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THE REFERENCE STABILITY OF A MACRO-ECONOMIC SYSTEM WITH A MINIMUM-VARIANCE CONTROLLER

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Summary: In this short paper some results are summarized with respect to the so-called reference stability of a closed loop system, obtained by applying a minimum-variance controller to a linear dynamic system with a non-zero reference trajectory

Kurzfassung: Diese Langfassung beschreibt einige Ergebnisse in betreff die Referenz Stabilität eines 'closed loop' Systems, die resultiert aus der Anwendung einer Minimum-Varianz Kontroll Gleichung.

1. Introduction

In macro-economic regulation we are often faced with a non-zero reference target value or reference trajectory (e.g. a constant growth path) whereas in engineering the reference target value (set point) usually is taken to be zero with no external (exogenous) input other than (white) noise. In economics a lot has been done on optimal control with non-zero reference values, see e.g. Chow (1975) but to our knowledge little research has been done on studying the (asymptotic) stability of the closed-loop system.

One of the problems in studying the stability is that the control gains must reach steady-state time-invariant values, which for time-varying reference values and exogenous inputs may fail to exist. Instead of studying the rather complicated control equations as given for instance by Chow (1975), a simple minimum-variance (MV)-controller will be applied to a linear system with exogenous inputs and white noise and the resulting closed loop system studied for its stability. The short paper is organized as follows. After formulating the (process) system we define the concepts of reference stability and admissibility of reference trajectories. The MV-controller will be

given and the reference stability will be studied with help of the echelon or phase canonical form. It turns out that only under rather severe restrictions the reference trajectory can be reached.

2. Minimum-Variance Control, Reference Stability and Phase Canonical Forms

Consider the system

$$\underline{y}(k+1) = A\underline{y}(k) + Bu(k) + z(k+1) + \underline{w}(k+1), \quad k = 0, 1, 2, \dots \quad (1)$$

where $\underline{y}(k)$ is an n -dimensional target vector (stochastic vectors are underlined); $u(k)$ is a q -dimensional control input vector with $q \leq n$; $z(k+1)$ is a vector of deterministic, exogenous uncontrollable inputs and the sequence $\{\underline{w}(k)\}$ white noise with covariance $V(w)$. The initial value $\underline{y}(0)$ is assumed to be given. The rank of B is q (notation: $r(B) = q$) and it is further assumed that (A, B) is controllable, i.e. $r(B, AB, \dots, A^{n-1}B) = n$. Let the reference trajectory be given by

$$\underline{y}^*(k+1) = A^* \underline{y}^*(k), \quad k = 0, 1, 2, \dots \quad (2)$$

with $\underline{y}^*(0)$ and A^* known. Subtracting equation (2) from (1) leads to the error equation

$$\underline{e}(k+1) = A\underline{e}(k) + Bu(k) + x(k+1) + \underline{w}(k+1) \quad (3)$$

where $\underline{e}(k) \triangleq \underline{y}(k) - \underline{y}^*(k)$ and $x(k+1) \triangleq (A - A^*)\underline{y}^*(k) + z(k+1)$.

Definition (1): The system (1) is said to be (Lyapunov) reference stable if with $\|E\{\underline{e}(0)\}\|^2 < \infty$ there exists a k_0 and $\delta > 0$ such that $\|E\{\underline{e}(k)\}\|^2 < \delta$ for $k \geq k_0$.

Definition (2): The system (1) is said to be asymptotically reference stable if with $\|E\{\underline{e}(0)\}\|^2 < \infty$ $\|E\{\underline{e}(k)\}\|^2 \rightarrow 0$ for $k \rightarrow \infty$.

Definition (3): The reference trajectory $\{\underline{y}^*(k)\}$ is said to be weakly admissible if there exists a control sequence $\{u(k)\}$ such that with $\|E\{\underline{e}(0)\}\|^2 < \infty$ there exists a k_0 such that $\|E\{\underline{e}(k)\}\|^2 < \delta$ for $k \geq k_0$ and $\delta > 0$.

The reference trajectory is strongly admissible if $\|E\{\underline{e}(k)\}\|^2 = 0$ for $k \geq k_0$. The reference trajectory is admissible in the large if $\|E\{\underline{e}(k)\}\|^2 \rightarrow 0$ for $k \rightarrow \infty$.

Here $E\{\cdot\}$ denotes the expectation operator and $\|\cdot\|$ the Euclidean norm. Consider the following minimum one-period ahead cost functionals

$$J(k) \triangleq \min_{u(k)} E\{\underline{e}'(k+1)Q\underline{e}(k+1)\}, \quad k = 0, 1, 2, \dots$$

where $\underline{e}(k)$ is given by equation (3). The weighting matrix Q is assumed to be symmetric positive definite and can be factorized as $Q = D'D$ where the inverse of D exists.

Premultiplying equation (3) by D leads to the equation

$$D\bar{e}(k+1) = DA\bar{e}(k) + DBu(k) + Dx(k+1) + Dw(k+1)$$

or

$$\bar{e}(k+1) = \bar{A}\bar{e}(k) + \bar{B}u(k) + \bar{x}(k+1) + \bar{w}(k+1) \quad (4)$$

Minimizing the cost functional

$$J(k) = E\{\bar{e}'(k+1)D'\bar{e}(k+1)\} = E\{\bar{e}(k+1)'\bar{e}(k+1)\}$$

with respect to $u(k)$ leads to the MV-controller

$$\bar{u}(k) = -\tilde{B}^+[\tilde{A}\bar{e}(k) + \bar{x}(k+1)] = -(B'QB)^{-1}B'Q[A\bar{e}(k) + x(k+1)]$$

where $\tilde{B}^+ = (\tilde{B}'\tilde{B})^{-1}\tilde{B}'$ is a pseudo-inverse, see for details Aalders and co-workers (1984). Substitution of the MV-control equation into equation (4) yields the closed loop system

$$\bar{e}(k+1) = F\bar{e}(k) + Mx(k+1) + \bar{w}(k+1) \quad (5)$$

where $F \triangleq (I - \tilde{B}\tilde{B}^+)\tilde{A}$ and $M \triangleq (I - \tilde{B}\tilde{B}^+)\tilde{B}^+x(k+1)$. From equation (5) it is seen that the closed loop system is Lyapunov reference stable if the sequence $\{x(k+1)\}$ is bounded and F is stable. In order to investigate the stability of F we use the so-called echelon or phase canonical form, see e.g. Luenberger (1967) and a theorem given by Wonham (1974).

Theorem: (phase canonical form). If (A, B) is controllable and $r(B) = q$ then there exists a non-singular transformation matrix S such that $\bar{A} = SAS^{-1}$ and $\bar{B} = SB$ with

$$\bar{A} = \begin{bmatrix} \kappa_1 & \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & * \end{bmatrix} \\ \kappa_2 & \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & * \end{bmatrix} \\ \vdots & \vdots \\ \kappa_q & \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & * \end{bmatrix} \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_q \geq 1$ with $\sum_{i=1}^q \kappa_i = n$ are the Kronecker or controllability indices with κ_1 as 'the' controllability index and $*$ denotes a 'free' parameter.

Premultiplying equation (3) by S and defining $\bar{e}(k+1) \triangleq S\bar{e}(k+1)$ we have

$$\bar{e}(k+1) = \bar{A}\bar{e}(k) + \bar{B}\bar{u}(k) + \bar{x}(k+1) + \bar{w}(k+1) \quad (6)$$

where $\bar{A} = SAS^{-1}$; $\bar{B} = SB$; $\bar{x}(k+1) = Sx(k+1)$ and $\bar{w}(k+1) = Sw(k+1)$.
Let the cost functional for equation (6) be given by

$$J(k) = E\{\bar{e}'(k+1)\bar{e}(k+1)\} = E\{\bar{e}'(k+1)S'Se(k+1)\}$$

that is the cost functional for equation (3) with weighting matrix $Q = S'S$ then the MV-controller for equation (6) is given by

$$\bar{u}(k) = -(\bar{B}'\bar{B})^{-1}\bar{B}'[\bar{A}\bar{e}(k) + \bar{x}(k+1)]$$

and the closed loop system by

$$\bar{e}(k+1) = \bar{F}\bar{e}(k) + \bar{M}\bar{x}(k+1) + \bar{w}(k+1) \quad (7)$$

where $\bar{F} = (I - \bar{B}\bar{B}^+)\bar{A}$ and

$$\bar{M} = \text{diag}(1 \dots 1 \quad 0 \overset{\kappa_1}{1} \dots 1 \quad 0 \overset{\kappa_2}{2} \dots 1 \quad 0 \overset{\kappa_q}{q} \dots 1)$$

By simple calculation it is seen that

$$\bar{F} = \text{diag}(D_1, \dots, D_q)$$

is nilpotent with index κ_1 , i.e. $\bar{F}^{\kappa_1} = 0$ and where

$$D_i \underset{(\kappa_i \times \kappa_i)}{A} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & & & \ddots & 1 \\ 0 & & & \dots & 0 \end{bmatrix}$$

which is consistent with Wonham (1974, pp. 122-126).

The expected closed loop system is

$$E\{\bar{e}(k+1)\} = \bar{F}E\{\bar{e}(k)\} + \bar{M}\bar{x}(k+1) \quad (8)$$

Evaluating equation (8) we have with starting value $e(0) \neq 0$ after κ_1 steps

$$E\{\bar{e}(\kappa_1)\} = \bar{F}^{\kappa_1}e(0) + \sum_{i=0}^{\kappa_1-1} \bar{F}^i \bar{M}\bar{x}(\kappa_1-i)$$

or in general

$$E\{\bar{e}(k)\} = \sum_{i=0}^{\kappa_1-1} \bar{F}^i \bar{M}\bar{x}(k-i) \quad k \geq \kappa_1 \quad (9)$$

From equation (9) the following conclusions can be drawn:

- (i) The system is Lyapunov reference stable for $k \geq \kappa_1$ where κ_1 is the controllability index if the sequence $\{\bar{x}(k)\}$ is bounded for all k which is fulfilled if the sequence of exogenous inputs $\{z(k)\}$ and the sequence of reference values $\{y^*(k)\}$ are bounded (no exponential growth is allowed).
- (ii) Under the condition that the sequence of exogenous inputs

$\{z(k)\}$ is bounded for all k all reference trajectories such that $\{y^*(k)\}$ is bounded for all k are weakly admissible for the MV-controller with $q < n$.

(iii) Because the controllability indices obey the equations $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_q \geq 1$ and $\sum_{i=1}^q \kappa_i = n$ we have that if $q \rightarrow n$ then $\kappa_1 \rightarrow 1$ with as limit for $q = n$ $\kappa_1 = 1$. In that case it can be seen from equation (9) that $E\{\underline{e}(k)\} = 0$ for all $k \geq 1$ and the reference trajectory is strongly admissible for the system and no conditions are imposed on the sequence of reference values and exogenous inputs.

(iv) In case of $q < n$, that is the number of control inputs is smaller than the number of outputs the reference trajectory is strongly admissible if the following equation holds for all k :

$$z(k+1) + (A-A^*)y^*(k) = 0$$

with as special case of no exogenous inputs ($z(k) = 0$ for all k): $A = A^*$ that is the reference transition matrix A^* equals the process transition matrix A .

In concluding this short paper it is noticed that MV-control is a special case of the so-called stability improvement by state feedback, see Kwakernaak (1972, theorem 6.27). This theorem tells us that any initial expected error $E\{\underline{e}(0)\}$ can be reduced to zero within n steps (with no exogenous inputs) through the feed-back equation

$$u(k) = -Fy(k)$$

where F is chosen such that $(A-BF)$ has all its characteristic values of the origin, that is F is nilpotent with index n . In our case $F^{K_1} = 0$, with $\kappa_1 \leq n$ where κ_1 is the controllability index.

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