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Jansen, M.J.M.; Tijs, S.

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On Characterizing Properties of the Value Sets and the Equilibrium Point Sets of Non-Cooperative Two-Person Games

M. J. M. JANSEN¹ and S. H. TIJS¹

Summary: In this paper, for certain subfamilies of the family of bounded measurable two-person games in normal form, the value sets are characterized by three properties, called the maximum property, the minimum property and the adjunction property. Furthermore, one of the papers of VILKAS is critically discussed. Finally, for bimatrix games, two systems of characterizing properties for the equilibrium point sets are given.

1. Introduction

In a beautiful paper [4], VILKAS gave an axiomatic definition of the value of a matrix game. Extensions to other classes of zero-sum games were given by TIJS in [2], pp. 60–65 and [3] and by HAZEN in [1]. An attempt to extend his results to finite non-zero sum games was made by VILKAS in [5] (cf. VOROBEV [6], p. 96). We will show in section 4 that this attempt was not successful. In section 3 of this paper, we consider, for bounded measurable non-cooperative two-person games, two properties, called the adjunction property and the maximum property. We show that the value function on classes of games, which are closed with respect to adjunction of one strategy, is the smallest multifunction with the two above-mentioned properties. In section 5, two systems of characterizing properties for the equilibrium point sets of finite bimatrix games are given.

2. Some Properties of Non-Cooperative Two-Person Games in Normal Form

In the following we look at *bounded measurable two-person games* $\Gamma = \langle X, Y, K, L \rangle$, where the *strategy spaces* X and Y are non-empty measurable spaces, with σ -algebras of subsets $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, and where the *payoff functions* $K : X \times Y \rightarrow \mathbf{R}$, $L : X \times Y \rightarrow \mathbf{R}$ are bounded measurable functions with respect to the product σ -algebra of $\mathfrak{X}$ and $\mathfrak{Y}$. We suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ contain the one point sets of X and Y , respectively. The class of these games is denoted by BM .

For $\Gamma = \langle X, Y, K, L \rangle \in BM$ let $P_X(P_Y)$ denote the family of probability meas-

¹ University of Nijmegen, Department of Mathematics. Toernooiveld, 6525 ED Nijmegen, The Netherlands.

ures on $X(Y)$. A point $(\bar{\mu}, \bar{\nu}) \in P_X \times P_Y$ is called an *equilibrium point* of Γ if

$$E_K(\bar{\mu}, \bar{\nu}) = \max_{\mu \in P_X} E_K(\mu, \bar{\nu}) \quad \text{and} \quad E_L(\bar{\mu}, \bar{\nu}) = \max_{\nu \in P_Y} E_L(\bar{\mu}, \nu),$$

where

$$E_K(\mu, \nu) = \int \int K(x, y) d\mu(x) d\nu(y), \quad \text{for } \mu \in P_X, \quad \nu \in P_Y$$

and

$$E_L(\mu, \nu) = \int \int L(x, y) d\mu(x) d\nu(y), \quad \text{for } \mu \in P_X, \quad \nu \in P_Y.$$

We denote the set of equilibrium points of Γ by $E(\Gamma)$ and we call the subset $V(\Gamma) = \{(E_K(\bar{\mu}, \bar{\nu}), E_L(\bar{\mu}, \bar{\nu})) \in \mathbf{R}^2; (\bar{\mu}, \bar{\nu}) \in E(\Gamma)\}$ of $\mathbf{R}^2$, the *value set* of the game Γ .

In the following two theorems we derive two important properties of the *value multifunction* $V: BM \rightarrow \mathbf{R}^2$.

Theorem 1: (*Maximum property of V*): Let $\Gamma = \langle X, Y, K, L \rangle \in BM$. Suppose $(\bar{x}, \bar{y}) \in X \times Y$ satisfies

$$(i) \quad K(\bar{x}, \bar{y}) = \max_{x \in X} K(x, \bar{y}), \quad \text{and}$$

$$(ii) \quad L(\bar{x}, \bar{y}) = \max_{y \in Y} L(\bar{x}, y).$$

Then $(K(\bar{x}, \bar{y}), L(\bar{x}, \bar{y})) \in V(\Gamma)$.

Proof: Let $e_{\bar{x}}(e_{\bar{y}})$ be the probability measure with mass 1 in $\bar{x} \in X$ ($\bar{y} \in Y$). Then (i) and (ii) imply that $(e_{\bar{x}}, e_{\bar{y}}) \in E(\Gamma)$ and consequently, $(K(\bar{x}, \bar{y}), L(\bar{x}, \bar{y})) = (E_K(e_{\bar{x}}, e_{\bar{y}}), E_L(e_{\bar{x}}, e_{\bar{y}})) \in V(\Gamma)$. ■

Before we can formulate the second property of V , we need a

Definition 1: Let $\Gamma = \langle X, Y, K, L \rangle$ be an element of BM and let $\alpha \in P_X, \beta \in P_Y$. Then the two-person game in normal form

$$\Gamma_\alpha = \langle X \cup \{x_\infty\}, Y, K_\alpha, L_\alpha \rangle \quad \text{with } x_\infty \notin X,$$

$$K_\alpha(x, y) = K(x, y), \quad L_\alpha(x, y) = L(x, y), \quad \text{for all } x \in X, \quad y \in Y$$

and

$$K_\alpha(x_\infty, y) = \int_X K(x, y) d\alpha(x), \quad L_\alpha(x_\infty, y) = \int_X L(x, y) d\alpha(x),$$

for all $y \in Y$, will be called the *one-point-adjunction of Γ with respect to α* . The *one-point-adjunction of Γ with respect to β* will be the game $\Gamma^\beta = \langle X, Y \cup \{y_\infty\}, K^\beta, L^\beta \rangle$, where $y_\infty \notin Y$,

$$K^\beta(x, y) = K(x, y), \quad L^\beta(x, y) = L(x, y), \quad \text{for all } x \in X, \quad y \in Y$$

and

$$K^\beta(x, y_\infty) = \int_Y K(x, y) d\beta(y), \quad L^\beta(x, y_\infty) = \int_Y L(x, y) d\beta(y),$$

for all $x \in X$.

It is straightforward to show that for $\Gamma \in BM$ also Γ_α and Γ^β are elements of BM , where we provide $X \cup \{x_\infty\}$ with the σ -algebra $\mathfrak{X} \cup \{S \cup \{x_\infty\}; S \in \mathfrak{X}\}$ and $Y \cup \{y_\infty\}$

with the σ -algebra $\mathfrak{Y} \cup \{T \cup \{y_\infty\}; T \in \mathfrak{Y}\}$. In the following we will denote $(\Gamma_\alpha)^\beta = (\Gamma^\beta)_\alpha$ by

$$\Gamma(\alpha, \beta) = \langle X \cup \{x_\infty\}, Y \cup \{y_\infty\}, K(\alpha, \beta), L(\alpha, \beta) \rangle.$$

Theorem 2: Let $\Gamma = \langle X, Y, K, L \rangle \in BM$, $\alpha \in P_X$ and $\beta \in P_Y$. Then

- (i) $V(\Gamma_\alpha) = V(\Gamma)$,
- (ii) $V(\Gamma^\beta) = V(\Gamma)$ and
- (iii) $V(\Gamma(\alpha, \beta)) = V(\Gamma)$ (Adjunction property of V).

Proof: (i) Suppose $(\mu, \nu) \in E(\Gamma)$ and let $\tilde{\mu} \in P_{X \cup \{x_\infty\}}$ be the probability measure with $\tilde{\mu}(U) = \mu(U - \{x_\infty\})$, for all measurable subsets U of $X \cup \{x_\infty\}$. Then $(\tilde{\mu}, \nu) \in E(\Gamma_\alpha)$, $E_{K_\alpha}(\tilde{\mu}, \nu) = E_K(\mu, \nu)$ and $E_{L_\alpha}(\tilde{\mu}, \nu) = E_L(\mu, \nu)$. This implies that $V(\Gamma) \subset V(\Gamma_\alpha)$.

Conversely, suppose $(\mu, \nu) \in E(\Gamma_\alpha)$ and let $\hat{\mu}$ be the probability measure $\mu' + \mu(\{x_\infty\})\alpha$ on X , where μ' is the restriction of μ to $\mathfrak{X}$. Then it follows that $(\hat{\mu}, \nu) \in E(\Gamma)$ and $E_K(\hat{\mu}, \nu) = E_{K_\alpha}(\mu, \nu)$ and $E_L(\hat{\mu}, \nu) = E_{L_\alpha}(\mu, \nu)$. So $V(\Gamma_\alpha) \subset V(\Gamma)$ and (i) is proved. (ii) can be proved similarly as (i), and (iii) is an immediate consequence of (i) and (ii). ■

3. Characterization of the Value Multifunction

Definition 2: A subfamily $\mathfrak{C}$ of BM will be called *closed with respect to one-point-adjunction*, if, for all $\Gamma = \langle X, Y, K, L \rangle \in \mathfrak{C}$ and all $\alpha \in P_X$, $\beta \in P_Y$, we have $\Gamma_\alpha \in \mathfrak{C}$ and $\Gamma^\beta \in \mathfrak{C}$.

Now we formulate our main result.

Theorem 3: Let $\mathfrak{C}$ be a subfamily of BM , which is closed with respect to one-point-adjunction and let $f: \mathfrak{C} \rightarrow \mathbf{R}^2$ be a multifunction with the following properties:

P.1 (Maximum property) For each $\Gamma = \langle X, Y, K, L \rangle \in \mathfrak{C}$ and $(\bar{x}, \bar{y}) \in X \times Y$ with

$$K(\bar{x}, \bar{y}) = \max_{x \in X} K(x, \bar{y}) \quad \text{and} \quad L(\bar{x}, \bar{y}) = \max_{y \in Y} L(\bar{x}, y) \quad (3.1)$$

we have $(K(\bar{x}, \bar{y}), L(\bar{x}, \bar{y})) \in f(\Gamma)$.

P.2 (Adjunction property) For each $\Gamma = \langle X, Y, K, L \rangle \in \mathfrak{C}$, $\alpha \in P_X$, $\beta \in P_Y$, we have $f(\Gamma(\alpha, \beta)) = f(\Gamma)$.

P.3 (Minimum property) If $g: \mathfrak{C} \rightarrow \mathbf{R}^2$ is a multifunction, satisfying the maximum property and the adjunction property, then

$$g(\Gamma) \supset f(\Gamma), \quad \text{for each } \Gamma \in \mathfrak{C}.$$

Then $f = V$.

Proof: V satisfies, in view of theorems 1 and 2, the maximum property and the adjunction property. So, in view of P.3, we have $V(\Gamma) \supset f(\Gamma)$, for each $\Gamma \in \mathfrak{C}$. Conversely, let $(u, v) \in V(\Gamma)$, where $\Gamma = \langle X, Y, K, L \rangle \in \mathfrak{C}$. Then there exist $\alpha \in P_X$ and $\beta \in P_Y$ such that $(\alpha, \beta) \in E(\Gamma)$ and $E_K(\alpha, \beta) = u$, $E_L(\alpha, \beta) = v$. But then P.1 can be

applied with $\Gamma(\alpha, \beta)$ in the role of Γ and (x_∞, y_∞) in the role of $(\bar{x}, \bar{y})$. So

$$(u, v) = (K(\alpha, \beta)(x_\infty, y_\infty), L(\alpha, \beta)(x_\infty, y_\infty)) \in f(\Gamma(\alpha, \beta)).$$

Then $(u, v) \in f(\Gamma)$ in view of P.2. Hence $V(\Gamma) \subset f(\Gamma)$, for each $\Gamma \in \mathfrak{G}$. ■

In view of theorem 3, subclasses of BM , which are closed with respect to one-point-adjunction are interesting. We mention some.

1. The class $\mathfrak{G}_2$ of two-person games with finite strategy spaces.
2. The class of two-person games, where at least one of the strategy spaces is finite (cf. [2], p. 60).
3. The class of games $\langle X, Y, K, L \rangle$, where X and Y are compact metric spaces and where the payoff functions K and L are continuous (cf. [1], p. 28). [If d is the metric on X , then we can take an arbitrary $a \in X$ and take d_a , defined by

$$d_a(x, x') = d(x, x') \text{ if } (x, x') \in X \times X,$$

$$d_a(x_\infty, x) = d(a, x) + 1, \text{ for all } x \in X$$

and

$$d_a(x_\infty, x_\infty) = 0,$$

as metric on $X \cup \{x_\infty\}$. Then $K_\alpha, L_\alpha, K^\beta, L^\beta$ are bounded and continuous for all $\alpha \in P_X$ and $\beta \in P_Y$.]

4. The class $\mathfrak{Z}$ of zero-sum games $\langle X, Y, K, -K \rangle$ in BM .

5. The class $\mathfrak{Z}\mathfrak{B}$ of zero-sum games in BM with a real value. For $\mathfrak{Z}\mathfrak{B}$ we have also the following result.

Theorem 4: Let $f: \mathfrak{Z}\mathfrak{B} \rightarrow \mathbf{R}$ be a (single-valued) function with the following properties:

Z.1 For each $\Gamma = \langle X, Y, K, -K \rangle \in \mathfrak{Z}\mathfrak{B}$ and $(\bar{x}, \bar{y}) \in X \times Y$, satisfying (3.1) ($L = -K$), we have $K(\bar{x}, \bar{y}) = f(X, Y, K, -K)$.

Z.2 For each $\Gamma = \langle X, Y, K, -K \rangle \in \mathfrak{Z}\mathfrak{B}$, $\alpha \in P_X$ and $\beta \in P_Y$, we have $f(\Gamma(\alpha, \beta)) = f(\Gamma)$. Then $f(\Gamma)$ is the value $\text{val}(\Gamma)$ of the game Γ , for each $\Gamma \in \mathfrak{Z}\mathfrak{B}$.

Proof: Similarly, as in the proof of theorem 3, it follows that $\text{val}(\Gamma) \subset f(\Gamma)$. Since val and f are functions, $\text{val} = f$. ■

4. Discussion of a Paper of VILKAS

Let $\mathfrak{G}_n$ be the family of n -person games in normal form with finite strategy spaces for all players. In his paper [5], VILKAS studies the family $\mathfrak{F}_n$ of multifunctions F from $\mathfrak{G}_n$ into $\mathbf{R}^n$, which possess the following three properties:

(1) (*monotony*) If Γ and Γ' are games in $\mathfrak{G}^n$ for which the payoff functions of all players, except the i -th player, are equal and for which $K_i \cong K'_i$, where $K_i(K'_i)$ is the payoff function for player i in the game $\Gamma(\Gamma')$, then there exists an $f \in F(\Gamma)$ and an $f' \in F(\Gamma')$, such that $f_i \cong f'_i$.

(2) (*dominance*) If a game Γ differs from a game Γ' only by a dominated strategy of the i -th player, then $F_i(\Gamma) = F_i(\Gamma')$.

(3) (*objectivity*) If Γ is an element of $\mathfrak{G}^n$ for which the i -th payoff function is equal to a constant $c_i \in \mathbf{R}$, then $f_i = c_i$, for all $f \in F(\Gamma)$.

Let $V: \mathfrak{G}_n \rightarrow \mathbf{R}^n$ be the multifunction, which assigns to $\Gamma \in \mathfrak{G}_n$ the set of equilibrium payoffs. VILKAS suggests in [5], that $\mathfrak{F}_n = \{V\}$. In fact, he proves in theorem 1 on p. 523 that $\mathfrak{F}_n \subset \{V\}$.

Let us now restrict our attention to the case $n=2$. The following examples will show that $\mathfrak{F}_2 = \emptyset$, because V does not satisfy the monotony property (examples 1 and 2) and the dominance property (example 3). Hence $V \notin \mathfrak{F}_2$.

Examples

1. Let $\Gamma(t)$ be the bimatrix game with payoff matrices A and $B(t)$ for player 1 and player 2, respectively, where

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 0 & -1 \\ -1 & t \end{bmatrix} \quad \left(-\frac{1}{2} \leq t \leq \frac{1}{2} \right)$$

and let $\Gamma^*(t)$ be the bimatrix game $(A^*, B(t))$, where $A^* = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ and $B(t)$ as above. Then $V(\Gamma(t))$ contains one element $v = (3, 0)$ and $V^*(\Gamma(t))$ contains also one element $v^* = (2, t)$. Now it follows from

$$A^* \geq A \quad \text{and} \quad v_1^* = 2 < 3 = v_1,$$

that V does not satisfy the monotony property. Note further, that $v_2^* < v_2$, $v_2^* = v_2$, $v_2^* > v_2$ for $t \in \left[-\frac{1}{2}, 0\right)$, $t = 0$ and $t \in \left(0, \frac{1}{2}\right]$, respectively.

2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, $A' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & 10 \\ 10 & -10 \end{bmatrix}$. Then A' differs from A only in one cell, $A' \geq A$ and the first coordinate of the unique element of $V(A, B)$ equals $\frac{1}{2}$, which is greater than the first coordinate $\frac{1}{3}$ of the unique element of $V(A', B)$. This shows again that the monotony property fails for non-zero-sum games.

3. To show that $V: \mathfrak{G}_2 \rightarrow \mathbf{R}^2$ does not satisfy the dominance property, we use the bimatrix game $\Gamma = (A, B)$ with $A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, appearing on p. 525 of [5].

Let Γ' be the bimatrix game (A', B') with $A' = [2 \ 1]$, $B' = [0 \ 1]$, which we obtain by excluding the second pure strategy of player 1. Note that the second row of A is dominated by the first row. Then

$$V(A, B) = \{(1, 1)\} \cup \left\{ (2, t); \frac{2}{3} \leq t \leq 2 \right\} \quad \text{and} \\ V(A', B') = \{(1, 1)\},$$

which shows that V does not have the dominance property of VILKAS.

5. Characterizing Properties of the Equilibrium Point Sets of Bimatrix Games

For an $m \times n$ -bimatrix game (A, B) , mixed strategies can be identified with elements of $S^m = \left\{ p \in \mathbf{R}^m; p \geq 0, \sum_{i=1}^m p_i = 1 \right\}$ and mixed strategies for player 2 with elements of S^n . The set of equilibrium points of (A, B) is denoted by $E(A, B)$.

For an $m \times n$ -bimatrix game $\Gamma = (A, B)$ and $p \in S^m, q \in S^n$ we denote the $(m+1) \times (n+1)$ -bimatrix game $(A(p, q), B(p, q))$, where

$$A(p, q) = \begin{bmatrix} A & Aq \\ pA & pAq \end{bmatrix}, \quad B(p, q) = \begin{bmatrix} B & Bq \\ pB & pBq \end{bmatrix}$$

by $\Gamma(p, q)$.

In the following $S = \bigcup_{m, n \in \mathbf{N}} S^m \times S^n$ and $\mathfrak{M}_2$ is the family of bimatrix games. Note that the multifunction $E : \mathfrak{M}_2 \rightarrow S$ satisfies the following properties.

E.1 For each $\Gamma = (A, B) \in \mathfrak{M}_2$ we have

$$\text{if } a_{i_0 j_0} = \max_i a_{ij_0}, \quad b_{i_0 j_0} = \max_j b_{ij_0}, \quad \text{then } (e_{i_0}, e_{j_0}) \in E(\Gamma).$$

E.2 For each $m, n \in \mathbf{N}$ and each $m \times n$ -bimatrix game $\Gamma = (A, B)$ we have

$$(\bar{p}, \bar{q}) \in E(\Gamma) \quad \text{if and only if} \quad (e_{m+1}, e_{n+1}) \in E(\Gamma(\bar{p}, \bar{q})).$$

Now we give characterizing properties for the multifunction $E : \mathfrak{M}_2 \rightarrow S$.

Theorem 5: Let $g : \mathfrak{M}_2 \rightarrow S$ be a multifunction, such that for each $m \times n$ -bimatrix game $\Gamma \in \mathfrak{M}_2$ we have $g(\Gamma) \subset S^m \times S^n$. Suppose, furthermore, that g satisfies the following conditions:

Q.1 (Maximum property) If for a game $\Gamma = (A, B) \in \mathfrak{M}_2$ we have

$$a_{i_0 j_0} = \max_i a_{ij_0}, \quad b_{i_0 j_0} = \max_j b_{ij_0}, \quad \text{then } (e_{i_0}, e_{j_0}) \in g(\Gamma).$$

Q.2 (Adjunction property) For each $m, n \in \mathbf{N}$ and each $m \times n$ -bimatrix game $\Gamma = (A, B)$ we have

$$(\bar{p}, \bar{q}) \in g(\Gamma) \quad \text{if and only if} \quad (e_{m+1}, e_{n+1}) \in g(\Gamma(\bar{p}, \bar{q})).$$

Q.3. (Minimum property) If $h : \mathfrak{M}_2 \rightarrow S$ is a multifunction with

$$h(\Gamma) \subset S^m \times S^n,$$

if Γ has size $m \times n$, and which satisfies the maximum property and the adjunction property, then

$$g(\Gamma) \subset h(\Gamma), \quad \text{for each } \Gamma \in \mathfrak{M}_2.$$

Then

$$g(\Gamma) = E(\Gamma), \quad \text{for each } \Gamma \in \mathfrak{M}_2.$$

Proof: The multifunction $E : \mathfrak{M}_2 \rightarrow S$ satisfies, in view of E.1 and E.2, the maximum property and the adjunction property. Hence, by Q.3, we have $g(\Gamma) \subset E(\Gamma)$, for each $\Gamma \in \mathfrak{M}_2$.

Take an $m \times n$ -bimatrix game $\Gamma = (A, B) \in \mathfrak{M}_2$ and let $(\bar{p}, \bar{q}) \in E(\Gamma)$. Then, by E.2, we have $(e_{m+1}, e_{n+1}) \in E(\Gamma(\bar{p}, \bar{q}))$, which, in view of Q.1, implies that $(e_{m+1}, e_{n+1}) \in g(\Gamma(\bar{p}, \bar{q}))$. But then $(\bar{p}, \bar{q}) \in g(\Gamma)$ by property Q.2. So $E(\Gamma) \subset g(\Gamma)$, for each $\Gamma \in \mathfrak{M}_2$. ■

Now we want to replace Q.1 by three properties, which have some similarity with the axioms of VILKAS.

R.1 (*Pseudo monotony property*) For each pair of bimatrix games (A, B) and $(\bar{A}, B)$ with $A \leq \bar{A}$, $a_{i_0 j_0} = \bar{a}_{i_0 j_0}$ and $(e_{i_0}, e_{j_0}) \in g(\bar{A}, B)$, we have $(e_{i_0}, e_{j_0}) \in g(A, B)$.

R.2 (*Weak dominance property*) For each pair of bimatrix games (A, B) and (A', B') , where (A', B') is obtained by eliminating a dominated row of A or a dominated column of B , the set $g(A', B')$ is (essentially) a subset of $g(A, B)$: if $(p, q) \in g(A', B')$ and, for instance, the last row of A is dominated, then $(p', q) \in g(A, B)$, where $p' = (p, 0)$.

R.3 (*Objectivity*) For each 1×1 -bimatrix game Γ , $g(\Gamma) = \{(e_1, e_1)\}$.

Note that the multifunction E satisfies the properties R.1, R.2 and R.3. We have the following theorem.

Theorem 6: Let $g : \mathfrak{M}_2 \rightarrow S$ be a multifunction such that, for each $m \times n$ -bimatrix game $\Gamma \in \mathfrak{M}_2$, we have $g(\Gamma) \subset S^m \times S^n$, and suppose that g satisfies R.1, R.2 and R.3, Q.2 and Q.3. Then $g = E$.

Proof: Again, it is sufficient to show that $E(\Gamma) \subset g(\Gamma)$, for each $\Gamma \in \mathfrak{M}_2$. (a) Let $\Gamma = (A, B)$ be an $m \times n$ -bimatrix game with $(e_{i_0}, e_{j_0}) \in E(A, B)$. Let $\bar{A} = [\bar{a}_{ij}]_{i=1, j=1}^{m, n}$ be the $m \times n$ -matrix, such that for all j

$$\bar{a}_{i_0 j} = \max_i a_{ij} \quad \text{and} \quad \bar{a}_{ij} = a_{ij} \quad \text{if} \quad i \neq i_0.$$

By using R.2, repeatedly, and by using R.3, we obtain

$$\begin{aligned} g(\bar{A}, B) &\supset g([\bar{a}_{i_0 j}]_{j=1}^n, [b_{i_0 j}]_{j=1}^n) \supset g([\bar{a}_{i_0 j_0}], [b_{i_0 j_0}]) = \\ &= \{(e_{i_0}, e_{j_0})\}. \end{aligned}$$

Now applying R.1, gives $(e_{i_0}, e_{j_0}) \in g(A, B)$.

(b) Now let $(\bar{p}, \bar{q})$ be an equilibrium point of $\Gamma = (A, B)$. Then $(e_{m+1}, e_{n+1}) \in E(\Gamma(\bar{p}, \bar{q}))$. In view of part (a) of this proof, we obtain $(e_{m+1}, e_{n+1}) \in g(\Gamma(\bar{p}, \bar{q}))$ and by Q.2 we may conclude that $(\bar{p}, \bar{q}) \in g(\Gamma)$. Hence, $E(\Gamma) \subset g(\Gamma)$, for each $\Gamma \in \mathfrak{M}_2$. ■

Note that property R.3 can be omitted from theorem 6, if we suppose that the multifunction g satisfies $g(\Gamma) \neq \emptyset$, for each $\Gamma \in \mathfrak{M}_2$. [By NASH's theorem $E(\Gamma) \neq \emptyset$, for each $\Gamma \in \mathfrak{M}_2$.]

We conclude this paper by noting that theorems 3, 5 and 6 can be extended in an obvious way to n -person games ($n > 2$).

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Zusammenfassung

In dieser Arbeit werden die Wertmengen für gewisse Teilklassen der Klasse der beschränkten meßbaren Zweipersonenspiele in Normalform charakterisiert durch drei Eigenschaften, die als Maximumeigenschaft, Minimumeigenschaft und Adjungierungseigenschaft bezeichnet werden.

Weiter wird eine der Arbeiten von VILKAS kritisch diskutiert. Zum Schluß werden für Bimatrixspiele zwei Systeme charakterisierender Eigenschaften für die Gleichgewichtspunktmengen angegeben.

Резюме

В настоящей работе множества значений для некоторых частных класса ограниченных измеримых игр для двух лиц в нормальной форме характеризуются по средством трех свойств называемых свойства максимума, свойства минимума и свойства присоединения. Дальше критически обсуждается одна из работ Вилькаса. В конце для биматричных игр даны две системы характеризующих свойств для множеств точек равновесия.

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