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Tijs, S.H.

Published in:
Naval Research Logistics

Publication date:
1981

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
Tijs, S. H. (1981). A characterization of the value of zero-sum two person games. *Naval Research Logistics*, 28, 153-156.

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A CHARACTERIZATION OF THE VALUE OF ZERO-SUM TWO-PERSON GAMES

S. H. Tijs

*Department of Mathematics
Catholic University
Nijmegen, The Netherlands*

ABSTRACT

For the family D , consisting of those zero-sum two-person games which have a value, the value-function on D is characterized by four properties called objectivity, monotony, symmetry and sufficiency.

INTRODUCTION

In a beautiful paper, E. I. Vilkas gave a characterization of the value-function, defined on the class of all finite matrix games [2]. In [1], pp. 60-65 this result was extended to the class of all finite and semi-infinite matrix games.

The purpose of this paper is to deduce characterizing properties for the value-function on the set of all determined two-person games. The organization of the paper is as follows: the necessary notation and definitions are given in sections 1 and 2; in section 3, properties for the value-function are presented, which are shown in section 4 to be characteristic of this function.

1. A (zero-sum) two-person game is an ordered triple $\langle X, Y, K \rangle$, in which X and Y are nonempty sets (called the *pure strategy spaces* of player I and player II, respectively) and $K : X \times Y \rightarrow \mathbb{R}$ is a real-valued function on the Cartesian product of X and Y (called the *pay off function* of player I).

2. Let $\langle X, Y, K \rangle$ be a two-person game. For each $x \in X$ ($y \in Y$) let us denote the probability measure on X (Y) with mass 1 in x (y) by e_x (e_y). Let P_X be the set of all convex combinations of elements of $\{e_x; x \in X\}$; likewise let P_Y be the convex hull of $\{e_y; y \in Y\}$. Then the two-person game $\langle P_X, P_Y, E_K \rangle$ with

$$E_K(\mu, \nu) := \int \int K(x, y) d\mu(x) d\nu(y) \text{ for each } (\mu, \nu) \in P_X \times P_Y$$

is called the *c-mixed extension* of the game $\langle X, Y, K \rangle$. The lower value $\sup_{\mu \in P_X} \inf_{\nu \in P_Y} E_K(\mu, \nu)$ of the game $\langle P_X, P_Y, E_K \rangle$ is denoted by $\underline{v}(X, Y, K)$ and the upper value $\inf_{\nu \in P_Y} \sup_{\mu \in P_X} E_K(\mu, \nu)$ is denoted by $\bar{v}(X, Y, K)$. Note that

$$-\infty \leq \underline{v}(X, Y, K) \leq \bar{v}(X, Y, K) \leq \infty.$$

If $\underline{v}(X, Y, K) = \bar{v}(X, Y, K)$ for a game $\langle X, Y, K \rangle$, then we say that the game is a *determined game*. In that case, the common value is denoted by $v(X, Y, K)$ and called the *value* of (the c-mixed extension of) $\langle X, Y, K \rangle$. The family of determined games is denoted by D .

3. In this section we want to look at some distinguished properties of the value-function $v : D \rightarrow [-\infty, \infty]$. For this purpose we need some definitions.

DEFINITION 1: The *transpose* of a two-person game $\langle X, Y, K \rangle$ is the two-person game $\langle Y, X, -K' \rangle$ where

$$K'(y, x) := K(x, y) \text{ for each } (y, x) \in Y \times X.$$

DEFINITION 2: Let $\langle X, Y, K \rangle$ be a two-person game and let S be a nonempty subset of X . Then we say that S is *sufficient for player I in the game* $\langle X, Y, K \rangle$ if for each $x \in X - S$ there exists a $\mu \in P_S$ such that

$$E_K(\mu, e_y) \geq K(x, y) \text{ for each } y \in Y.$$

DEFINITION 3: Let $\langle X, Y, K \rangle$ be a two-person game and let T be a nonempty subset of Y . We say that T is *sufficient for player II in the game* $\langle X, Y, K \rangle$ if T is sufficient for player I in the game $\langle Y, X, -K' \rangle$.

THEOREM 1:

(P.1) ["Objectivity"] Let $\langle X, Y, K \rangle$ be a two-person game and suppose that $X = \{a\}$, $Y = \{b\}$. Then $\langle X, Y, K \rangle \in D$ and $v(X, Y, K) = K(a, b)$.

(P.2) ["Monotonicity"] Let $\langle X, Y, K \rangle \in D$ and $\langle X, Y, L \rangle \in D$ and suppose that $L \geq K$ (i.e. $L(x, y) \geq K(x, y)$ for each $(x, y) \in X \times Y$). Then $v(X, Y, L) \geq v(X, Y, K)$.

(P.3) ["Symmetry"] Let $\langle X, Y, K \rangle \in D$. Then $\langle Y, X, -K' \rangle \in D$ and $v(Y, X, -K') = -v(X, Y, K)$.

(P.4) ["Sufficiency"] Let $\langle X, Y, K \rangle$ be a two-person game, and $\emptyset \neq S \subset X$ and let $K' : S \times Y \rightarrow \mathbb{R}$ be the restriction of K to $S \times Y$. Suppose that S is sufficient for player I in the game $\langle X, Y, K \rangle$. Then $\langle S, Y, K' \rangle \in D$ iff $\langle X, Y, K \rangle \in D$ and

$$v(X, Y, K) = v(S, Y, K') \text{ if } \langle S, Y, K' \rangle \in D.$$

PROOF: (P.1) and (P.2) are obvious. (P.3) follows from the fact that

$$-E_K(\mu, \nu) = E_{-K'}(\nu, \mu) \text{ for each } (\mu, \nu) \in P_X \times P_Y.$$

Now let us prove (P.4). First we note that P_S can be seen (in an obvious manner) as a subset of P_X , and that $E_{K'}$ is the restriction of E_K to $P_S \times P_Y$.

Take $\alpha \in P_X$. Then there exist $n \in \mathbb{N}$, $x^1, x^2, \dots, x^n \in X$ and $p_1, p_2, \dots, p_n \in [0, \infty)$ such that $\sum_{i=1}^n p_i = 1$ and $\alpha = \sum_{i=1}^n p_i e_{x^i}$. Since S is sufficient for player I in the game $\langle X, Y, K \rangle$ for each $i \in \{1, \dots, n\}$, there exists an $\alpha_i \in P_S$ such that

$$E_K(\alpha_i, e_y) \geq K(x^i, y) \text{ for each } y \in Y.$$

[If $x^i \in S$, then we can take $\alpha_i = e_{x^i}$.] Let $\bar{\alpha} := \sum_{i=1}^n p_i \alpha_i$. Then $\bar{\alpha} \in P_S$ and

$$E_K(\bar{\alpha}, e_y) = \sum_{i=1}^n p_i E_K(\alpha_i, e_y) \geq \sum_{i=1}^n p_i K(x^i, y) = E_K(\alpha, e_y) \text{ for each } y \in Y.$$

But then,

$$(i) \quad E_K(\bar{\alpha}, \nu) \geq E_K(\alpha, \nu) \text{ for each } \alpha \in P_X \text{ and each } \nu \in P_Y.$$

This implies that

$$\sup_{\mu \in P_S} E_{K'}(\mu, \nu) = \sup_{\alpha \in P_X} E_K(\alpha, \nu) \text{ for each } \nu \in P_Y$$

and thus

$$(ii) \quad \bar{v}(S, Y, K') = \bar{v}(X, Y, K).$$

From (i) we may also conclude that

$$\inf_{\nu \in P_Y} E_{K'}(\bar{\alpha}, \nu) \geq \inf_{\nu \in P_Y} E_K(\alpha, \nu) \text{ for each } \alpha \in P_X$$

and then

$$(iii) \quad \underline{v}(S, Y, K') = \underline{v}(X, Y, K).$$

Now (P.4) follows from (ii) and (iii). ||

4. The following theorem shows that the properties (P.1)-(P.4) characterize the value-function $v : D \rightarrow [-\infty, \infty]$.

THEOREM 2: Let $f : D \rightarrow [-\infty, \infty]$ be a function with the following four properties:

(Q.1) If $X = \{a\}$, $Y = \{b\}$ and if K is a real-valued function on $X \times Y$, then $f(X, Y, K) = K(a, b)$.

(Q.2) For each $\langle X, Y, K \rangle \in D$, $\langle X, Y, L \rangle \in D$ with $L \geq K$: $f(X, Y, L) \geq f(X, Y, K)$.

(Q.3) For each $\langle X, Y, K \rangle \in D$: $f(Y, X, -K') = -f(X, Y, K)$.

(Q.4) For each $\langle X, Y, K \rangle \in D$ and $\langle S, Y, K' \rangle \in D$, where $S \subset X$, K' is the restriction of K to $S \times Y$ and where S is sufficient for player I in the game $\langle X, Y, K \rangle$, we have $f(S, Y, K') = f(X, Y, K)$.

Then $f(X, Y, K) = v(X, Y, K)$ for each $\langle X, Y, K \rangle \in D$.

PROOF: First we note that (Q.3) and (Q.4) imply

(Q.5) For each $\langle X, Y, K \rangle \in D$ and $\langle X, T, K'' \rangle \in D$, where $T \subset Y$, K'' is the restriction of K to $X \times T$ and where T is sufficient for player II in the game $\langle X, Y, K \rangle$, we have $f(X, T, K'') = f(X, Y, K)$.

Now take an $\langle X, Y, K \rangle \in D$ with $v(X, Y, K) \in (-\infty, \infty]$ and take a real number t such that $v(X, Y, K) > t$. We want to prove that $f(X, Y, K) \geq t$. For this purpose we introduce the following five two-person games:

(1) $\langle X \cup \{a\}, Y, L \rangle$ where $a \notin X$ and where $L(x, y) := K(x, y)$ for each $(x, y) \in X \times Y$ and $L(a, y) := t$ for each $y \in Y$.

(2) $\langle X \cup \{a\}, Y, M \rangle$ where $M(x, y) := \text{minimum } \{K(x, y), t\}$ for each $(x, y) \in (X \cup \{a\}) \times Y$.

(3) $\langle X \cup \{a\}, Y \cup \{b\}, N \rangle$ where $b \notin Y$ and where $N(x, y) := M(x, y)$ for each $(x, y) \in (X \cup \{a\}) \times Y$ and $N(x, b) := t$ for each $x \in X \cup \{a\}$.

(4) $\langle \{a\}, Y \cup \{b\}, N' \rangle$ where N' is the restriction of N to $\{a\} \times (Y \cup \{b\})$.

(5) $\langle \{a\}, \{b\}, N'' \rangle$ where N'' is the restriction of N' to $\{a\} \times \{b\}$.

Since $v(X, Y, K) > t$, there exists a $\mu \in P_X$ such that

$$E_L(\mu, e_y) = E_K(\mu, e_y) \geq t = L(a, y) \text{ for each } y \in Y.$$

Hence, X is sufficient for player I in the game $\langle X \cup \{a\}, Y, L \rangle$. By (P.4) and (Q.4) we may conclude that

$$(Q.6) \quad \langle X \cup \{a\}, Y, L \rangle \in D \text{ and } f(X \cup \{a\}, Y, L) = f(X, Y, K).$$

It follows from (Q.1) that

$$(Q.7) \quad f(\{a\}, \{b\}, N'') = t.$$

In the game $\langle \{a\}, Y \cup \{b\}, N' \rangle$ the set $\{b\}$ is sufficient for player II because

$$E_{N'}(e_a, e_b) = N'(a, y) = t \text{ for each } y \in Y.$$

Then $\langle \{a\}, Y \cup \{b\}, N' \rangle \in D$ in view of (P.3) and (P.4); now (Q.5) implies

$$(Q.8) \quad f(\{a\}, Y \cup \{b\}, N') = f(\{a\}, \{b\}, N'').$$

In the game $\langle X \cup \{a\}, Y \cup \{b\}, N \rangle$ the set $\{a\}$ is sufficient for player I because for each $x \in X$:

$$t = E_N(e_a, e_y) \geq N(x, y) \text{ for each } y \in Y \cup \{b\}.$$

By (P.4) and (Q.4) we obtain: $\langle X \cup \{a\}, Y \cup \{b\}, N \rangle \in D$ and

$$(Q.9) \quad f(X \cup \{a\}, Y \cup \{b\}, N) = f(\{a\}, Y \cup \{b\}, N').$$

It is easy to see that Y is sufficient for player II in the game $\langle X \cup \{a\}, Y \cup \{b\}, N \rangle$. Hence, by (P.3) and (P.4): $\langle X \cup \{a\}, Y, M \rangle \in D$; and then by (Q.5):

$$(Q.10) \quad f(X \cup \{a\}, Y, M) = f(X \cup \{a\}, Y \cup \{b\}, N).$$

Now $L \geq M$ and then by (Q.2) we have

$$(Q.11) \quad f(X \cup \{a\}, Y, L) \geq f(X \cup \{a\}, Y, M).$$

Combining (Q.6)-(Q.11) we obtain: $f(X, Y, K) \geq t$. Thus, we have proved that $f(X, Y, K) \geq t$ for each $\langle X, Y, K \rangle \in D$ with $v(X, Y, K) \in (-\infty, \infty]$ and each $t < v(X, Y, K)$. But then

$$(Q.12) \quad f(X, Y, K) \geq v(X, Y, K) \text{ for each } \langle X, Y, K \rangle \in D.$$

It follows from (Q.3), (Q.12) and (P.3) that

$$(Q.13) \quad f(X, Y, K) = -f(Y, X, K') \leq -v(Y, X, K') = v(X, Y, K) \text{ for each } \langle X, Y, K \rangle \in D.$$

Properties (Q.12) and (Q.13) imply the conclusion of the theorem. $\parallel$

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