-1} .

It is easily verified that up to a minus sign this row coincides with row number $n_1 + \dots + n_j$ of F_4 , which demonstrates the final part of the proof. $\square$

5. An example

The theory of the preceding sections can be conveniently illustrated by considering the non-controllable plant governed by the state and output equations

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & -2 & 0 \\ -1 & \frac{1}{2} & 0 & 2 & 0 \\ 1 & -1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(k), \quad \mathbf{x}(0) = \mathbf{x}$$

$$\mathbf{y}(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Throughout this section the notation introduced in the preceding sections will be used. Let the problem be to design a minimal-time deadbeat-output state-feedback controller with respect to $\mathbb{C}_g$, where $\mathbb{C}_g$ is the open unit disk. According to the presented theory, the first thing to do is to calculate a basis such that the system is described by a recurrence equation of the type considered in Theorem 1.

To determine this basis, first $V_g^*(\ker \mathbf{C})$ needs to be calculated. Simple calculations show that $\ker \mathbf{C} = \text{span} \{(0 \ 1 \ 1 \ 0 \ 0)^T\}$ and that $\ker \mathbf{C} \subset \langle \mathbf{A} | \text{Im } \mathbf{B} \rangle$. So, $V^*(\ker \mathbf{C}) = \ker \mathbf{C}$.

A feedback matrix $\mathbf{K}$ that makes $V^*(\ker \mathbf{C})$ $(\mathbf{A} + \mathbf{BK})$ -invariant is

$$\mathbf{K} = \begin{bmatrix} 0 & 2 & -2\frac{1}{2} & 0 & 0 \\ 0 & -1 & -\frac{1}{4} & 0 & 0 \end{bmatrix}$$

(see Wonham (1979) for a constructive approach).

The following characterization of $V_g^*(\ker \mathbf{C})$ is well-known:

$$V_g^*(\ker \mathbf{C}) = X_g(\mathbf{A} + \mathbf{BK}) \cap V^*(\ker \mathbf{C}) + R^*(\ker \mathbf{C})$$

Here $X_g(\mathbf{A} + \mathbf{BK})$ is the stable subspace of $\mathbf{A} + \mathbf{BK}$ and $R^*(\ker \mathbf{C})$ is the largest controllability subspace contained in $\ker \mathbf{C}$. Using the invariant controllability subspace algorithm ($R_0 := \ker \mathbf{C}$, $R_{i+1} := \ker \mathbf{C} \cap \mathbf{A}^{-1}(R_i + \text{Im } \mathbf{B})$, see Wonham (1979)), it is easily verified that $V_g^*(\ker \mathbf{C}) = \ker \mathbf{C}$. Choosing the following for X_i :

$$X_1 : \{(0 \ 1 \ 1 \ 0 \ 0)^T\} := \{\underline{b}_1\}$$

$$X_2 : \{0\}$$

$$X_3 : \{(0 \ -1 \ 1 \ 0 \ 0)^T, (1 \ 0 \ 0 \ 0 \ 0)^T, (0 \ 0 \ 0 \ 1 \ 0)^T\} =: \{\underline{b}_2, \underline{b}_3, \underline{b}_4\}$$

$$X_4 : \{(0 \ 0 \ 0 \ 0 \ 1)^T\} = \{\underline{b}_5\}$$

yields then the state-space decomposition

$$\begin{bmatrix} x_1(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -3 & \frac{1}{2} & 0 & 0 \\ 0 & 2\frac{1}{2} & -1\frac{1}{2} & 2 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

(with

$$\mathbf{x}(k) = \tilde{S}_1 \begin{bmatrix} x_1(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$x_1, x_2 \in \mathbb{R}$ and $x_3 \in \mathbb{R}^3$; and $\tilde{S}_1 = (\underline{b}_1 \underline{b}_2 \underline{b}_3 \underline{b}_4 \underline{b}_5)$.)

Obviously

$$\sigma[\mathbf{A} | \mathbb{R}^n / (\langle \mathbf{A} | \text{Im } \mathbf{B} \rangle + V_g^*(\ker \mathbf{C}))] = \{0\}$$

So, according to Theorem 2 the problem has a solution.

To construct a solution, consider the algorithm mentioned in Lemma 5. First, transformation matrices S_1 and T must be found such that

$$\bar{A}_{33} = S_1 \begin{bmatrix} 2\frac{1}{2} & -1\frac{1}{2} & 2 \\ -2 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} S_1^{-1}$$

and

$$\bar{B}_3 := S_1 \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} T$$

are in state-space canonical form. Following the algorithm described by Luenberger (1967) we have that, with

$$S_1 := \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1\frac{1}{2} \\ -2 & 0 & -3 \end{bmatrix}$$

and

$$T := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\bar{A}_{33} := \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 1 & -1\frac{3}{4} \\ 7 & 0 & 5\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \bar{B}_3 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, application of the similarity transformation

$$(x_1(k) \ x_3(k) \ x_4(k))^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 1\frac{1}{2} & 0 \\ 0 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{x}(k) \quad (\bar{x}(k) = \tilde{S}_2 x(k))$$

and input transformation $u(k) = T\bar{u}(k)$ yields the transformed state and output equations

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_3(k+1) \\ \bar{x}_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 9 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & -1\frac{3}{4} & 1\frac{1}{2} \\ 0 & 7 & 0 & 5\frac{1}{2} & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_3(k) \\ \bar{x}_4(k) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \bar{u}(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -6 & 0 & -1 & 0 \\ 0 & 0 & -4 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_3(k) \\ \bar{x}_4(k) \end{bmatrix} \quad (y(k) = \bar{C}\bar{x}(k))$$

According to theorem 4 a minimal-time deadbeat-output state-feedback gain matrix

with respect to $\mathbb{C}_g$ for this system is

$$\mathbf{F} = \left[\begin{array}{c|ccc} 0 & \frac{1}{2} & -1 & 1\frac{3}{4} \\ 0 & -7 & 0 & -5\frac{1}{2} \end{array} \middle| \begin{array}{c} -1\frac{1}{2} \\ 3 \end{array} \right] \quad (= (0|F_3|F_4))$$

It is easily verified that using the state feedback $\bar{\mathbf{u}}(k) = \mathbf{F}\bar{\mathbf{x}}(k)$, the closed-loop system becomes

$$\bar{\mathbf{x}}(k+1) = \begin{bmatrix} \frac{1}{2} & 5\frac{1}{2} & 1 & -\frac{3}{4} & 1\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{x}}(k)$$

$$\mathbf{y}(k) = \bar{\mathbf{C}}\bar{\mathbf{x}}(k)$$

and is deadbeat-output with minimal settling time 2.

The corresponding MV-controller, yielding the same result, is (according to the algorithm described in Theorem 5) obtained if the weight matrix $\mathbf{Q}$ is chosen as follows:

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1\frac{1}{2} \\ 0 & \alpha & \frac{1}{2} & 1\frac{1}{2} & \beta \end{bmatrix}, \quad \text{with } \alpha, \beta \in \mathbb{R} \text{ arbitrarily}$$

In terms of the original system, $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, this means that if this system is controlled using MV-control and the weight matrix is chosen as $(\tilde{\mathbf{S}}_2\tilde{\mathbf{S}}_1^{-1})^T\mathbf{Q}(\tilde{\mathbf{S}}_2\tilde{\mathbf{S}}_1^{-1})$, then the system becomes deadbeat-output with minimal settling time 2.

6. Conclusions

In this paper the deadbeat-output state-feedback control problem with internal stability has been solved. The obtained results extend in a straightforward way to the existing literature on this subject, where restrictions were made on the considered system. Here all these assumptions were dropped and a geometric approach was followed to solve the problem. To obtain an expression for the minimal settling time and a minimal time controller the analysis performed by Kimura and Tanaka (1981) turned out to be very useful. Using a similar argument, an algorithm was derived for the construction of a minimal-time controller. By considering a particular case their result is obtained.

The big advantage of this geometric approach is that these results can be used to relate least-squares minimization problems to the construction of minimal-time deadbeat state-feedback controllers.

Starting from the transformed state-space description of the system, it was shown, using Luenberger's phase canonical form, that it is always possible to construct a minimal-time output-deadbeat state-feedback controller which can be reinterpreted

in a least-squares setting. Furthermore, it was shown that this type of controller has also some transient minimal-time output-deadbeat properties. Its minimal settling time turned out to be directly computable from the transformed system.

A subject which remains for future research is to solve this kind of problem if the state of the system cannot be completely observed. That is, based on just partial state observations one wishes to control a number of target variables as fast as possible to zero.

Appendix

Proof of Lemma 4

It is easily verified that for $m = 0$ and $m = 1$ then the claim holds whatever n is. So without loss of generality one assumes $r \geq 2$.

Now consider A^k for some $2 \leq k \leq r$. Obviously this matrix has the structure

$$\begin{bmatrix} J_1^k & D_k \\ 0 & J_2^k \end{bmatrix}$$

Since $A^k = AA^{k-1}$, it is not difficult to see that the rows of the matrix D_k are recursively obtained as follows:

$$i\text{th row } D_k = (i+1)\text{th row } D_{k-1} + i\text{th row } D_{\rightarrow k-1}, \quad 1 \leq i \leq r \quad (\text{A } 1)$$

Here $D_{\rightarrow k} \in \mathbb{R}^{n \times m}$ is the matrix whose first k columns are zero and the other columns consist of the $m-k$ first columns of D . Using this relationship D_k equals (as can be shown by induction)

$$\begin{bmatrix} * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \\ f_1 & f_2 + d_{n-11} & f_3 + d_{n-12} + d_{n-21} & \dots & f_m + \sum_{i=1}^{k-1} d_{n-im-i} \\ 0 & & & & \vdots \\ \vdots & & & & \vdots \\ & & f_3 + d_{n-12} + d_{n-21} & f_{m-k-3} + d_{n-1m-k+2} + d_{n-2m-k+1} \\ & & f_2 + d_{n-11} & f_{m-k+2} + d_{n-1m-k+1} \\ 0 & \dots & 0 & f_1 & f_2 & f_{m-k+1} \end{bmatrix} \quad (\text{A } 2)$$

Here $*$ denotes an arbitrary real number. To derive the final result two cases are considered: $n \geq m$ and $n < m$, respectively. If $n \geq m$, it follows from (A 2) that

$$A^r = \left[\begin{array}{c|c} J_1^r & D_r \\ \hline 0_{m \times n} & 0_{m \times m} \end{array} \right]$$

with

$$D_{\underline{r}} := \left[\begin{array}{cccc} * & \cdots & & * \\ \vdots & & & \vdots \\ * & \cdots & & * \\ f_1 & \cdots & & f_m + \sum_{i=1}^{m-1} d_{n-im-i} \\ 0 & \cdots & & \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & f_1 \end{array} \right]$$

Again, using the relationship $\mathbf{A}^{r+k} = \mathbf{A}\mathbf{A}^{r+k-1}$ it is now inductively easily verified that for $1 \leq k \leq n - m$

$$\mathbf{A}^{r+k} = \left[\begin{array}{c|c} J_1^{r+k} & D_{\underline{r}+k} \\ \hline 0_{m \times n} & 0_{m \times m} \end{array} \right]$$

with

$$D_{\underline{r}+k} := \left\{ \begin{array}{l} n-m-k \\ \vdots \\ * \end{array} \right\} \left[\begin{array}{cccc} * & \cdots & & * \\ \vdots & & & \vdots \\ * & \cdots & & * \\ f_1 & \cdots & & f_m + \sum_{i=1}^{m-1} d_{n-im-i} \\ 0 & & & \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & f_1 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{array} \right] \left\{ \begin{array}{l} k \\ \vdots \\ 0 \end{array} \right.$$

By substituting $k = n - m$, the result is obtained. The case $n < m$ is proved similarly. First note that

$$\mathbf{A}^r = \left[\begin{array}{c|c} 0_{n \times n} & D_{\underline{r}} \\ \hline 0_{m \times m} & J_2^m \end{array} \right]$$

with

$$D_{\underline{r}} = \left[\begin{array}{cccccc} f_1 & f_2 + d_{n-11} & \cdots & & f_m + \sum_{i=1}^{n-1} d_{n-im-i} \\ 0 & & & & \\ \vdots & & f_2 + d_{n-11} & & \vdots \\ 0 & \cdots & 0 & f_1 & f_2 & \cdots & f_{m-n+1} \end{array} \right]$$

Applying the relationship $\mathbf{A}^{r+k} = \mathbf{A}\mathbf{A}^{r+k-1}$ yields for $1 \leq k \leq m - n$

$$\mathbf{A}^{r+k} = \left[\begin{array}{c|c} 0_{n \times n} & D_{r+k} \\ \hline 0_{m \times n} & J_2^{r+k} \end{array} \right]$$

with

$$D_{r+k} = \begin{bmatrix} 0 & \cdots & 0 & f_1 & f_2 + d_{n-11} & \cdots & f_{m-k} + \sum_{i=1}^{n-1} d_{n-im-k-i} \\ \vdots & & \vdots & & f_2 + d_{n-11} & & \vdots \\ 0 & \cdots & 0 & & f_1 & f_2 & \cdots & f_{m-k-n+1} \end{bmatrix}$$

k

Taking $k = m - n$ then yields the stated result. □

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