Tilburg University

Extensions and modifications of the t-value for cooperative games

Driessen, T.S.H.; Tijs, S.H.

Published in:

Selected topics in operations research and mathematical economics

Publication date:

1984

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Driessen, T. S. H., & Tijs, S. H. (1984). Extensions and modifications of the t-value for cooperative games. In G. Hammer, & D. Pallaschke (Eds.), *Selected topics in operations research and mathematical economics: Proceedings of the 8th symposium on operations research, held at the University of Karlsruhe, West Germany, August 22-25, 1983* (pp. 252-261). (Lecture notes in economics and mathematical systems; No. 226). Springer.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

EXTENSIONS AND MODIFICATIONS OF THE τ -VALUE
FOR COOPERATIVE GAMES

T.S.H. DRIESSEN and S.H. TIJS
Dept. of Mathematics, Catholic University,
Nijmegen, The Netherlands

ABSTRACT

Tijs (1981) defined the τ -value only for a subclass of cooperative games in characteristic function form. In this paper an extension of the τ -value to the space of all games is given. Another solution concept, the core-value, is also introduced, which appears to coincide with the τ -value for games with a non-empty core. Properties of the τ -value and the core-value are given. Modifications of the τ -value with respect to the upper vector and the coalition structure are also treated.

Keywords: game in characteristic function form, core, τ -value, core-value.

1. INTRODUCTION

A cooperative n-person game in characteristic function form is a pair (N, v) , where $N := \{1, 2, \dots, n\}$ with $n \geq 2$ and v is a real-valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. N is the set of players and v the characteristic function of the game. A subset S of N is called a coalition and $v(S)$ the worth of the coalition S in the game. The function v itself will also be called an n -person game. The set of all n -person games is denoted by G^n .

For a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the i -th coordinate x_i represents the pay-off to player i . For $S \subset N$, we write often $x(S)$ instead of $\sum_{i \in S} x_i$, where $x(\emptyset) = 0$. For a game $v \in G^n$, the set $X(v)$ of feasible pay-off vectors is defined by

$$X(v) := \{x \in \mathbb{R}^n; x(N) = v(N)\}.$$

The core $C(v)$ of a game $v \in G^n$ is defined as the set of all feasible pay-off vectors that can not be improved upon by any coalition. That is,

$$C(v) := \{x \in X(v); x(S) \geq v(S) \text{ for all } S \subset N\}. \quad (1.1)$$

The set of n -person games with a non-empty core will be denoted by B^n .

In order to recall the definition of the τ -value (Tijs, 1981), we introduce for any game $v \in G^n$ the vectors b^v , λ^v and the map $g^v : 2^N \rightarrow \mathbb{R}$ as follows:

$$b_i^v := v(N) - v(N - \{i\}) \quad \text{for all } i \in N, \quad (1.2)$$

$$g^v(S) := \sum_{i \in S} b_i^v - v(S) = b^v(S) - v(S) \quad \text{for all } S \subset N, \quad (1.3)$$

$$\lambda_i^v := \min_{S; i \in S} g^v(S) \quad \text{for all } i \in N. \quad (1.4)$$

The vector b^v is called the upper vector of the game v because it is an upper bound for the core of the game v . That is,

$$x_i \leq b_i^v \quad \text{for each } x \in C(v) \text{ and all } i \in N. \quad (1.5)$$

1984

The map g^v is called the gap function of the game v and $g^v(S)$ the gap of the coalition S in v . The vector $b^v - \lambda^v$ is a lower bound for the core of the game v , i.e.

$$b_i^v - \lambda_i^v \leq x_i \quad \text{for each } x \in C(v) \text{ and all } i \in N. \quad (1.6)$$

A proof of this result can be found in Tijs and Driessen (1983). In view of (1.6), the vector $b^v - \lambda^v$ is called the lower vector of the game v and the vector λ^v the concession vector of v .

The subclass Q^n of G^n is defined as the set of those n -person games v , for which the lower vector $b^v - \lambda^v$ lies below the upper vector b^v as well as the lower (upper) vector lies below (above) the hyperplane $X(v)$ of feasible pay-off vectors. Thus,

$$Q^n := \{v \in G^n; g^v(S) \geq 0 \text{ for all } S \subset N \text{ and } \lambda^v(N) \geq g^v(N)\}. \quad (1.7)$$

In Tijs (1981), the τ -value $\tau(v)$ of a game $v \in Q^n$ is defined as the unique feasible pay-off vector for the game v , lying on the line segment $[b^v - \lambda^v, b^v]$ with end points the lower vector $b^v - \lambda^v$ and the upper vector b^v . For a game $v \in Q^n$, the τ -value $\tau(v)$ is given by the formula

$$\begin{aligned} \tau(v) &= b^v & \text{if } g^v(N) &= 0 \\ &= b^v - g^v(N) (\lambda^v(N))^{-1} \lambda^v & \text{if } g^v(N) > 0. \end{aligned} \quad (1.8)$$

In Tijs and Lipperts (1982), it is proved that Q^n is a full-dimensional cone in the $(2^n - 1)$ -dimensional linear space G^n . Nevertheless, there are many n -person games, for which the τ -value is not defined. For instance, simple n -person games without veto players do not belong to Q^n (Tijs, 1981).

In this paper we extend and modify the τ -value concept in several ways. First we give two extensions of the τ -value to the space G^n of all n -person games. In section 2, we define the τ -value of a game $v \in G^n$ with the aid of those ε -games v_ε , which belong to Q^n . In section 3, we introduce the core-value of a game $v \in G^n$, which is defined with the aid of those ε -games, which have a non-empty core. We will show that the core-value on G^n can be seen as an extension of the τ -value, restricted to the class B^n of n -person games with a non-empty core. In section 4, some properties of the τ -value and the core-value are mentioned. Secondly, we give two modifications of the τ -value. In section 5, we define the τ -value with respect to a coalition structure. In the last section 6, we replace the upper vector b^v by another utopia vector in order to define a new value, which is similar to the τ -value.

2. THE τ -VALUE ON G^n

In this section we extend the τ -value from Q^n to G^n . For that purpose, we recall a generalization of the core introduced by Shapley and Shubik (1963, 1966). Let $v \in G^n$. For any $\varepsilon \in \mathbb{R}$, the strong ε -core $C_\varepsilon(v)$ of the game v is defined by

$$C_\varepsilon(v) := \{x \in X(v); x(S) \geq v(S) - \varepsilon \text{ for all } S \neq N, \emptyset\}. \quad (2.1)$$

It can be interpreted as the set of all feasible pay-off vectors that can not be

improved upon by any coalition if one imposes a cost of ϵ in all cases where a coalition $S \neq N, \emptyset$ is formed. Clearly, by (1.1) and (2.1), $C_0(v) = C(v)$ and $C_\epsilon(v) = C(v_\epsilon)$ where the n -person game v_ϵ is defined by

$$\begin{aligned} v_\epsilon(S) &= v(S) - \epsilon & \text{if } S \neq N, \emptyset \\ &= v(S) & \text{if } S = N \text{ or } S = \emptyset. \end{aligned} \quad (2.2)$$

The game v_ϵ is called the ϵ -game, corresponding with v and ϵ . For notational reasons, we write also $v(\epsilon)$ instead of v_ϵ . In the following, we are interested in the smallest non-negative ϵ such that v_ϵ belongs to Q^n .

LEMMA 2.1. Let $v \in G^n$ and $\epsilon \in \mathbb{R}$. Then

- (i) $b_i^{v(\epsilon)} = b_i^v + \epsilon$ for all $i \in N$
- (ii) $g^{v(\epsilon)}(S) = g^v(S) + (|S| + 1)\epsilon$ for all $S \neq N, \emptyset$
 $g^{v(\epsilon)}(N) = g^v(N) + n\epsilon$
- (iii) $g^{v(\epsilon)} \geq 0$ iff $\epsilon \geq -\min_{S \neq N, \emptyset} (g^v(S) / (|S| + 1))$
- (iv) $\lambda^{v(\epsilon)}(N) \geq g^{v(\epsilon)}(N)$ if $\epsilon \geq \max\{0, (g^v(N) - \lambda^v(N)) / n\}$.

PROOF. (i) By (1.2) and (2.2), $b_i^{v(\epsilon)} = v_\epsilon(N) - v_\epsilon(N - \{i\}) = v(N) - v(N - \{i\}) + \epsilon = b_i^v + \epsilon$.

(ii) Let $S \neq \emptyset$. By (1.3) and (i), $g^{v(\epsilon)}(S) = b^{v(\epsilon)}(S) - v_\epsilon(S) = b^v(S) + |S|\epsilon - v_\epsilon(S)$. If $S \neq N$, then $g^{v(\epsilon)}(S) = b^v(S) + |S|\epsilon - v(S) + \epsilon = g^v(S) + (|S| + 1)\epsilon$. Also $g^{v(\epsilon)}(N) = b^v(N) + n\epsilon - v(N) = g^v(N) + n\epsilon$.

(iii) follows immediately from (ii) by noting that $g^w(\emptyset) = 0$ and $g^w(N) = g^w(N - \{j\})$ for any $w \in G^n$ and all $j \in N$.

(iv) Let $\epsilon \geq 0$. Then, by (ii), $g^{v(\epsilon)}(S) \geq g^v(S) + 2\epsilon$ for all $S \neq \emptyset$. Thus, by (1.4), $\lambda^{v(\epsilon)} \geq \lambda^v + 2\epsilon$ for all $i \in N$. Hence, $\lambda^{v(\epsilon)}(N) \geq \lambda^v(N) + 2n\epsilon$. If, in addition, $\epsilon \geq (g^v(N) - \lambda^v(N)) / n$, then $\lambda^{v(\epsilon)}(N) \geq \lambda^v(N) + 2n\epsilon \geq g^v(N) + n\epsilon = g^{v(\epsilon)}(N)$, where the equality follows from (ii). So, (iv) holds. $\square$

From lemma 2.1 (iii), (iv) and (1.7) we may conclude that $v_\epsilon \in Q^n$ if ϵ is large enough. For a game $v \in G^n$, the following critical value of ϵ will be of interest:

$$\begin{aligned} \epsilon^*(v) &:= \min\{\epsilon; \epsilon \geq 0, g^{v(\epsilon)} \geq 0 \text{ and } \lambda^{v(\epsilon)}(N) \geq g^{v(\epsilon)}(N)\} \\ &= \min\{\epsilon; \epsilon \geq 0, v_\epsilon \in Q^n\}. \end{aligned} \quad (2.3)$$

It is obvious that $v \in Q^n$ iff $\epsilon^*(v) = 0$. Now we give the extension of the τ -value from Q^n to G^n .

DEFINITION 2.2. The τ -value $\tau(v)$ of a game $v \in G^n$ is defined as the τ -value of the game $v_{\epsilon^*(v)} \in Q^n$.

We conclude this section with an example of a simple three-person game without veto players.

EXAMPLE 2.3. Let v be the three-person game with $v(\{i\}) = 0$ for all $i \in N$ and $v(S) = 1$ if $|S| = 2$ or 3 . Then $b^v = (0, 0, 0)$. Thus $g^v(\{i\}) = 0$ for all $i \in N$ and $g^v(S) = -1$ if $|S| = 2$ or 3 . Hence $v \notin Q^3$. By lemma 2.1 (iii), $g^{v(\epsilon)} \geq 0$ iff $\epsilon \geq 1/3$. For $\epsilon = 1/3$, $g^{v(\epsilon)}(N) = g^v(N) + n\epsilon = 0$, so $\lambda^{v(\epsilon)} = (0, 0, 0)$ and $\lambda^{v(\epsilon)}(N) = 0 = g^{v(\epsilon)}(N)$. Hence $\epsilon^*(v) = 1/3$ and $\tau(v) = \tau(v_{1/3}) = (1/3, 1/3, 1/3)$.

3. THE CORE-VALUE

In the previous sections, we paid special attention to games belonging to Q^n . Note that Q^n contains the set B^n of games with a non-empty core, because of (1.5) and (1.6). Without proof, we remark that the inclusion $B^n \subset Q^n$ is strict iff $n \geq 4$. Hence, for $n \geq 4$, Q^n contains games with an empty core. In this section, we ignore such games and consider only ϵ -games with a non-empty core in order to define a new value on G^n in a similar way as is done for the τ -value.

For a game $v \in G^n$, let $\epsilon_0(v)$ be the smallest ϵ such that the strong ϵ -core is not empty, i.e.

$$\epsilon_0(v) := \min\{\epsilon; C_\epsilon(v) \neq \emptyset\} = \min\{\epsilon; C(v_\epsilon) \neq \emptyset\}.$$

The smallest non-empty strong ϵ -core of a game is treated formally for the first time by Maschler, Peleg and Shapley (1979). Note that the critical value $\epsilon_0(v)$ may be negative and that $\epsilon_0(v) > 0$ iff $C(v) = \emptyset$.

We are interested in the smallest non-negative ϵ such that the strong ϵ -core is not empty. So, we define for any $v \in G^n$

$$\epsilon_1(v) := \min\{\epsilon; \epsilon \geq 0, C_\epsilon(v) \neq \emptyset\} = \min\{\epsilon; \epsilon \geq 0, C(v_\epsilon) \neq \emptyset\}. \quad (3.1)$$

Then $\epsilon_1(v) = \max\{0, \epsilon_0(v)\}$ and $\epsilon_1(v) = 0$ iff $C(v) \neq \emptyset$. Now we introduce a new value.

DEFINITION 3.1. The core-value $c(v)$ of a game $v \in G^n$ is defined as the τ -value of the game $v_{\epsilon_1(v)} \in B^n$ ($\subset Q^n$).

PROPOSITION 3.2. Let $v \in G^n$. Then

- (i) $\epsilon^*(v) \leq \epsilon_1(v)$.
- (ii) If $\epsilon^*(v) = \epsilon_1(v)$, then $c(v) = \tau(v)$.
- (iii) If the core $C(v)$ is non-empty, then $c(v) = \tau(v)$.

PROOF. (i) follows immediately from (2.3), (3.1) and $B^n \subset Q^n$.

(ii) is a direct consequence of the definitions 2.2 and 3.1.

(iii) Let $C(v) \neq \emptyset$. Then $\epsilon_1(v) = 0$, so $c(v) = \tau(v)$ by definition 3.1. $\square$

We noted already that the inclusion $B^n \subset Q^n$ is strict iff $n \geq 4$. Hence, for any two- or three-person game v , the core-value $c(v)$ coincides with the τ -value $\tau(v)$.

In case $n \geq 4$, even for games in Q^n with an empty core, the core-value not necessarily coincides with the τ -value. The following example illustrates this.

EXAMPLE 3.3. Let v be the four-person game with $v(\{i\}) = 0$ for all $i \in N$,

$v(\{2,4\}) = v(\{3,4\}) = 2$, $v(\{1,2,3\}) = 4$, $v(N) = 5$ and $v(S) = 3$ otherwise. Then $b^v = (2,2,2,1)$ and $g^v(\{1,4\}) = 0$, $g^v(S) = 1$ if $|S| = 2$, $S \neq \{1,4\}$, $g^v(\{4\}) = 1$, $g^v(S) = 2$ otherwise. Hence, $\lambda^v = (0,1,1,0)$. Because $g^v \geq 0$ and $\lambda^v(N) = 2 = g^v(N)$, we have $v \in Q^4$. By (1.8), the τ -value $\tau(v) = b^v - \lambda^v = (2,1,1,1)$. It is easy to see that $C(v) = \emptyset$. By simple calculations, we find $\epsilon_0(v) = 1/2$ and $c(v) = (7/4, 5/4, 5/4, 3/4) \neq \tau(v)$.

In view of this example, the core-value on G^n can not be seen as an extension of the τ -value on Q^n . But, by proposition 3.2.(iii), the core-value on G^n can be seen as an extension of the τ -value on B^n .

4. PROPERTIES OF THE τ -VALUE AND THE CORE-VALUE

In this section we give some properties of the τ -value and the core-value. These properties are derived from the properties of the τ -value on Q^n .

First of all, we recall some notions. Let $v \in G^n$. For any permutation $\theta : N \rightarrow N$, the n -person game θv is defined by

$$(\theta v)(\theta S) := v(S) \quad \text{for all } S \subset N.$$

A player $i \in N$ is called a dummy in the game v if

$$v(S \cup \{i\}) - v(S) = v(\{i\}) \quad \text{for all } S \subset N - \{i\}. \quad (4.1)$$

A game $w \in G^n$ is said to be strategically equivalent to v if there exist $k > 0$ and $d \in \mathbb{R}^n$ such that $w(S) = k v(S) + d(S)$ for all $S \subset N$. We write shortly $w = kv + d$. Particularly, $w = v$ means $w(S) = v(S)$ for all $S \subset N$.

THEOREM 4.1. (cf. Tijs, 1981). Let $v \in Q^n$. Then the τ -value $\tau(v)$ has the following properties

- (i) Individual rationality, i.e. $\tau_i(v) \geq v(\{i\})$ for all $i \in N$.
- (ii) Efficiency, i.e. $\sum_{i \in N} \tau_i(v) = v(N)$.
- (iii) Symmetry, i.e. $\theta v \in Q^n$ and $\tau_{\theta(i)}(\theta v) = \tau_i(v)$ for all $i \in N$ and any permutation $\theta : N \rightarrow N$.
- (iv) Dummy player property, i.e. $\tau_i(v) = v(\{i\})$ if i is a dummy in v .
- (v) S-equivalence property, i.e. $w \in Q^n$ and $\tau(w) = k\tau(v) + d$ whenever $w = kv + d$ with $k > 0$ and $d \in \mathbb{R}^n$.

PROOF. The proofs of (ii), (iii) and (v) are omitted because they are straightforward.

To prove (i) and (iv), note that for all $j \in N$

$$b_j^v - \lambda_j^v \leq \tau_j(v) \leq b_j^v \quad \text{and} \quad 0 \leq \lambda_j^v \leq g^v(\{j\}) = b_j^v - v(\{j\}). \quad (4.2)$$

Hence, $\tau_j(v) \geq b_j^v - \lambda_j^v \geq v(\{j\})$ for all $j \in N$ and (i) is proved. Let $i \in N$ be a dummy in v . Then by (4.1), $b_i^v = v(N) - v(N - \{i\}) = v(\{i\})$. It follows from (4.2) that $\lambda_i^v = 0$ and $\tau_i(v) = b_i^v = v(\{i\})$. $\square$

Now we list some properties of the τ -value on G^n .

THEOREM 4.2. Let $v \in G^n$. Then the τ -value $\tau(v)$ has the following properties

- (i) $\epsilon^*(v)$ -Individual rationality, i.e. $\tau_i(v) \geq v(\{i\}) - \epsilon^*(v)$ for all $i \in N$.
- (ii) Efficiency, i.e. $\sum_{i \in N} \tau_i(v) = v(N)$.
- (iii) Symmetry, i.e. $\tau_{\theta(i)}(\theta v) = \tau_i(v)$ for all $i \in N$ and any permutation $\theta : N \rightarrow N$.
- (iv) $\epsilon^*(v)$ -Dummy player property, i.e. $\tau_i(v) \leq v(\{i\}) + \epsilon^*(v)$ if i is a dummy in v .
- (v) S-equivalence property, i.e. if $w = kv + d$ with $k > 0$, $d \in \mathbb{R}^n$ then

$$\tau(w) = k\tau(v) + d.$$

PROOF. In this proof, we write $\epsilon(v)$ instead of $\epsilon^*(v)$. Then, by definition 2.2.,

$\tau(v) = \tau(v_{\epsilon(v)})$ where $v_{\epsilon(v)} \in Q^n$.

(i) For any $j \in N$, $\tau_j(v) = \tau_j(v_{\epsilon(v)}) \geq v_{\epsilon(v)}(\{j\}) = v(\{j\}) - \epsilon(v)$ where the inequality follows from theorem 4.1.(i).

(ii) By theorem 4.1.(ii), $\sum_{i \in N} \tau_i(v) = \sum_{i \in N} \tau_i(v_{\epsilon(v)}) = v_{\epsilon(v)}(N) = v(N)$.

(iii) Let $\theta : N \rightarrow N$ be a permutation. It is straightforward to show that $(\theta v)_{\epsilon} = \theta(v_{\epsilon})$ for any $\epsilon \in \mathbb{R}$. Hence, $\epsilon^*(\theta v) = \epsilon^*(v)$ by noting that, for any $w \in G^n$ and any permutation $\pi : N \rightarrow N$, we have $\pi^{-1}(\pi w) = w$ and $w \in Q^n$ iff $\pi w \in Q^n$. It follows by theorem 4.1.(iii) that for all $i \in N$

$$\tau_{\theta(i)}(\theta v) = \tau_{\theta(i)}((\theta v)_{\epsilon(\theta v)}) = \tau_{\theta(i)}((\theta v)_{\epsilon(v)}) = \tau_{\theta(i)}(\theta(v_{\epsilon(v)})) = \tau_i(v_{\epsilon(v)}) = \tau_i(v).$$

(iv) Let i be a dummy in v . Then, by (4.1), $b_i^v = v(\{i\})$. Thus,

$$\tau_i(v) = \tau_i(v_{\epsilon(v)}) \leq b_i^{v_{\epsilon(v)}} = b_i^v + \epsilon(v) = v(\{i\}) + \epsilon(v).$$

(v) Let $w = kv + d$ with $k > 0$ and $d \in \mathbb{R}^n$. Then also, for any $\epsilon \in \mathbb{R}$, $w_{\epsilon} = k v_{\epsilon/k} + d$.

Hence, $\epsilon^*(w) = k \epsilon^*(v)$ by noting that $w_1 \in Q^n$ iff $w_2 \in Q^n$ whenever w_1 and w_2 are strategically equivalent. It follows by theorem 4.1.(v) that

$$\tau(w) = \tau(w_{\epsilon(w)}) = \tau(k v_{\epsilon(w)/k} + d) = \tau(k v_{\epsilon(v)} + d) = k \tau(v_{\epsilon(v)}) + d = k \tau(v) + d. \quad \square$$

Similar properties for the core-value are stated in the following theorem. The proof of this theorem is omitted because it is similar to the proof of theorem 4.2.

THEOREM 4.3. Let $v \in G^n$. Then the core-value $c(v)$ has the following properties

- (i) $\epsilon_1(v)$ -Individual rationality, i.e. $c_i(v) \geq v(\{i\}) - \epsilon_1(v)$ for all $i \in N$.
- (ii) Efficiency, i.e. $\sum_{i \in N} c_i(v) = v(N)$.
- (iii) Symmetry, i.e. $c_{\theta(i)}(\theta v) = c_i(v)$ for all $i \in N$ and any permutation $\theta : N \rightarrow N$.
- (iv) $\epsilon_1(v)$ -Dummy player property, i.e. $c_i(v) \leq v(\{i\}) + \epsilon_1(v)$ if i is a dummy in v .
- (v) S-equivalence property, i.e. if $w = kv + d$ with $k > 0$, $d \in \mathbb{R}^n$ then

$$c(w) = k c(v) + d.$$

By theorem 4.1.(iv), the τ -value on Q^n satisfies the dummy player property. This property is not valid for the τ -value (and the core-value) on G^n as we can learn from the following example.

EXAMPLE 4.4. Let v be the four-person game with $v(N) = v(\{1,2,3\}) = 5$, $v(\{2,3\}) = v(\{2,3,4\}) = 6$ and $v(S) = 0$ otherwise. Then player 4 is a dummy in the game v . For this game $\epsilon^*(v) = 1/2$ and $\tau(v) = \tau(v_{1/2}) = (-1/2, 121/46, 121/46, 11/46)$. We

see that $\tau_4(v) \neq v(\{4\})$ although player 4 is a dummy. Furthermore, $\varepsilon_0(v) = 1/2 = \varepsilon^*(v)$, so $c(v) = \tau(v)$ by proposition 3.2. (ii).

The following example shows that the properties mentioned in the theorems 4.2 and 4.3 can be useful in order to determine values in an easy way.

EXAMPLE 4.5. Let v be the four-person game with $v(\{i\}) = 0$ for all $i \in N$, $v(N) = 5$ and $v(S) = 3$ otherwise. Then $\theta v = v$ for any permutation $\theta : N \rightarrow N$. By the symmetry and the efficiency, $c(v) = (5/4, 5/4, 5/4, 5/4) = \tau(v)$. Note that $v \in Q^4$, but $C(v) = \emptyset$. So $\varepsilon^*(v) = 0 < \varepsilon_1(v)$, although $c(v) = \tau(v)$. Thus, the converses of proposition 3.2 (ii) and (iii) do not hold.

REMARK 4.6. The τ -value and the core-value are defined with the aid of a suitable ε -game. Instead of ε -games, we could have used Δ -games. A Δ -game is obtained if one imposes costs $|S|\Delta$ in all cases where a coalition $S \neq N$ is formed. That is, for any $v \in G^N$ and $\Delta \in \mathbb{R}$, the Δ -game v_Δ is defined by $v_\Delta(S) := v(S) - |S|\Delta$ if $S \neq N$ and $v_\Delta(N) := v(N)$. The core of v_Δ is called the weak Δ -core of v .

5. THE τ -VALUE WITH RESPECT TO COALITION STRUCTURES

Many game-theoretic solution notions can be defined not only with reference to the all-player coalition, but also with reference to an arbitrary coalition structure (Aumann and Dreze, 1974). In this section, we define the τ -value with respect to a coalition structure.

Fix N and a coalition structure $\mathcal{B} = (B_1, \dots, B_p)$ on N , i.e. a partition of N . Let $v \in G^N$. First we define the upper vector $b^{v, \mathcal{B}}$, the gap function $g^{v, \mathcal{B}}$ and the concession vector $\lambda^{v, \mathcal{B}}$ of the game v w.r.t. $\mathcal{B}$ by

$$\begin{aligned} b_i^{v, \mathcal{B}} &:= v(B_\ell) - v(B_\ell - \{i\}) && \text{if } i \in B_\ell, \\ g^{v, \mathcal{B}}(S) &:= b^{v, \mathcal{B}}(S) - v(S) && \text{for all } S \subset N, \\ \lambda_i^{v, \mathcal{B}} &:= \min_{S: i \in S} g^{v, \mathcal{B}}(S) && \text{for all } i \in N. \end{aligned}$$

Furthermore, the set $X(v, \mathcal{B})$ of feasible pay-off vectors for the game v w.r.t. $\mathcal{B}$ is given by

$$X(v, \mathcal{B}) := \{x \in \mathbb{R}^N; x(B_\ell) = v(B_\ell) \text{ for each } B_\ell \in \mathcal{B}\}.$$

For games v with a non-negative gap function $g^{v, \mathcal{B}}$, we have $\lambda^{v, \mathcal{B}} \geq 0$ and hence, $b^{v, \mathcal{B}} - \lambda^{v, \mathcal{B}} \leq b^{v, \mathcal{B}}$. If, in addition, $(b^{v, \mathcal{B}} - \lambda^{v, \mathcal{B}})(B_\ell) \leq v(B_\ell)$ for each $B_\ell \in \mathcal{B}$, then we define the τ -value $\tau(v, \mathcal{B})$ of the game v w.r.t. $\mathcal{B}$ as the unique feasible pay-off vector for v w.r.t. $\mathcal{B}$, which lies, for each $B_\ell \in \mathcal{B}$, on the line segment with end points the restrictions of the vectors $b^{v, \mathcal{B}} - \lambda^{v, \mathcal{B}}$ and $b^{v, \mathcal{B}}$ to B_ℓ . Thus, we consider the subclass

$$Q^N(\mathcal{B}) := \{v \in G^N; g^{v, \mathcal{B}} \geq 0 \text{ and } \lambda^{v, \mathcal{B}}(B_\ell) \geq g^{v, \mathcal{B}}(B_\ell) \text{ for each } B_\ell \in \mathcal{B}\}.$$

DEFINITION 5.1. For a game $v \in Q^n(B)$, its τ -value $\tau(v, B)$ with respect to B is given by

$$\begin{aligned}\tau_i(v, B) &= b_i^{v, B} && \text{if } i \in B_\ell \text{ and } g^{v, B}(B_\ell) = 0 \\ &= b_i^{v, B} - g^{v, B}(B_\ell) (\lambda^{v, B}(B_\ell))^{-1} \lambda_i^{v, B} && \text{if } i \in B_\ell \text{ and } g^{v, B}(B_\ell) > 0.\end{aligned}$$

In case $B = (N)$, we are dealing with the τ -value $\tau(v)$ on Q^n .

The τ -value $\tau(v, B)$ w.r.t. B can be extended to G^n in a similar way as in section 2.

An ϵ -game $v_{\epsilon, B}$ w.r.t. B is then defined by

$$\begin{aligned}v_{\epsilon, B}(S) &= v(S) - \epsilon && \text{if } S \neq \emptyset, B_1, \dots, B_p \\ &= v(S) && \text{otherwise.}\end{aligned}$$

Properties, similar to those of theorem 4.2., can also be given for the τ -value $\tau(v, B)$ with respect to the coalition structure B .

6. UTOPIA VECTORS

The set $U(v)$ of utopia vectors for the game $v \in G^n$ is defined by

$$U(v) := \{x \in \mathbb{R}^n; x(S) \geq v(S) \text{ for all } S \subset N\}. \quad (6.1)$$

That is, $U(v)$ consists of those vectors, which can not be improved upon by any coalition. Note that $b^v \in U(v)$ if $v \in Q^n$ and that $U(v) \cap X(v) = C(v)$. In this section we pay special attention to the vector $r(v)$, which appears in the definition of the reasonable set $R(v)$, introduced by Milnor (1952). For any $v \in G^n$, this set is defined by

$$R(v) := \{x \in X(v); x_i \leq r_i(v) \text{ for all } i \in N\}$$

where

$$r_i(v) := \max_{S; i \notin S} (v(S \cup \{i\}) - v(S)) \quad \text{for all } i \in N. \quad (6.2)$$

Thus, $R(v)$ consists of all feasible pay-off vectors, that give no player more than the largest amount he can contribute to a coalition. Note that $r_i(v) \geq b_i^v$ for all $i \in N$ and hence, by (1.5), $C(v) \subset R(v)$.

The τ -value of a game $v \in Q^n$ is based on the utopia vector b^v . Now we define in a similar way a new value, based on the vector $r(v)$.

First of all, we define for any $v \in G^n$ and any $x \in \mathbb{R}^n$, the gap function $g^{v, x}$ and the concession vector $\lambda^{v, x}$ by

$$\begin{aligned}g^{v, x}(S) &:= x(S) - v(S) && \text{for all } S \subset N, \\ \lambda_i^{v, x} &:= \min_{S; i \in S} g^{v, x}(S) && \text{for all } i \in N.\end{aligned}$$

The following lemma states that $r(v)$ is always a utopia vector and that its corresponding gap function $g^{v, r(v)}$ is increasing.

LEMMA 6.1. Let $v \in G^n$. Then

(i) $r(v) \in U(v)$.

- (ii) $g^{v,r(v)}$ is increasing, i.e. $g^{v,r(v)}(S) \leq g^{v,r(v)}(T)$ if $S \subset T \subset N$.
- (iii) $\lambda_i^{v,r(v)} = g^{v,r(v)}(\{i\})$ for all $i \in N$.
- (iv) $\lambda^{v,r(v)}(N) \geq g^{v,r(v)}(N)$ iff $v(N) \geq \sum_{i \in N} v(\{i\})$.

PROOF. (i) Let $S \neq \emptyset$. Write $S = \{i_1, \dots, i_s\}$, where $s = |S|$. Then

$$v(S) = v(S) - v(\emptyset) = \sum_{t=1}^s [v(\{i_1, \dots, i_{t-1}, i_t\}) - v(\{i_1, \dots, i_{t-1}\})] \leq \sum_{t=1}^s r_{i_t}(v) = \sum_{j \in S} r_j(v),$$

where the inequality follows from (6.2). Thus, by (6.1), $r(v) \in U(v)$.

(ii) It is sufficient to show that for each $S \subset N$ and all $j \notin S$,
 $g^{v,r(v)}(S) \leq g^{v,r(v)}(S \cup \{j\})$. But this inequality is equivalent to
 $v(S \cup \{j\}) - v(S) \leq r_j(v)$, which holds by (6.2).

(iii) follows immediately from the definition of $\lambda^{v,r(v)}$ and (ii).

(iv) is a direct consequence of (iii). $\square$

For any game v with $v(N) \geq \sum_{i \in N} v(\{i\})$, we define the m-value $m(v)$ of the game v as the unique feasible pay-off vector, lying on the line segment with end points $(v(\{1\}), v(\{2\}), \dots, v(\{n\}))$ and $r(v)$. Now we immediately obtain

COROLLARY 6.2. Let $v \in G^n$ be zero-normalized (i.e. $v(\{i\}) = 0$ for all $i \in N$) and $v(N) > 0$. Then the m-value $m(v)$ of the game v is given by $v(N) (\sum_{i \in N} r_i(v))^{-1} r(v)$.

For games $v \in Q^n$, the τ -value $\tau(v)$ is a reasonable outcome because
 $\tau_i(v) \leq b_i^v \leq r_i(v)$ for all $i \in N$. In general, the τ -value of $v \in G^n$ is not in the reasonable set. For instance, for the game v of example 4.4, we have $\tau_4(v) > 0 = r_4(v)$.

We conclude this paper with some remarks on convex games. These games are introduced by Shapley (1971), who also showed that the core of a convex game is non-empty and that its structure is very regular. By proposition 3.2.(iii), the core-value $c(v)$ of a convex game v coincides with its τ -value $\tau(v)$. Furthermore, by the definition of convexity of the game v , it can easily be shown that $r(v) = b^v$ and hence, by lemma 6.1.(iii), $b^v - \lambda^v = (v(\{1\}), v(\{2\}), \dots, v(\{n\}))$. The vectors $b^v - \lambda^v$ and b^v appear to be sharp lower and upper bounds, respectively, for the core of a convex game, but its core-value (= τ -value), which is defined with the aid of those vectors, is not necessarily a core element (cf. Driessen and Tijs (1982), Tijs and Lipperts (1982)).

REFERENCES

- Aumann, R.J. and Dreze, J.H. (1974). Cooperative games with coalition structures. *Int. J. Game Theory* 3, 217-237.
- Driessen, T.S.H. and Tijs, S.H. (1982). Semiconvex games and the τ -value. Report 8228, Department of Mathematics, Catholic University, Nijmegen, The Netherlands.
- Driessen, T.S.H. and Tijs, S.H. (1983). The τ -value, the nucleolus and the core for a subclass of games. *Methods Oper. Res.* (to appear).
- Maschler, M., Peleg, B. and Shapley, L.S. (1979). Geometric properties of the kernel, nucleolus and related solution concepts. *Math. Oper. Res.* 4, 303-338.
- Milnor, J.W. (1952). Reasonable outcomes for n-person games. RM-916, The Rand Corporation, Santa Monica, CA.

- Shapley, L.S. (1971). Cores of convex games. *Int. J. Game Theory* 1, 11-26.
- Shapley, L.S. and Shubik, M. (1963). The core of an economy with nonconvex preferences. RM-3518, The Rand Corporation, Santa Monica, CA.
- Shapley, L.S. and Shubik, M. (1966). Quasi-cores in a monetary economy with nonconvex preferences. *Econometrica* 34, 805-827.
- Tijs, S.H. (1981). Bounds for the core and the τ -value. In: *Game Theory and Mathematical Economics* (Eds. O. Moeschlin and D. Pallaschke), North-Holland Publ. Cie, 123-132.
- Tijs, S.H. and Driessen, T.S.H. (1983). The τ -value as a feasible compromise between utopia and disagreement. Report 8312, Department of Mathematics, Catholic University, Nijmegen, The Netherlands.
- Tijs, S.H. and Lipperts, F.A.S. (1982). The hypercube and the core cover of n -person cooperative games. *Cah. Cent. Etud. Rech. Oper.* 24, 27-37.