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On the structure of the set of perfect equilibria in bimatrix games

P. E. M. Borm¹, M. J. M. Jansen^{2,3}, J. A. M. Potters², and S. H. Tijs¹

¹ Department of Econometrics, P.O. Box 90153, 5000 LE Tilburg, The Netherlands,

² Department of Mathematics/NICI, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands

³ Open University, P.O. Box 2960, 6401 DL Heerlen, The Netherlands

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Summary. In this paper attention is focussed on the structure of the set of perfect equilibria. It turns out that the structure of this set resembles the structure of the Nash equilibrium set. Maximal Selten subsets are introduced to take the role of maximal Nash subsets. It is found that the set of perfect equilibria is the finite union of maximal Selten subsets. Furthermore it is shown that the dimension relation for maximal Nash subsets can be extended to faces of such sets. As a result a dimension relation for maximal Selten subsets is derived.

Zusammenfassung. Die vorliegende Arbeit ist der Struktur der Menge perfekter Gleichgewichte gewidmet. Es stellt sich heraus, daß die Struktur dieser Menge der Struktur der Menge der Nash Gleichgewichte ähnlich ist. Maximale Selten Mengen werden eingeführt, um die Rolle der maximalen Nash Menge zu übernehmen. Es wird gezeigt, daß die Menge perfekter Gleichgewichte aus endlich vielen maximalen Selten Mengen zusammengestellt ist. Außerdem wird die Dimension maximaler Selten Mengen beschrieben.

Key words: Bimatrix game, perfect equilibrium, undominated equilibrium, (maximal) Selten set

Schlüsselwörter: Bimatrixspiel, perfektes Gleichgewicht, undominiertes Gleichgewicht (maximale) Selten Menge

1. Introduction

This paper deals with the structure of a special subset of Nash equilibria for bimatrix games. Problems concerning the structure of the set of Nash equilibria itself were first studied for matrix games in the articles of Shapley and Snow (1950), Bohnenblust et al. (1950) and Gale and Sherman (1950).

Shapley and Snow gave a characterization of extreme optimal strategies in terms of square submatrices of the payoff matrix. Bohnenblust, Karlin and Shapley and also

Gale and Sherman derived a dimension relation for the optimal strategy sets of the two players. Furthermore, Gale and Sherman solved the construction problem by giving necessary and sufficient conditions on two polytopes – with the dimension relation as the most important one – for being the sets of optimal strategies of a matrix game.

A generalization of some of these results to the class of bimatrix games is given by several authors: Vorobev (1958), Kuhn (1961), Millham (1972, 1974), Heuer and Millham (1976), Winkels (1979) and Jansen (1981). For the description of the structure of the set of Nash equilibria in bimatrix games attention is focussed on *maximal Nash subsets*, a term which was introduced by Heuer and Millham. Nash (1951) called them subsolutions. It appears that the set of Nash equilibria of a bimatrix game is the (not necessarily disjoint) union of the finite number of maximal Nash subsets (cf. Winkels 1979; Jansen 1981). Furthermore a dimension relation for maximal Nash subsets was derived (cf. Jansen 1981).

In this paper we investigate the structure of the set of *perfect equilibria* (Selten 1975). In the process we will introduce *maximal Selten subsets* in analogy to maximal Nash subsets. It is found that the set of perfect equilibria is the not necessarily disjoint union of the finite number of maximal Selten subsets and that maximal Selten subsets satisfy the dimension relation which was found for maximal Nash subsets.

The organization of this paper is as follows. In Sect. 2 we first recall some basic facts about bimatrix games and maximal Nash subsets. In Sect. 3 we investigate the (sets of) undominated strategies. In Sect. 4 we deal with the structure of the set of perfect equilibria. Finally, in Sect. 5, the dimension relation for maximal Nash subsets is generalized.

Notation. $\mathbb{N} := \{1, 2, \dots\}$ is the set of positive integers. $\mathbb{R}^t$ is the vector space of t -tuples of real numbers and $\Delta_t := \{p \in \mathbb{R}^t \mid p \geq 0, \sum_{i=1}^t p_i = 1\}$. The unit vectors in $\mathbb{R}^t$ are denoted by $e_1, e_2, \dots, e_t$. For $A \subset \mathbb{R}^t$ we denote by $\text{conv}(A)$ the convex hull of A . For a convex set $C \subset \mathbb{R}^t$ we

denote by $\text{relint}(C)$ the relative interior of C and by $\dim(C)$ the dimension of C .

2. The Nash equilibrium set

In this paper we consider (mixed extensions of) $m \times n$ -bimatrix games (A, B) with $A = [a_{ij}]_{i=1}^m_{j=1}^n$ and $B = [b_{ij}]_{i=1}^m_{j=1}^n$ ($m, n \in \mathbb{N}$). Here (A, B) is a short notation for the two-person game $\langle \Delta_m, \Delta_n, K, L \rangle$ in strategic form with (mixed) strategy spaces Δ_m and Δ_n for player 1 and player 2, respectively and payoff functions $K: \Delta_m \times \Delta_n \rightarrow \mathbb{R}$ and $L: \Delta_m \times \Delta_n \rightarrow \mathbb{R}$, respectively with $K(p, q) = pAq$ and $L(p, q) = pBq$ for all $(p, q) \in \Delta_m \times \Delta_n$.

A matrix game A is identified with the bimatrix game $(A, -A)$.

A Nash equilibrium of a bimatrix game (A, B) is a pair $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_n$ such that $\hat{p}A\hat{q} \geq pA\hat{q}$ and $\hat{p}B\hat{q} \geq \hat{p}Bq$ for all $p \in \Delta_m, q \in \Delta_n$. In other words, unilateral deviation from an equilibrium does not pay. The set of all equilibria of a bimatrix game (A, B) is denoted by $E(A, B)$.

For a strategy $p \in \Delta_m$ we denote by $C(p) := \{i \in \{1, \dots, m\} | p_i > 0\}$ the carrier of p , by $PB_2(p) := \{j \in \{1, \dots, n\} | pBe_j = \max_{k \in \{1, \dots, n\}} pBe_k\}$ the pure best answers of player 2 to p and by $B_2(p) := \text{conv}(PB_2(p))$ the best answers of player 2 to p . For $q \in \Delta_n$, $C(q)$, $PB_1(q)$ and $B_1(q)$ are defined analogously.

It is a well-known fact that if $(p, q) \in \Delta_m \times \Delta_n$, then $(p, q) \in E(A, B)$ if and only if $C(p) \subset PB_1(q)$ and $C(q) \subset PB_2(p)$.

Nash (1951) showed that $E(A, B)$ is nonempty for all games (A, B) . Jansen (1981) showed that $E(A, B)$ is the finite union of maximal Nash subsets, where $T \subset E(A, B)$ is a Nash subset if every pair of elements in T is interchangeable, i.e. $(p, q), (x, y) \in T$ implies $(p, y), (x, q) \in T$ and a Nash subset T is called maximal if it is not properly contained in another Nash subset. Moreover Jansen showed that maximal Nash subsets are the Cartesian product of two convex polytopes and for each of these polytopes a dimension relation was formulated.

A theoretical way to determine the Nash equilibrium set is given by means of extreme equilibria. Here, an equilibrium is called extreme if it is an extreme point of some maximal Nash subset. A characterization of extreme equilibria in terms of square submatrices of the payoff matrices can be found in Jansen (1981) (cf. Vorobev 1958; Kuhn 1961). This characterization provides a systematical way to determine all extreme equilibria.

3. Perfect equilibria and undominatedness

Selten (1975) defined perfect equilibria by requiring stability against some mistakes the players can make in choosing their strategies. Formally

Definition. Let (A, B) be an $m \times n$ -bimatrix game. An equilibrium $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_n$ is called perfect if there exists a sequence $\{(p^k, q^k)\}_{k \in \mathbb{N}}$ of completely mixed strategy pairs converging to $(\hat{p}, \hat{q})$, such that, for all $k \in \mathbb{N}$

$$\hat{p} \in B_1(q^k) \quad \text{and} \quad \hat{q} \in B_2(p^k).$$

The set of all perfect equilibria of the game (A, B) is denoted by $PF(A, B)$.

In Selten (1975) it is shown that $PF(A, B) \neq \emptyset$ for all bimatrix games (A, B) . For such games van Damme (1983) showed that an equilibrium is perfect if and only if both equilibrium strategies are undominated.

Definition. For an $m \times n$ -bimatrix game (A, B) a strategy $p \in \Delta_m$ ($q \in \Delta_n$) is called undominated (in (A, B)) if for all $\tilde{p} \in \Delta_m$ ($\tilde{q} \in \Delta_n$)

$$\tilde{p}A \geq pA \text{ implies } \tilde{p}A = pA \quad (B\tilde{q} \geq Bq \text{ implies } B\tilde{q} = Bq).$$

Van Damme's characterization will be useful because we can provide a full description of both sets of undominated strategies by introducing the notion of undominated (sub)simplex of the strategy spaces, i.e. a subsimplex consisting of undominated strategies only. For $I \subset T := \{1, \dots, t\}$ the subsimplex Δ_I of the $(t-1)$ -dimensional simplex Δ_t is given by $\Delta_I := \text{conv}\{e_i \in \mathbb{R}^t | i \in I\}$.

Lemma 1. If $p \in \Delta_m$ is an undominated strategy for player 1 in the $m \times n$ -bimatrix game (A, B) and $I \subset C(p)$, then Δ_I is an undominated subsimplex of Δ_m .

Proof. Let $p' \in \Delta_I$. Then there is a real number $t \in (0, 1)$ such that $tp' \leq p$. Define $p'' := (1-t)^{-1}(p - tp')$ and we find $p = tp' + (1-t)p''$. Now suppose p' is dominated by the strategy $\hat{p}$, i.e. $\hat{p}A \geq p'A$ and $\hat{p}A \neq p'A$. Then, $(t\hat{p} + (1-t)p'')A \geq (tp' + (1-t)p'')A = pA$ and there is no equality. This however, would imply that the strategy p is dominated. $\square$

Let us call a subsimplex Δ_I of Δ_m ($I \subset \{1, \dots, m\}$) maximal undominated for player 1 in the $m \times n$ -bimatrix game (A, B) if it is undominated and if there is no $K \subset \{1, \dots, m\}$ properly containing I such that Δ_K is undominated. Setting

$$\mathcal{I} := \{I \subset \{1, \dots, m\} | \Delta_I \text{ is maximal undominated}\},$$

Lemma 1 gives the following result

Lemma 2. For an $m \times n$ -bimatrix game (A, B) the set U_1 of undominated strategies of player 1 is the finite union of the maximal undominated subsimplices of Δ_m . In formula

$$U_1 = \bigcup_{I \in \mathcal{I}} \Delta_I.$$

Of course, a similar statement can be made with respect to the undominated strategies of player 2.

4. The set of perfect equilibria

Before giving the definition of a (maximal) Selten subset, we illustrate the underlying line of thought in

Example 1. Let (A, B) be the 3×4 -bimatrix game defined by

$$(A, B) = \begin{bmatrix} (-3, 3) & (-3, 0) & (-3, 0) & (-3, 2) \\ (3, 0) & (3, 0) & (3, 0) & -(6, 0) \\ (-3, 0) & (-3, 0) & (-3, 3) & (-3, 2) \end{bmatrix}.$$

This game has six maximal Nash subsets. One of these maximal Nash subsets is

$$T = T_1 \times T_2 := \{e_2\} \times \left\{ q \in \Delta_4 \mid q_4 \leq \frac{2}{3} \right\}.$$

It easily follows that $U_1 = \Delta_3$ and $U_2 = \Delta_{\{1,4\}} \cup \Delta_{\{3,4\}}$.

Fig. 1. (••••: U_2 , ---: $U_2 \cap T_2$)

The set of undominated equilibria inside the maximal Nash subset T consists of two components, as we can see in the figure. These components we like to call maximal Selten subsets. Now we formally introduce (maximal) Selten subsets.

Definition. Let (A, B) be an $m \times n$ -bimatrix game and S a non empty subset of $PF(A, B)$. S is called a *Selten subset* if there are convex sets $S_1 \subset \Delta_m$ and $S_2 \subset \Delta_n$ such that $S = S_1 \times S_2$. A Selten subset S is called *maximal* if it is not properly contained in another Selten subset.

Note that, in comparison with the definition of Nash subsets, we have added a convexity requirement. If not, we should find in Example 1 a Selten subset $\{e_2\} \times \{e_1, e_3\}$, not contained in any of the two components we like to call a maximal Selten subset. This difficulty arises because in Example 1 the convex hull $\{e_2\} \times \text{conv}\{e_1, e_3\}$ is no longer a subset of $PF(A, B)$ in contrast with the corresponding property for Nash subsets.

It directly follows that every (maximal) Selten subset is a Nash subset and therefore contained in a maximal Nash subset. In Example 1 we have seen that a maximal Nash subset can contain several (even disjoint) maximal Selten

subsets. Even in a matrix game the equilibrium set (the unique maximal Nash subset) can contain two maximal Selten subsets, as can be seen in

Example 2. Let the 3×4 -matrix game A be given by

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}.$$

Then we have $v(A) = 0$, $E(A, -A) = \Delta_3 \times \{e_4\}$, and

$$PF(A, -A) = \text{conv}\{e_1, e_3\} \times \{e_4\} \cup \text{conv}\{e_2, e_3\} \times \{e_4\}.$$

Remark. For an $m \times n$ -bimatrix game (A, B) the maximal Selten subsets correspond to maximal convex subsets of $PF(A, B)$. For, if $S \subset PF(A, B)$ is convex, then $S \subset T$ for some maximal Nash subset $T = T_1 \times T_2$. Letting

$$\tilde{S}_1 := \{p \in T_1 \mid \text{there is a } q \in T_2 \text{ such that } (p, q) \in S\}$$

$$\tilde{S}_2 := \{q \in T_2 \mid \text{there is a } p \in T_1 \text{ such that } (p, q) \in S\},$$

we find that $S \subset \tilde{S} := \tilde{S}_1 \times \tilde{S}_2 \subset E(A, B)$ and that $\tilde{S}$ is a convex subset of $U_1 \times U_2$. This implies that $\tilde{S}$ is a Selten subset containing S .

The structure of the set of perfect equilibria is now described in

Theorem 1. Let (A, B) be an $m \times n$ -bimatrix game. The $PF(A, B)$ is the finite union of the maximal Selten subsets.

Proof. According to the observations made in Sect. 2, we can find polytopes $P_1, \dots, P_u \subset \Delta_m$ and $Q_1, \dots, Q_u \subset \Delta_n$ such that

$$E(A, B) = \bigcup_{k=1}^u P_k \times Q_k.$$

By Lemma 2 the set $U := U_1 \times U_2$ of undominated strategy pairs is of the form

$$U = \bigcup_{i=1}^s V_i \times \bigcup_{j=1}^t W_j = \bigcup_{i=1}^s \bigcup_{j=1}^t V_i \times W_j,$$

where $V_1, \dots, V_s \subset \Delta_m$ and $W_1, \dots, W_t \subset \Delta_n$ are the maximal undominated subsimplices. By van Damme's result we have

$$PF(A, B) = E(A, B) \cap U = \bigcup_{i,j,k} (P_k \cap V_i) \times (Q_k \cap W_j), \quad (1)$$

where we may suppose that no set of this union is properly contained in any of the other sets. Then we have written $PF(A, B)$ as a finite union of Selten subsets. In fact each of these Selten subsets is maximal as we will show now. Let S be one of the Selten subsets in (1) and suppose that S is not maximal. Then S is properly contained in a Selten subset, say $S^* = S_1^* \times S_2^*$. Since S^* is contained in a maximal Nash

subset, say $P_k \times Q_k$ for some k , $S_1^* \subset V_i$ for some i and $S_2^* \subset W_j$ for some j ,

$$S \subset S^* \subset (P_k \cap V_i) \times (Q_k \cap W_j),$$

which is a contradiction. Hence $PF(A, B)$ is the finite union of maximal Selten subsets. $\square$

Since a maximal undominated subsimplex is a face of one of the strategy spaces, the representation (1) implies

Corollary 1. *Let (A, B) be a bimatrix game. Let $S_1 \times S_2$ be a maximal Selten subset contained in a maximal Nash subset $T_1 \times T_2$. Then, for $i \in \{1, 2\}$, S_i is a face of T_i .*

Defining an *extreme perfect equilibrium* as an extreme point of some maximal Selten subset, (1) also implies

Corollary 2. *For any bimatrix game an extreme perfect equilibrium is also an extreme equilibrium.*

Hence, in determining the set of perfect equilibria, one can first trace all extreme equilibria (e.g. by using the characterization mentioned in Sect. 2) and then find out which of these extreme equilibria are undominated. An algorithm to decide on this issue has been provided by van Damme (1983). So, at least theoretically, we can compute the set of perfect equilibria in a systematic way.

5. A dimension relation for maximal Selten subsets

The purpose of this section is to extend the dimension relation as given by Jansen (1981) to faces of a maximal Nash subset. In view of Corollary 1 such an extension leads to a dimension relation for maximal Selten subsets. First we need a result of Jansen et al. (1993).

Lemma 3. *Let F be a face of a maximal Nash subset and let $(\dot{p}, \dot{q}) \in \text{relint}(F)$. Then $(p, q) \in F$ if and only if $C(p) \subset C(\dot{p})$, $PB_2(p) \supset PB_2(\dot{p})$, $C(q) \subset C(\dot{q})$ and $PB_1(q) \supset PB_1(\dot{q})$.*

Consequently, if (A, B) is a bimatrix game and $F = F_1 \times F_2$ is a face, then the sets $C(F_1) := C(\dot{p})$ and $C(F_2) := C(\dot{q})$, and the matrices

$$A(F_1) := [a_{ij}]_{i \in PB_1(\dot{q}), j \in C(\dot{q})} \quad \text{and} \quad B(F_2) := [b_{ij}]_{i \in C(\dot{p}), j \in PB_2(\dot{p})}$$

do not depend on the choice of the point $(\dot{p}, \dot{q}) \in \text{relint}(F)$. Using Lemma 3 the proof of Theorem 5 of Jansen (1981) can easily be adapted in order to prove

Theorem 2. *Let (A, B) be an $m \times n$ -bimatrix game with $A > 0$ and $B > 0$. Let $F = F_1 \times F_2$ be a face of a maximal Nash subset. Then*

$$\dim(F_1) = |C(F_1)| - \text{rank } B(F_1) \quad \text{and}$$

$$\dim(F_2) = |C(F_2)| - \text{rank } A(F_2).$$

As a consequence of Corollary 1 we have the following

Corollary 3. *Let $S = S_1 \times S_2$ be a maximal Selten subset for a bimatrix game (A, B) with $A > 0$ and $B > 0$. Then*

$$\dim(S_1) = |C(S_1)| - \text{rank } B(S_1) \quad \text{and}$$

$$\dim(S_2) = |C(S_2)| - \text{rank } A(S_2).$$

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