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A Relation between Perfect Equilibria in Extensive Form Games and Proper Equilibria in Normal Form Games ¹⁾

By *E. van Damme*, Delft²⁾

Abstract: The concept of quasi-perfect equilibria for games in extensive form is introduced. It is shown that a proper equilibrium of a normal form game induces a quasi-perfect equilibrium in every extensive form game having this normal form.

1. Introduction

In *Selten* [1975] the concept of perfect equilibria was introduced for games in extensive form with perfect recall in order to exclude the possibility that disequilibrium behavior is prescribed at unreached parts of the game tree. Selten achieves equilibrium behavior everywhere by assuming that every player will make mistakes with a small probability and, consequently, will take every choice with a positive probability. Hence, in such a “perturbed game” there are no unreached parts and, therefore, every player will make an equilibrium choice everywhere. A perfect equilibrium is then defined as an equilibrium which can be obtained as the limit of a sequence of equilibria, associated with a sequence of perturbed games for which the mistake parameter goes to zero. Selten showed that every finite game in extensive form with perfect recall has at least one perfect equilibrium, so indeed the perfectness concept is a useful refinement of the Nash equilibrium concept.

Selten [1975] also introduced the perfectness concept for finite games in normal form. However, the relation between perfect equilibria in extensive form games and perfect equilibria in normal form games is not as nice as one (perhaps) would like it to be. Namely, a perfect equilibrium of the extensive form need not be perfect in the normal form (example 4) and a perfect equilibrium of the normal form need not be perfect in the extensive form (cf. the examples 1 – 3).

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In *Myerson* [1978] another undesirable property of perfect equilibria of normal form games is noted, namely, that adding strictly dominated strategies may transform imperfect equilibria into perfect ones. Therefore, Myerson introduced a refinement of the perfectness concept: the properness concept. Furthermore, he showed that every normal form game has at least one proper equilibrium.

In this paper, the concept of quasi-perfect equilibria for games in extensive form is introduced. The difference between this concept and Selten's perfectness concept is that the latter requires that each player at every information set takes a choice which is optimal against mistakes of all players (including the player himself), whereas the quasi-perfectness concept requires that at every information set a choice is taken which is optimal against mistakes of the other players. Both the perfectness and the quasi-perfectness concept are refinements of the sequential equilibrium concept [*Kreps/Wilson*], but a quasi-perfect equilibrium need not be perfect (cf. example 3) and a perfect equilibrium need not be quasi-perfect (cf. example 4). In section 4 of the paper, we relate the properness concept of the normal form to the quasi-perfectness concept of the extensive form. Our main result is that a proper equilibrium of a normal form game induces a quasi-perfect equilibrium in every extensive form game having this normal form. Hence, proper equilibria of the normal form induce sensible behavior in the extensive form.

In section 3, we review the reason why a perfect equilibrium of the normal form does not induce sensible behavior in the extensive form. The examples given in this section also illustrate that the result of section 4 is the best we can obtain: we cannot expect that a choice prescribed by a proper equilibrium will be a best reply against mistakes of the player himself.

Throughout the paper, the notation used is the same as in *Selten* [1975]. For the convenience of the reader some notation and some definitions from that paper are reproduced in section 2.

2. Preliminaries

In this paper we consider noncooperative finite n -person games in extensive form and the normal form games associated with them. A finite game in extensive form is determined by:

- (i) A game tree K , a tree with finitely many vertices and edges,
- (ii) for each vertex of K , a specification of which player has to move at this vertex,
- (iii) a specification of the information a player has, when he has to move; this specification is provided by the information sets,
- (iv) a description of the possible choices at each move,
- (v) a specification of the probabilities that are involved with the chance moves, and
- (vi) a specification of the rewards for each player.

We will restrict ourselves to games with *perfect recall* [Kuhn], which means that at any instant of the game each player knows what he previously has known and what he previously has done. Below, some definitions with respect to a finite game Γ in extensive form with perfect recall are given.

Let u be an information set of player i in Γ . A local strategy b_{iu} at u is a probability distribution on C_u , where C_u denotes the set of choices at u . The probability that b_{iu} assigns to c is denoted by $b_{iu}(c)$ and B_{iu} denotes the set of all local strategies at u . We view C_u as a subset of B_{iu} .

A *behavior strategy* b_i of player i is a mapping that assigns to every information set of player i a local strategy. U_i denotes the set of all information sets of player i and B_i is the set of all behavior strategies of this player. A behavior strategy b_i is *completely mixed* if $b_{iu}(c) > 0$ for all $u \in U_i$ and $c \in C_u$. If $b_i \in B_i$ and $b'_{iu} \in B_{iu}$, then b_i/b'_{iu} is used to denote the behavior strategy which results from b_i if b_{iu} is changed to b'_{iu} , whereas all other local strategies remain unchanged.

A *pure strategy* π_i of player i is a mapping which assigns an element of C_u to every $u \in U_i$. We write π_{iu} for the choice which π_i assigns to u and Π_i denotes the set of all pure strategies of player i .

A *mixed strategy* q_i of player i is a probability distribution on Π_i . The probability that q_i assigns to π_i is denoted by $q_i(\pi_i)$ and q_i is *completely mixed* if $q_i(\pi_i) > 0$ for all $\pi_i \in \Pi_i$. The set of all mixed strategies of player i is denoted by Q_i .

Let $S_i = B_i \cup Q_i$ and $S = \prod_{i=1}^n S_i$. Elements of S are called *strategy combinations*. A

strategy combination is *completely mixed* if all of its components are completely mixed. If $s \in S$ and $s'_i \in S_i$, then s/s'_i denotes the strategy combination which results from s if s_i is replaced by s'_i and all other components remain unchanged. If b is a behavior strategy combination and $b'_{iu} \in B_{iu}$, then b/b'_{iu} is used to denote $b/(b_i/b'_{iu})$. For $s \in S$ and every vertex x of the tree we can compute the probability $\rho(x; s)$ that x will be reached when s is played. Similarly, we write $\rho(u; s)$ for the probability that the information set u will be reached when s is played. Following Kuhn [1953], we call an information set $u \in U_i$ *relevant* when playing s_i , if u can be reached when s_i is played (i.e. if $\rho(u; s/s_i) > 0$ for some $s \in S$). The set of all $u \in U_i$ which are relevant for s_i is denoted by $\text{Rel}(s_i)$. Furthermore, we define for $u \in U_i$ and $c \in C_u$:

$$\text{Rel}(u) := \{\pi_i \in \Pi_i; u \in \text{Rel}(\pi_i)\},$$

$$\text{Rel}(c) := \{\pi_i \in \text{Rel}(u); \pi_{iu} = c\}.$$

From the assumption of perfect recall it follows that the players can restrict themselves to behavior strategies [Kuhn]. To be more precise: for every player i and for every

mixed strategy q_i there exists a behavior strategy b_i which is *realization equivalent* to q_i , i.e. which satisfies

$$\rho(x; s/q_i) = \rho(x; s/b_i) \quad \text{for every } s \text{ and } x.$$

If all information sets are relevant for q_i (which is the case if q_i is completely mixed) there is a unique behavior strategy which is realization equivalent to q_i . This behavior strategy is given by

$$b_{iu}(c) = \frac{\sum_{\pi_i \in \text{Rel}(c)} q_i(\pi_i)}{\sum_{\pi_i \in \text{Rel}(u)} q_i(\pi_i)} \quad \text{for all } u \in U_i, c \in C_u, \quad (2.1)$$

and is called the behavior strategy *induced* by q_i .

Associated with every endpoint z of K , there is a payoff for each player; $h_i(z)$ denotes the payoff to player i when z is reached. If the behavior strategy combination b is played, the *expected payoff* to player i is $H_i(b) = \sum_z \rho(z; b) h_i(z)$. The *normal form* associated with Γ is the normal form $G = (\Pi_1, \dots, \Pi_n, H_1, \dots, H_n)$.

If $\rho(u; b) > 0$, then the conditional probability $\rho(x; u, b)$ that x will be reached, given that u has been reached and b is played is well-defined for every vertex x of K . We denote player i 's expected payoff given that u is reached and b is played by $H_{iu}(b)$, hence, $H_{iu}(b) = \sum_z \rho(z; u, b) h_i(z)$.

We now turn to the definitions of perfect and proper equilibria. Let Γ be an extensive form game and let $\epsilon > 0$. A behavior strategy combination b is an *ϵ -perfect equilibrium* of Γ if b is completely mixed and satisfies

$$\text{if } H_{iu}(b/c) < H_{iu}(b/c'), \text{ then } b_{iu}(c) \leq \epsilon \quad \text{for all } i, u; c, c' \in C_u. \quad (2.2)$$

A behavior strategy combination is a *perfect equilibrium* of Γ if it is a limit point (as ϵ tends to 0) of ϵ -perfect equilibria.

Let G be a normal form game and let $\epsilon > 0$. A mixed strategy combination q is an *ϵ -perfect equilibrium* of G if it is completely mixed and satisfies

$$\text{if } H_i(q/\pi_i) < H_i(q/\pi'_i), \text{ then } q_i(\pi_i) \leq \epsilon \quad \text{for all } i, \pi_i, \pi'_i. \quad (2.3)$$

A mixed strategy combination is a *perfect equilibrium* of G if it is a limit point of ϵ -perfect equilibria of G .

Let G be a normal form game and let $\epsilon > 0$. A mixed strategy combination q is an *ϵ -proper equilibrium* of G if it is completely mixed and satisfies

$$\text{if } H_i(q/\pi_i) < H_i(q/\pi'_i), \text{ then } q_i(\pi_i) \leq \epsilon q_i(\pi'_i) \quad \text{for all } i, \pi_i, \pi'_i. \quad (2.4)$$

A mixed strategy combination is a *proper equilibrium* of G if it is a limit point of ϵ -proper equilibria of G .

Remarks:

1. The definition of perfectness given above is somewhat different from Selten's original definition. We have given an equivalent definition, due to Myerson [1978]. (It follows from Lemma 5 and Theorem 4 of Selten [1975] that indeed also for games in extensive form this definition is equivalent.)
2. Since an ϵ_1 -proper equilibrium is also ϵ_2 -proper if $\epsilon_1 < \epsilon_2$, we have that, if q is a proper equilibrium of G , there exists for every $\epsilon > 0$ some ϵ -proper equilibrium q^ϵ of G such that $q = \lim_{\epsilon \downarrow 0} q^\epsilon$.
3. Let Γ be an extensive form game with normal form G and let q be a proper equilibrium of G . Furthermore, let q^ϵ be an ϵ -proper equilibrium of G with $\lim_{\epsilon \downarrow 0} q^\epsilon = q$ and let b^ϵ be the behavior strategy combination induced by q^ϵ . Since Γ is finite, there exists a limit point (as ϵ tends to 0) of $\{b^\epsilon\}_\epsilon$. Such a limit point is called a *limit behavior strategy combination* induced by q . Note that, if b is a limit behavior strategy combination induced by q , then b_i is realization equivalent to q_i , for every i .

In order to facilitate comparison between the perfectness concept and the quasi-perfectness concept, we give the following characterization of perfect equilibria:

Lemma 1: A behavior strategy combination b is a perfect equilibrium of an extensive form game Γ if and only if there exists a sequence $\{b^k\}_{k \in \mathbb{N}}$ of completely mixed behavior strategy combinations, converging to b , such that b_{iu} is a *local best reply* against b^k for every i, u and k , i.e.

$$H_{iu}(b^k/b_{iu}) = \max_{b'_{iu} \in B_{iu}} H_{iu}(b^k/b'_{iu}) \quad \text{for all } i, u, k.$$

Proof. This immediately follows from the theorems 4 and 7 of Selten [1975]. $\square$

3. Examples

In this section, we give some examples to illustrate the discrepancy between perfectness in the normal form and perfectness in the extensive form [also see Selten, section 13]. The discussion of the examples will exhibit the relation between properness in the normal form and quasi-perfectness in the extensive form. In example 3 we illustrate what kind of information cannot be regained by considering proper equilibria.

Example 1.

extensive form Γ

	L_2	R_2
$L_1 l$	3 1	1 0
$L_1 r$	0 0	0 1
$R_1 l$	2 2	2 2
$R_1 r$	2 2	2 2

Corresponding normal form G

The unique subgame perfect (hence, unique perfect) equilibrium of Γ is $(L_1 l, L_2)$. This equilibrium is also a perfect equilibrium of G , but it is not the only one: also $(R_1 r, R_2)$ is perfect in G . Namely, for every $\epsilon \in (0, 1/3)$, we have that $q^\epsilon = (q_1^\epsilon, q_2^\epsilon)$ with $q_1^\epsilon = (\epsilon, \epsilon, \epsilon, 1 - 3\epsilon)$ and $q_2^\epsilon = (\epsilon, 1 - \epsilon)$ is a 3ϵ -perfect equilibrium of G with $\lim_{\epsilon \downarrow 0} q^\epsilon = (R_1 r, R_2)$.

In order to investigate why a perfect equilibrium of G need not be perfect in Γ , let us compare ϵ -perfect equilibria of Γ with ϵ -perfect equilibria of G . Note that (2.2) expresses that, in an ϵ -perfect equilibrium of Γ , every non-optimal choice has to be chosen with a small probability. However, if q^ϵ is an ϵ -perfect equilibrium of G close to $(R_1 r, R_2)$, then

$$q_1^\epsilon(L_1 r) \geq q_1^\epsilon(L_1 l), \quad \text{if } \epsilon \text{ is small,} \quad (3.1)$$

since R_2 must be a best reply against q_1^ϵ if ϵ is small. Therefore, we have for the behavior strategy b_1^ϵ induced by q_1^ϵ :

$$b_{1v}^\epsilon(r) \geq b_{1v}^\epsilon(l) \quad \text{if } \epsilon \text{ is small.} \quad (3.2)$$

Hence, player 1 is required to choose r with a positive probability at his second information set, which obviously is irrational since l strictly dominates r . The reason that irrational behavior is prescribed at player 1's second information set is that the local strategy at this information set is completely determined by the mistake probabilities $q_1^\epsilon(L_1 l)$ and $q_1^\epsilon(L_1 r)$, which may be chosen arbitrary in an ϵ -perfect equilibrium.

Notice that, if q^ϵ is an ϵ -proper equilibrium of G , then (3.1) cannot be satisfied and, therefore, the only proper equilibrium of G is (L_1l, L_2) . Hence, in this example, rational behavior in the extensive form can be detected already in the normal form. However, in the examples 2 and 3 we will show that things are not always as nice as in this example.

Example 2

Lr is a proper equilibrium of G , which is not perfect in Γ (only Ll is perfect in Γ). Hence, a proper equilibrium of the normal form may prescribe irrational (nonmaximizing) behavior at an information set which is not relevant when this equilibrium is played. However, rational behavior at such an information set can be obtained by considering limit behavior strategies (cf. remark 3 of section 2). Namely, if q^ϵ is an ϵ -proper equilibrium with $\lim_{\epsilon \downarrow 0} q^\epsilon = Lr$, then

$$q^\epsilon(Rr) \leq \epsilon q^\epsilon(Rl)$$

and, therefore, if b^ϵ is induced by q^ϵ , then $\lim_{\epsilon \downarrow 0} b^\epsilon = Ll$. Hence, the proper equilibrium Lr induces a unique limit behavior strategy, viz. Ll , the unique perfect equilibrium of Γ .

Example 3

The unique perfect equilibrium of Γ is Ll (player 1 has to choose L since he has to incorporate the possibility of making mistakes at his second information set). A proper equilibrium of G is Rl and the unique limit behavior strategy induced by Rl is Rl itself. Hence, a proper equilibrium may prescribe a choice which is not optimal against mistakes made by the player himself. However, if player 1 is sure that he will make no mistakes, he can safely play R at his first information set. Therefore, we call Rl a quasi-perfect equilibrium of Γ .

For a general n -person game in extensive form with perfect recall, a quasi-perfect equilibrium is defined as a behavior strategy combination which prescribes at every information set a choice which is optimal against mistakes of the other players. Before giving the formal definition, we introduce some notation. For an information set u , we write $Z(u)$ for the set of all endpoints of the tree coming after u and if $u, v \in U_i$, we write $u \leq v$ if $Z(u) \supset Z(v)$. As usual, $u < v$ stands for $u \leq v$ and $u \neq v$ (hence, $u < v$ means v comes after u). Note that the relation $\leq$ partially orders U_i , since the game has perfect recall. If $b_i, b'_i \in B_i$ and $u \in U_i$, then we use $b_i /_u b'_i$ to denote the behavior strategy b''_i defined by

$$b''_{iv} = \begin{cases} b'_{iv} & \text{if } v \geq u, \\ b_{iv} & \text{otherwise.} \end{cases}$$

Furthermore, if $b \in B$ and $b'_i \in B_i$, then $b /_u b'_i$ denotes the behavior strategy combination $b / (b_i /_u b'_i)$.

Definition 1. Let Γ be an n -person game in extensive form with perfect recall. A behavior strategy combination b is a *quasi-perfect equilibrium* of Γ if there exists a sequence $\{b^k\}_{k \in \mathbb{N}}$ of completely mixed behavior strategy combinations, converging to b , such that for all i, u and k

$$H_{iu}(b^k /_u b_i) = \max_{b'_i \in B_i} H_{iu}(b^k /_u b'_i). \quad (3.3)$$

In example 3, we have already seen that a quasi-perfect equilibrium need not be perfect. The next example shows that a perfect equilibrium need not be quasi-perfect.

The unique quasi-perfect equilibrium of Γ is $(L_1 l, L_2)$. This equilibrium is also perfect, but it is not the only one: another perfect equilibrium is $(R_1 l, L_2)$. Note that this example also shows that a perfect equilibrium of Γ need not be perfect in G (the unique perfect equilibrium of G is $(L_1 l, L_2)$).

Example 4

 extensive form Γ

	L_2	R_2
$L_1 \ell$	1 1	1 1
$L_1 r$	0 0	0 0
$R_1 \ell$	1 1	0 0
$R_1 r$	1 1	0 0

 normal form G

It can easily be shown that every quasi-perfect equilibrium of the extensive form induces a perfect equilibrium in the normal form (the converse is not true, cf. example 1). The following proposition is easy to show either [cf. *Kreps/Wilson*, Proposition 5].

Proposition 1. Every quasi-perfect equilibrium is a sequential equilibrium.

The converse of this proposition is false, since for normal form games the set of perfect equilibria coincides with the set of quasi-perfect equilibria and since for such a game not every sequential (= Nash) equilibrium is perfect.

4. The Main Results

In this section, we will prove that every proper equilibrium of a normal form game induces a quasi-perfect equilibrium in every extensive form game having this normal form. Furthermore, several characterizations of quasi-perfect equilibria will be derived.

We start with a definition. Let b be a completely mixed behavior strategy combination and let b'_{iu} be a local strategy at the information set u of player i . We say that b'_{iu} is a *quasi-best reply* against b if there exists some $b'_i \in B_i$ which prescribes b'_{iu} at u such that

$$H_{iu}(b/_u b'_i) = \max_{b''_i \in B_i} H_{iu}(b/_u b''_i).$$

We have the following lemma:

Lemma 2. b is a quasi-perfect equilibrium of Γ if and only if there exists a sequence $\{b^k\}_{k \in \mathbb{N}}$ of completely mixed behavior strategy combinations converging to b such that b_{iu} is a quasi-best reply against b^k for every i, u and k .

Proof. It immediately follows from the definitions that a quasi-perfect equilibrium has the property of the lemma.

Conversely, assume that b and $\{b^k\}_{k \in \mathbb{N}}$ satisfy the condition of the lemma. For a player i and $u \in U_i$, define

$$S(u) := \{v \in U_i; v > u\},$$

$$S_1(u) := \{v \in S(u); \text{not there exists } w \in S(u) \text{ with } v \in S(w)\},$$

$$Z_1(u) := \{z \in Z; \text{not there exists } w \in S(u) \text{ with } z \in Z(w)\}.$$

$S(u)$ (resp. $S_1(u)$) is the set of all information sets coming after u (resp. coming immediately after u). Fix $k \in \mathbb{N}$. We will prove (3.3) by using backwards induction in the game tree.

If $S(u) = \emptyset$, then (3.3) follows immediately from the definition of quasi-best replies.

Next, assume (3.3) has already been proved for all information sets in $S(u)$. Then, for every $b'_i \in B_i$, we have⁴)

$$\begin{aligned} H_{iu}(b^k / u b'_i / b_{iu}) &= \\ &= \sum_{v \in S_1(u)} \rho(v; u, b^k / u b'_i / b_{iu}) H_{iv}(b^k / u b'_i / b_{iu}) + \\ &+ \sum_{z \in Z_1(u)} \rho(z; u, b^k / u b'_i / b_{iu}) h_i(z) \\ &= \sum_{v \in S_1(u)} \rho(v; u, b^k / u b_i) H_{iv}(b^k / v b'_i) + \\ &+ \sum_{z \in Z_1(u)} \rho(z; u, b^k / u b_i) h_i(z) \\ &\leq \sum_{v \in S_1(u)} \rho(v; u, b^k / u b_i) H_{iv}(b^k / v b_i) + \\ &+ \sum_{z \in Z_1(u)} \rho(z; u, b^k / u b_i) h_i(z) \\ &= H_{iu}(b^k / u b_i), \end{aligned}$$

where the inequality follows from the induction hypothesis. Combining this with the fact that b_{iu} is quasi-best reply against b^k , we see that for all $b'_i \in B_i$

$$H_{iu}(b^k / u b'_i) \leq H_{iu}(b^k / u b_i),$$

which proves (3.3) and completes the proof of the lemma. $\square$

We will need the following characterization of quasi-best replies:

⁴) $b^k / u b'_i / b_{iu} = b^k / u (b'_i / b_{iu}) = (b^k / u b'_i) / b_{iu}$, so no confusion can result.

Lemma 3. Let $\hat{b}$ be a completely mixed behavior strategy combination and let b_{iu} be a local strategy at $u \in U_i$. Then b_{iu} is a quasi-best reply against $\hat{b}$ if and only if for all $c \in C_u$:

if $b_{iu}(c) > 0$, then c is a quasi-best reply against $\hat{b}$.

Proof. This immediately follows from the observation that for every $b'_i \in B_i$

$$H_{iu}(\hat{b}/_u b'_i/b_{iu}) = \sum_{c \in C_u} H_{iu}(\hat{b}/_u b'_i/c) b_{iu}(c)$$

and the fact that $H_{iu}(\hat{b}/_u b'_i/c)$ depends only on what b'_i prescribes at the information sets which come after c (here again we use that the game has perfect recall). $\square$

Let us call a strategy combination b an ϵ -quasi-perfect-equilibrium ($\epsilon > 0$) if it is completely mixed and satisfies for all i, u and $c, c' \in C_u$:

$$\text{if } \max_{b'_i \in B_i} H_{iu}(b/_u b'_i/c) < \max_{b'_i \in B_i} H_{iu}(b/_u b'_i/c'), \text{ then } b_{iu}(c) \leq \epsilon.$$

Then it is an easy consequence of Lemma 2 and Lemma 3 that quasi-perfect equilibria can be characterized as follows:

Proposition 2. A behavior strategy combination is a quasi-perfect equilibrium if and only if it is a limit point (as ϵ tends to zero) of ϵ -quasi-perfect equilibria.

The main result of this section is:

Theorem 1. Let G be a normal form game and let q be a proper equilibrium of G . If Γ is an extensive form game with normal form G and if b is a limit behavior strategy combination induced by q in Γ , then b is a quasi-perfect equilibrium of Γ .

Proof. Let G and Γ be as in the formulation of the theorem. In view of proposition 2, it suffices to show that a behavior strategy combination b^ϵ , induced by an ϵ -proper equilibrium q^ϵ of G , is a $\sqrt{\epsilon}$ -quasi-perfect equilibrium of Γ , if ϵ is sufficiently small. Hence, let q^ϵ be an ϵ -proper equilibrium of G and let b^ϵ be induced by q^ϵ in Γ . First of all, notice that b^ϵ is completely mixed. Next, assume i, u, c and c' are such that

$$\max_{b'_i \in B_i} H_{iu}(b^\epsilon/_u b'_i/c) < \max_{b'_i \in B_i} H_{iu}(b^\epsilon/_u b'_i/c'). \quad (4.1)$$

Since the maximum in (4.1) is always attained by a pure strategy, (4.1) is equivalent to

$$\max_{\pi_i \in \text{Rel}(c)} H_{iu}(b^\epsilon/\pi_i) < \max_{\pi_i \in \text{Rel}(c')} H_{iu}(b^\epsilon/\pi_i). \quad (4.2)$$

Let $\hat{\pi}_i \in \text{Rel}(c')$ be such that $H_{iu}(b^\epsilon/\hat{\pi}_i)$ is equal to the right hand side of (4.2). Then

$$H_{iu}(b^\epsilon/\pi_i) < H_{iu}(b^\epsilon/\hat{\pi}_i) \quad \text{for all } \pi_i \in \text{Rel}(c),$$

and, therefore

$$H_i(b^\epsilon/\pi_i) < H_i(b^\epsilon/\pi_i/_u \hat{\pi}_i) \quad \text{for all } \pi_i \in \text{Rel}(c). \quad (4.3)$$

Since b^ϵ is realization equivalent to q^ϵ , (4.3) is equivalent to

$$H_i(q^\epsilon/\pi_i) < H_i(q^\epsilon/\pi_i/_u \hat{\pi}_i) \quad \text{for all } \pi_i \in \text{Rel}(c),$$

hence, since q^ϵ is an ϵ -proper equilibrium of G

$$q_i^\epsilon(\pi_i) \leq \epsilon q_i(\pi_i/_u \hat{\pi}_i) \quad \text{for all } \pi_i \in \text{Rel}(c). \quad (4.4)$$

Since $\pi_i/_u \hat{\pi}_i \in \text{Rel}(c')$ for all $\pi_i \in \text{Rel}(c)$, it follows from (4.4) that

$$q_i^\epsilon(\pi_i) \leq \epsilon \sum_{\pi_i' \in \text{Rel}(c')} q_i^\epsilon(\pi_i') \quad \text{for all } \pi_i \in \text{Rel}(c),$$

and therefore,

$$\sum_{\pi_i \in \text{Rel}(c)} q_i^\epsilon(\pi_i) \leq \epsilon |\text{Rel}(c)| \sum_{\pi_i' \in \text{Rel}(c')} q_i^\epsilon(\pi_i'), \quad (4.5)$$

where $|\text{Rel}(c)|$ denotes the number of elements in $\text{Rel}(c)$. Dividing both sides of (4.5) by $\sum_{\pi_i \in \text{Rel}(u)} q_i^\epsilon(\pi_i)$ yields

$$b_{iu}^\epsilon(c) \leq \epsilon |\text{Rel}(c)| b_{iu}^\epsilon(c') \leq \sqrt{\epsilon},$$

where the last inequality (which follows from the finiteness of the game) is valid for sufficiently small ϵ . $\square$

The converse of Theorem 1 is false: there exist quasi-perfect or perfect equilibria of Γ which are not induced by a proper equilibrium of G , as follows easily from the fact that in a normal form game a perfect equilibrium need not be proper. In fact, we proved a much stronger result than Theorem 1. Namely, from the proof one sees that in Theorem 1 one could replace “quasi-perfect” by “quasi-proper”, where a *quasi-proper equilibrium* is a limit point of ϵ -quasi-proper equilibria and where an ϵ -quasi-proper equilibrium is a completely mixed behavior strategy combination b which satisfies for all i, u, c and c' :

$$\text{if } \max_{b_i' \in B_i} H_{iu}(b/_u b_i'/c) < \max_{b_i' \in B_i} H_{iu}(b/_u b_i'/c'), \text{ then } b_{iu}(c) \leq \epsilon b_{iu}(c').$$

However, also the converse of this much stronger version of Theorem 1 is false: there exist quasi-proper equilibria of extensive form games which cannot be induced by proper equilibria of the associated normal form [see e.g. the game of figure 6.5.2 of *van Damme*].

Heights of Simple Games

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The concept of a simple game (cf. *Neumann, Morgenstern; Shapley, 1952*) has been used in a large variety of contexts, typically to formalize the power structure of representative bodies where the decisions made must reflect in some way the coalition structure.

At the intuitive level one should expect decision procedures which pay due respect to all relevant premises to be rather complicated. In the present paper we present a complexity measure, which we shall call the height, of a (proper) simple game. Loosely speaking, the height of a simple game is the number of iterations of a majority voting which must be performed before the final decision is reached. As such, it captures some but not all aspects of complexity, and it should be considered only as a first step towards a full treatment of complexity of decisions in representative bodies.

The operation of iterating weighted majority games is closely related to the compound operation on simple games introduced by *Shapley (1961)*. It turns out that every proper simple game can be obtained in this way. This is the topic of section 2 of the paper. In section 3, the height is introduced, and we give an upper and lower bound for $h(n)$, the maximal height of a simple game with n players. Finally, in section 4 we show that $h(n)$ grows approximately as $\log_2 n$ for large n .

The following special notation will be used: For C a finite set, $|C|$ denotes the cardinality of C . For x a nonnegative real number, $\lceil x \rceil$ denotes the smallest integer p such that $p \geq x$, and $\lfloor x \rfloor$ the greatest integer p such that $p \leq x$.