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Weakly strict equilibria in finite normal form games

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Abstract. In this paper we introduce the weakly strict equilibrium for (mixed extensions of) finite n -person games. For those games we prove that our new concept is a strict refinement of the undominated Nash equilibrium concept. We show that their set of weakly strict equilibria is non-empty and coincides with their set of strict equilibria (when the latter is also non-empty) and we study the relation between our refinement and other noncooperative concepts.

Zusammenfassung. In dieser Arbeit führen wir das schwach-strikte Gleichgewicht für die gemischte Erweiterung von endlichen n -Personen-Spielen ein. Für diese Spiele zeigen wir, daß dieses Konzept eine echte Verfeinerung des Nash-Gleichgewichts ist. Wir weisen nach, daß die Menge der schwach-strikten Gleichgewichte derartiger Spiele nicht leer ist und mit der Menge der strikten Gleichgewichte übereinstimmt, falls letztere nicht leer ist. Außerdem vergleichen wird das neue Konzept mit anderen in der Literatur vorgeschlagenen Gleichgewichtsbegriffen.

Key words: Nash equilibrium, strict equilibrium, weakly strict equilibrium, normal form games, random games

Schlüsselwörter: Nash-Gleichgewicht, Gleichgewicht, schwach-striktes Gleichgewicht, Spiele in Normalform, Zufallsspiele

1. Introduction

Kohlberg (1990) has brilliantly captured the core basis on which noncooperative game theory has been built. As he writes, the main idea of the theory "is to make a bold sim-

plification and, instead of asking how the process of deductions might unfold, ask where its rest-points might be". The problem, addressed by game theorists since 1950, when Nash published his seminal paper, is that it is extremely difficult to define satisfactorily what a rest-point must be in such way that all games, in a reasonably wide class, possess at least one of these rest-points.

In the papers of Nash (1950) and (1951), the first approach to this problem, the Nash equilibrium concept is proposed as a selfenforcing combination of strategies (or, in other words, as a rest-point) existing in every (mixed extended) finite normal form game. However, in Selten (1975) it was argued that not all Nash equilibria are always selfenforcing. Since then, many authors have tried to give an answer to the problem in the context of finite games. For instance, in Selten (1975), Myerson (1978) or Kalai and Samet (1984) there are different definitions of rest-points; Kohlberg and Mertens (1986), Mertens (1989) and (1991) or Hillas (1990), in view of the impossibility of introducing a universally adequate point-valued concept, proposed several definitions of *rest-sets*. In spite of the big amount of papers devoted to the matter (a good survey on which can be found in van Damme 1991), there is not a general agreement on what concept must be adopted as a plausible solution of a noncooperative game. That is the reason why new approaches to this problem are interesting.

In this paper we concentrate, as most authors, on (mixed extensions of) finite games. In section three we introduce, in this context, the weakly strict equilibrium, a new refinement of the undominated Nash equilibrium concept, and study some of its properties (for example, we prove that every (mixed extended) finite normal form game has a non-empty set of weakly strict equilibria and that this set coincides with the set of strict equilibria of the game if the latter is also non-empty). Previously, in the second section, we set up the notation and define and study the so-called *random games* which are used later for the definition of the weakly strict equilibrium.

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2. Finite random games

A finite random game consists of the following:

- initially, there is a set of finite games all having the same set of players and the same set of combination of strategies,
- one game is selected out of the set accordingly to the result of a random experiment,
- the players, not knowing the game randomly selected, have to choose their strategies simultaneously and independently (they can choose either pure or mixed strategies),
- the players receive the payoff corresponding to the combination of strategies chosen and to the game selected,
- all the elements of all games and all these rules are common knowledge.

We formally introduce a finite random game in the next definition.

Definition 1. A finite random game is a three-tuple

$$\langle (\Omega, \mathcal{A}, P), X(\Phi), H \rangle$$

where

1. $(\Omega, \mathcal{A}, P)$ is a probability space,
2. Φ (the set of combinations of pure strategies of the n players) is the cartesian product $\prod_{i=1}^n \Phi_i$, Φ_i being the finite set of pure strategies of player i , and $X(\Phi)$ (the set of combinations of strategies of the n -players) is the cartesian product $\prod_{i=1}^n X_i(\Phi_i)$, $X_i(\Phi_i)$ being the set of strategies of player i given by:

$$X_i(\Phi_i) = \left\{ x_i \in \mathbb{R}^{\Phi_i} \mid x_i(\varphi_i) \geq 0 \ \forall \varphi_i \in \Phi_i, \sum_{\varphi_i \in \Phi_i} x_i(\varphi_i) = 1 \right\},$$

and

3. H is the payoff function of the game given by:

$$H: X(\Phi) \times \Omega \rightarrow \mathbb{R}^n \\ (x, \omega) \rightarrow H(x, \omega) = (H_1(x, \omega), H_2(x, \omega), \dots, H_n(x, \omega))$$

where, for every $i \in \{1, \dots, n\}$, $x \in X(\Phi)$ and $\omega \in \Omega$, $H_i(x, \omega)$ is the payoff for the i -th player if x is played and the state of nature is ω ; it verifies

$$H_i(x, \omega) = \sum_{\varphi \in \Phi} H_i(\varphi, \omega) x(\varphi)$$

for every $i \in \{1, \dots, n\}$, $x \in X(\Phi)$ and $\omega \in \Omega$. H must be measurable as a function of ω (for all x).

Note that, for every $\omega \in \Omega$, $\langle X(\Phi), H_\omega \rangle$ is a finite normal form game (H_ω being the function defined from $X(\Phi)$ to $\mathbb{R}^n$ and given by $H_\omega(x) := H(x, \omega)$ for all $x \in X(\Phi)$). Besides, it is obvious that every finite normal form game can be thought as a finite random game. Interesting examples of finite random games are the so-called ε -randomizations of finite games. We use them to define the weakly strict equilibrium in Sect. 3.

Definition 2. Given a finite normal form game $G = \langle X(\Phi), H \rangle$, we define its ε -randomization G^ε as the fol-

lowing finite random game:

$$G^\varepsilon = \langle (\{-1, 0, 1\}^\Phi, \mathcal{A}, P), X(\Phi), H^\varepsilon \rangle$$

where

1. $\mathcal{A}$ is the class of subsets of $\{-1, 0, 1\}^\Phi$,
2. P is given by:

$$P(\omega) := \frac{1}{3^{|\Phi|}}$$

($|\Phi|$ denoting the cardinal of Φ) for every $\omega \in \{-1, 0, 1\}^\Phi$, and

3. H^ε is given by:

$$H_i^\varepsilon(x, \omega) = \sum_{\varphi \in \Phi} (H_i(\varphi) + \varepsilon \omega(\varphi)) x(\varphi)$$

for every $x \in X(\Phi)$, $\omega \in \{-1, 0, 1\}^\Phi$ and $i \in \{1, \dots, n\}$.

It is clear from the definition above that the ε -randomization of a finite game is simply a finite random game in which $3^{|\Phi|}$ different payoff-perturbed versions of the original game can be played with the same probability. Note that, in G^ε , every subset of $\{-1, 0, 1\}^\Phi$ is measurable. Then, it makes sense to define the function $N^\varepsilon: X(\Phi) \rightarrow [0, 1]$ given by:

$$N^\varepsilon(x) := P\{\omega \in \Omega \mid x \text{ is a Nash equilibrium of } \langle X(\Phi), H_\omega^\varepsilon \rangle\}$$

for all $x \in X(\Phi)$.

The next definition will be useful later on.

Definition 3. Given a finite normal form game $\langle X(\Phi), H \rangle$ we say that $x_i \in X_i(\Phi_i)$ is an ε -dominated strategy of player i if there exists $x'_i \in X_i(\Phi_i)$ such that

$$H_i(x \setminus x'_i) \geq H_i(x \setminus x_i) + 2\varepsilon \quad \forall x \in X(\Phi)$$

and

$$H_i(x^* \setminus x'_i) > H_i(x^* \setminus x_i) + 2\varepsilon \quad \text{for some } x^* \in X(\Phi).$$

In such a case we say that x'_i ε -dominates x_i , $x \in X(\Phi)$ is said to be ε -undominated if, for every i , x_i is not an ε -dominated strategy of player i .

Like about, in every ε -randomization of a finite game G^ε , as $\mathcal{A}$ is the class of subsets of $\{-1, 0, 1\}^\Phi$, it makes sense to define the function $U^\varepsilon: X(\Phi) \rightarrow [0, 1]$ given by:

$$U^\varepsilon(x) := P\{\omega \in \Omega \mid x \text{ is } \varepsilon\text{-undominated in } \langle X(\Phi), H_\omega^\varepsilon \rangle\}.$$

3. Weakly strict equilibria in finite games

Although we have remarked before that there is not a general agreement on what equilibrium concept is the most adequate solution of a finite game, everybody would admit that a strict equilibrium of a noncooperative game (finite or not) is, without any doubt, a rest-point of it. Remind that, if $G = \langle X(\Phi), H \rangle$ is a finite normal form game, $x \in X(\Phi)$ is a strict equilibrium of G if and only if, for all $i \in \{1, \dots, n\}$ and all $x'_i \in X_i(\Phi_i)$ (different from x_i), $H_i(x) > H_i(x \setminus x'_i)$. By $SE(G)$ we denote the set of strict equilibria of G . The main drawback of this concept is that not every finite normal form game possesses a strict equilibrium.

In this section we present a weak version of this concept which selects at least one point in every finite game and still maintains some interesting properties of the strict equilibrium. To that aim, we begin presenting an alternative characterization of the latter.

Let $G = \langle X(\Phi), H \rangle$ be a finite normal form game and, for every $\varepsilon > 0$, let G^ε be its ε -randomization and N^ε and U^ε the functions from $X(\Phi)$ to $[0, 1]$ as defined in Sect. 2. For every G^ε , we consider the function $S^\varepsilon: X(\Phi) \rightarrow [0, 1 + \varepsilon]$ given by:

$$S^\varepsilon(x) = U^\varepsilon(x) + \varepsilon N^\varepsilon(x),$$

for every $x \in X(\Phi)$. Now, we prove the following characterization of a strict equilibrium.

Theorem 1. *A combination of strategies $x \in X(\Phi)$ of a finite normal form game $G = \langle X(\Phi), H \rangle$ is a strict equilibrium of G if and only if there exist $\{\varepsilon^n\}$, a sequence of positive numbers converging to zero, and $\{x^n\}$, a sequence in $X(\Phi)$ converging to x , such that*

$$S^{\varepsilon^n}(x^n) = 1 + \varepsilon^n \quad (1)$$

for all $n \in \mathbb{N}$.

Proof. (Necessity) It is easy to see that, if x is a strict equilibrium of G , for every small $\varepsilon > 0$, x is an ε -undominated Nash equilibrium of the game $\langle X(\Phi), H_\omega^\varepsilon \rangle$ for all $\omega \in \Omega$. Hence, for any sequence of positive numbers converging to zero $\{\varepsilon^n\}$,

$$S^{\varepsilon^n}(x) = 1 + \varepsilon^n$$

for all n large enough (for instance, for all $n \geq N$). Thus, taking $\{x^n\}_{n \geq N}$ with $x^n = x$ for all $n \geq N$ and $\{\varepsilon^n\}_{n \geq N}$, the necessity is demonstrated.

(Sufficiency) Observe that, in G , only pure combinations of strategies can be strict equilibria, so $SE(G)$ has finitely many elements (if any). Thus, if $x \notin SE(G)$, there exists a neighbourhood V_x of x such that, for every $\hat{x} \in V_x$, $\hat{x} \notin SE(G)$. Now, take x such that there exist $\{\varepsilon^n\} \rightarrow 0$ and $\{x^n\} \rightarrow x$ with $S^{\varepsilon^n}(x^n) = 1 + \varepsilon^n$ for all n , and suppose that $x \notin SE(G)$. Then, taking into account the remarks above, $x^n \notin SE(G)$ for n large enough. Hence, $N^{\varepsilon^n}(x^n) < 1$ for n large enough and, consequently, $S^{\varepsilon^n}(x^n) < 1 + \varepsilon^n$. As this is impossible, $x \in SE(G)$ and the proof is concluded. $\square$

Note that, in Theorem 1, (1) is equivalent to

$$S^{\varepsilon^n}(x^n) = \max_{\hat{x} \in X(\Phi)} S^{\varepsilon^n}(\hat{x}) = 1 + \varepsilon^n.$$

However, for the ε^n -randomization there need not be strategy combinations for which the maximum value of $1 + \varepsilon^n$ is attained. Considering those strategy combinations at which the highest possible value is attained, we propose the following definition of weak strictness.

Definition 4. *Given a finite game $G = \langle X(\Phi), H \rangle$, we say that $x \in X(\Phi)$ is a weakly strict equilibrium of G if and only if there exist $\{\varepsilon^n\}$, a sequence of positive numbers converging to zero, and $\{x^n\}$, a sequence in $X(\Phi)$ converg-*

ing to x , such that

$$S^{\varepsilon^n}(x^n) = \max_{\hat{x} \in X(\Phi)} S^{\varepsilon^n}(\hat{x})$$

for all $n \in \mathbb{N}$.

Example 1. Consider the following bimatrix game.

	β_1	β_2
α_1	(1, 1)	(0, 0)
α_2	(0, 0)	(0, 0)

It is easy to see that, if ε is small enough, S^ε reaches a unique maximum at (α_1, β_1) . Hence, (α_1, β_1) is the only weakly strict equilibrium of this game.

The intuitive idea behind our new equilibrium concept is the following. In Theorem 1 we demonstrated that a strict equilibrium of a finite game $G = \langle X(\Phi), H \rangle$ satisfies that it is a Nash and, in some way, undominated equilibrium with probability one, even if we admit that, with a certain positive probability, the real payoffs of the game are slightly different of those indicated by the payoff function. This is a very appealing property but, unfortunately, is not true that every finite normal form game has an equilibrium satisfying it. A weakly strict equilibrium satisfies the following weaker property: it is a combination of strategies which has maximum probability of being a Nash and, in some way, undominated equilibrium when we admit that, with a certain positive probability, the real payoffs of the game are slightly different of those indicated by the payoff function. This is also an appealing property which has much to do with the idea underlying all refinements of the Nash condition: an equilibrium, to be selfenforcing, must keep being acceptable when the game is slightly perturbed.

Let us study now some properties of our concept. First, the existence.

Theorem 2. *Every finite normal form game has at least one weakly strict equilibrium.*

Proof. Let $G = \langle X(\Phi), H \rangle$ be a finite normal form game. Take $\{\varepsilon^n\}$ converging to zero and the corresponding sequence of ε^n -randomized versions of G , $\{G^{\varepsilon^n}\}$. As the function S^{ε^n} takes finitely many different values, it has a maximum x^n in $X(\Phi)$ (for every G^{ε^n}). Thus as $\{x^n\}$ is a sequence in a compact set, it has a subsequence which has a limit point x . Clearly, x is a weakly strict equilibrium of G . $\square$

In the following theorem we prove that our concept is a refinement of the undominated Nash equilibrium.

Theorem 3. *Every weakly strict equilibrium of a finite normal form game G is also an undominated Nash equilibrium of G .*

Proof. It is well known that every finite normal form game G has at least one undominated Nash equilibrium $\bar{x}$. Note that, for any G^ε ,

$$S^\varepsilon(\bar{x}) > 1 \quad (2)$$

because $U^\varepsilon(\bar{x}) = 1$ and $N^\varepsilon(\bar{x}) > 0$. Take now x , a weakly strict equilibrium of G , and suppose that it is dominated.

Then, as the set of undominated combinations of strategies of a finite game is closed, there exists an environment V_x of x such that, for every $\hat{x} \in V_x$, $\hat{x}$ is dominated. Hence, $U^\varepsilon(\hat{x}) \leq 1 - \frac{1}{3^{|\Phi|}}$ for all $\hat{x} \in V_x$ and then, when ε is small enough, $S^\varepsilon(\hat{x}) < 1$ for all $\hat{x} \in V_x$. But, in such a case, in view of (2), x cannot be weakly strict, which is a contradiction. Hence, x is undominated.

Suppose now that x is not a Nash equilibrium. As it is weakly strict, there exists $\{x^n\} \rightarrow x$ such that each x^n verifies that

$$S^{\varepsilon^n}(x^n) = \max_{\hat{x} \in X(\Phi)} S^{\varepsilon^n}(\hat{x}) \quad (3)$$

for some $\{\varepsilon^n\} \rightarrow 0$. As x is not a Nash equilibrium of G ,

$$H_i(x) < H_i(x \setminus x'_i)$$

for some $x'_i \in X_i(\Phi_i)$ and some i . Hence, there exists $\delta > 0$ such that, if n is large enough,

$$\delta + H_i^{\varepsilon^n}(x, \omega) < H_i^{\varepsilon^n}(x \setminus x'_i, \omega) \quad (4)$$

for all $\omega \in \Omega$. Now, it is clear that there exists a constant K such that, for every $x, y \in X(\Phi)$ and every $n \in \mathbb{N}$,

$$|H_i^{\varepsilon^n}(x, \omega) - H_i^{\varepsilon^n}(y, \omega)| = \left| \sum_{\varphi \in \Phi} ((H_i(\varphi) + \varepsilon \omega(\varphi))(x(\varphi))) \right| \leq K \|x - y\|_1 \quad (5)$$

for every $\omega \in \Omega$, where

$$\|x - y\|_1 = \sum_{\varphi \in \Phi} |x(\varphi) - y(\varphi)|.$$

Now, from (4) and (5), we can assure that, if n is large enough,

$$\delta < H_i^{\varepsilon^n}(x \setminus x'_i, \omega) - H_i^{\varepsilon^n}(x, \omega) \leq H_i^{\varepsilon^n}(x^n \setminus x'_i, \omega) + K \|x \setminus x'_i - x^n \setminus x'_i\|_1 - H_i^{\varepsilon^n}(x^n, \omega) - K \|x - x^n\|_1$$

for every $\omega \in \Omega$. But then, as $\{x^n\} \rightarrow x$, if n is large enough, we can assure that

$$0 < H_i^{\varepsilon^n}(x^n \setminus x'_i, \omega) - H_i^{\varepsilon^n}(x^n, \omega)$$

for all $\omega \in \Omega$.

Consequently, if n is large enough, $N^{\varepsilon^n}(x^n) = 0$, so $S^{\varepsilon^n}(x^n) \leq 1$. But this, in view of (2) and (3), is impossible. Hence, x has to be a Nash equilibrium. $\square$

The following theorem provides an interesting property of the weakly strict equilibrium and shows that this concept is a *strict* refinement of the undominated Nash equilibrium.

Theorem 4. *Every strict equilibrium of a finite normal form game G is weakly strict. Further, if G has strict equilibria, these are the only weakly strict ones.*

Proof. If is clear from Theorem 1 that every strict equilibrium is weakly strict. Now, if G has strict equilibria, only they maximize S^ε for ε small enough. Hence, as the set of strict equilibria is finite, the proof is concluded. $\square$

As a consequence of this theorem, we can assert that not every perfect equilibrium is weakly strict (we can even say that not every strictly perfect equilibrium, in the sense of Okada, 1981, is weakly strict). Namely, in the battle of the sexes, the completely mixed equilibrium is strictly perfect (and, hence, perfect) but not weakly strict (because it is not strict and there are two strict equilibria in the game).

Finally, we can wonder if every weakly strict equilibrium of a finite game is perfect. The answer, for bimatrix games, is *yes* (because, in that context, perfect and undominated Nash is the same) but, in general n -person finite form games, is *no*. For instance, in the 3-person game below $(\alpha_2, \beta_1, \gamma_1)$ is not perfect but it is weakly strict.

	β_1	β_2		β_1	β_2
α_1	(1, 1, 1)	(1, 0, 0)	α_1	(1, 0, 0)	(0, 0, 0)
α_2	(1, 1, 1)	(0, 0, 0)	α_2	(0, 0, 0)	(1, 0, 0)
	γ_1			γ_2	

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