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Benefit sharing in holding situations

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Abstract

A situation is studied where one agent has available an amount of storage facility and the other agents have some goods, part of which can be stored generating benefits. The problem of sharing the benefits produced by full cooperation between agents is tackled in this paper, by introducing a related cooperative game. This game turns out to be a big boss game with interesting theoretical properties. A solution concept, relying on optimal storage plans and associated holding prices, is also introduced, and its relationship with the core of the above holding game is explored in detail. The family of monotonic decreasing bijective mappings, defined on the set of non-negative real numbers, plays an important role in our approach.

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1. Introduction

We consider a situation in which $n + 1$ agents are involved. Agent 0 has a free storage facility of capacity h , whereas agents $1, 2, \dots, n$ have goods available which can be partially stored, generating certain benefits. In this context, two questions naturally arise: how to find an optimal storage plan, and how to distribute the gains derived from this optimal storage plan among the agents.

The first question leads us, under some natural assumptions on the benefit functions, to a constrained concave optimization problem. The second question is addressed here by considering a cooperative game associated with the holding situation. The principal aim of this paper is to analyze the properties and different solution concepts of this holding game, and its starting motivation came through the paper of Meca et al. (in press) on inventory games.

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The outline of the paper is as follows. In Section 2 we formally introduce holding problems and the associated cooperative games. The benefit functions are supposed to be elements of the class $\mathcal{R}$, a subclass of concave functions from $\mathbb{R}_+$ into itself. In this section we also introduce the class $\mathcal{D}$ of monotone decreasing bijective mappings on $\mathbb{R}_+$ with compact support. This class will play an important role in our paper due to the fact that it coincides with the set of generalized derivatives of the functions in $\mathcal{R}$. An optimality criterion for a feasible storage plan is given in terms of the generalized derivatives of the involved benefit functions. Aggregate benefit functions for coalitions of agents are also considered in this section. We prove that they also belong to $\mathcal{R}$, and the relationship between their subgradients and the generalized derivatives of the individual benefit functions is established and illustrated.

In Section 3 we show that the marginal contribution of a good holder is larger in small coalitions than in large coalitions. This is a key result to prove that holding games are strong big boss games, a class for which much is known about the classical cooperative solution concepts. Cooperative solutions of holding games are treated in Section 4. Historically, two methodologies have been used when looking for solutions for cooperative games. First, solutions obtained by a reductionist methodology, which is based on setting up some removal criteria from a bigger set of initial possible allocations. Solutions which have been obtained by applying this reductionist methodology are called *set solutions*, since removal criteria almost never produce a unique final allocation. The main set solutions are the core (Gillies, 1953), the stable sets (von Neumann and Morgenstern, 1947), the kernel (Davis and Maschler, 1965), and the bargaining set (Aumann and Maschler, 1964). The second group of solutions is based on that we could name a constructivist methodology: either an allocation is defined and proposed as a solution or certain desirable properties (axioms) which determine a unique allocation rule are established. These kind of solutions, which we call *values*, present a strong normative character and they have in common the fact of giving different answers to a natural question which arises in a spontaneous way in the context of cooperative games: what value should be assigned to each agent who takes part in a cooperative model? The most important values are: the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969) and the τ -value (Tijs, 1981). For a detailed description of these solution concepts we refer to Driessen (1988). A specific solution concept directly related to the optimality criterion given in Section 2, the so-called equilibrium allocation set, is studied in Section 4, where it is proved that this concept constitutes a refinement of the core.

To complete the paper, in Section 5 a dense survey is presented about well known algorithms, which can be applied to solve our constrained concave optimization problem, including their computational efficiency (rates of convergence) and we specially point out the procedures which exploit the separable structure of our objective function. Finally, Section 6 provides some concluding remarks.

2. Holding problems and related games

A *holding problem* can be described by means of an $(n+2)$ -tuple $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$, where $N = \{0, 1, 2, \dots, n\}$ is the set of involved agents or *players*, $h \in \mathbb{R}_+ \equiv [0, \infty)$ is the *holding capacity* of player 0, and for $i \in \{1, 2, \dots, n\}$ the function $B_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the *benefit function* of player i . For $x \in \mathbb{R}_+$, $B_i(x)$ is the benefit that agent i gets if this player is allowed to store an amount x .

We will assume that each B_i is an element of a class $\mathcal{R}$ of functions guaranteeing a smooth treatment of our questions concerning holding situations. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an element of $\mathcal{R}$ if the following properties are satisfied:

- (R.1) f is concave on $[0, \infty)$, and $f(0) = 0$.
- (R.2) f is constant for large values; i.e., there is a number $s(f) \in [0, \infty)$ such that f restricted to $[s(f), \infty)$ is constant.
- (R.3) The right-derivative at 0, $\lim_{k \searrow 0} k^{-1}(f(k) - f(0))$, is finite.

As it will be pointed out later in this section, these three properties play an important role in our approach, because they guarantee a convenient symmetric shape of the generalized derivatives of the functions in $\mathcal{R}$. Note that (R.2) is a natural condition for benefit functions in holding situations, where agents have only a limited amount of goods to store. (R.1) and (R.2) together entail that the elements of $\mathcal{R}$ are weakly increasing functions. Moreover, the continuity of $f \in \mathcal{R}$ on $[0, \infty)$ is implied by the concavity required in (R.1) together with (R.3).

Suppose that player 0 and the group $S \subset \{1, 2, \dots, n\}$ decide to cooperate in this holding situation, excluding the remaining players. When this group tries to maximize its global benefit, it will face the concave optimization problem

$$\max \left\{ \sum_{i \in S} B_i(x_i) \mid (x_i)_{i \in S} \in \mathbb{R}_+^S \text{ and } \sum_{i \in S} x_i = h \right\}.$$

The attainability of this maximum is guaranteed by the continuity of B_i , for each $i \in S$, and by the compactness of the feasible region. The non-empty optimal set will yield all the *optimal storage plans* for $S \cup \{0\}$. In general, we shall consider the *aggregate benefit function* of the group S which is the function $B_S : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$B_S(a) := \max \left\{ \sum_{i \in S} B_i(x_i) \mid (x_i)_{i \in S} \in \mathbb{R}_+^S \text{ and } \sum_{i \in S} x_i = a \right\},$$

for $a \in \mathbb{R}_+$. Obviously, $B_S(a)$ represents the maximal benefit that the agents in S can make if there is a storage capacity a available for them.

Now we introduce the *cooperative game* $(N, v_{\mathcal{H}})$ corresponding to the storage situation $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$. The characteristic function $v_{\mathcal{H}} : 2^N \rightarrow \mathbb{R}$ of this game is defined, for all non-empty $S \subset \{1, 2, \dots, n\}$, by

$$v_{\mathcal{H}}(S \cup \{0\}) := B_S(h) \quad \text{and} \quad v_{\mathcal{H}}(S) = 0.$$

Moreover, $v_{\mathcal{H}}(\{0\}) = 0$.

In this section we shall provide a necessary and sufficient condition for a feasible storage plan $(z_i)_{i \in S} \in \mathbb{R}_+^S$ to be optimal for $S \cup \{0\}$. To establish this result, identifying $\arg \max B_S(h)$, we need the notion of generalized derivative. Given $B_i \in \mathcal{R}$, its *generalized derivative* is the mapping $g_i : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ defined by

$$g_i(x) := \begin{cases} [g_i(x), \infty), & \text{for } x = 0, \\ [\underline{g}_i(x), \bar{g}_i(x)], & \text{for } x \in (0, \infty), \end{cases}$$

where $\underline{g}_i(x) := \lim_{k \searrow 0} k^{-1}(B_i(x+k) - B_i(x))$ is the right-derivative of B_i at $x \in [0, \infty)$, and $\bar{g}_i(x) := \lim_{k \searrow 0} k^{-1}(B_i(x) - B_i(x-k))$ is the left-derivative of B_i at $x \in (0, \infty)$. The existence of these one-sided derivatives in $[0, \infty)$ is guaranteed by the concavity of B_i (see, for instance, Hiriart-Urruty and Lemaréchal, 1993, Theorem 4.1.1 I), together with (R.3). The set $g_i(x)$ consists of the slopes of all the supporting straight lines to the hypograph of B_i , $\text{hypo } B_i := \{(y, \eta) \in \mathbb{R}^2 \mid \eta \leq B_i(y) \text{ and } y \geq 0\}$, through the point $(x, B_i(x))$, which motivates the term subgradient for $\lambda \in g_i(x)$, since $\lambda \in g_i(x)$ entails

$$B_i(y) \leq B_i(x) + \lambda(y - x), \quad \text{for all } y \in \mathbb{R}_+.$$

An alternative conventional notation in convex analysis for $g_i(x)$ is $\partial B_i(x)$.

Note that for each $B \in \mathcal{R}$, the generalized derivative g is an element of $\mathcal{D}$, the so-called class of *monotonic decreasing bijective mappings* on $\mathbb{R}_+$ with *compact support* (cf. Rockafellar, 1981, Chapter 7). $\mathcal{D}$ consists of those mappings $g : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ having the following three properties:

(D.1) *Monotonicity property*: for each $x_1, x_2 \in \mathbb{R}_+$ with $x_1 \leq x_2$ we have $\lambda_1 \geq \lambda_2$ for all $\lambda_1 \in g(x_1)$ and $\lambda_2 \in g(x_2)$.

(D.2) *Surjectivity property*: for each $\lambda \in \mathbb{R}_+$ there is an $x \in \mathbb{R}_+$ such that $\lambda \in g(x)$.

(D.3) *Compact support property*: $\{x \in \mathbb{R}_+ | g(x) \neq \{0\}\}$ is a bounded set.

Other interesting properties of $\mathcal{D}$ are the following:

(D.4) $\mathcal{D}$ is a convex cone: if $g_1, g_2 \in \mathcal{D}$ and $\alpha \in \mathbb{R}_+$, then $g_1 + g_2 \in \mathcal{D}$ and $\alpha g_1 \in \mathcal{D}$.

(D.5) $\mathcal{D}$ is closed with respect to taking inverses: if $g \in \mathcal{D}$, then $g^{-1} \in \mathcal{D}$.

(D.6) *Integrability of elements of $\mathcal{D}$* : if $s: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an arbitrary selection $s(x) \in g(x)$, then the function s is decreasing by (R.1) and, so, it has at most a countable set of discontinuities, entailing the existence of the integral

$$\int_0^a s(x) dx, \quad \text{for every } a \geq 0.$$

Actually, the concavity of any function $B \in \mathcal{R}$, implies that B is absolutely continuous on $[0, a]$, for every $a \geq 0$, and for each selection s of $g = \partial B$, we have

$$B(a) = \int_0^a s(x) dx, \quad \text{for every } a \geq 0. \quad (1)$$

Conversely, if $g \in \mathcal{D}$ and s is an arbitrary selection of g , one can easily realize that the function $B: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by means of the formula (1) belongs to $\mathcal{R}$, and that $g = \partial B$. The last equality comes from the fact that $B'(x) = g(x)$ provided that this generalized derivative is a single real number, together with the property stating that ∂B is a maximal monotone mapping (Rockafellar, 1970, Corollary 31.5.2) and the surjectivity of g . Thus, $\mathcal{D}$ is nothing else than the family of the generalized derivative mappings corresponding to all the functions in class $\mathcal{R}$.

Formally we write

$$(R.4) \quad B(a) = \int_0^a g(x) dx.$$

(D.7) *Inverse integral formula*: for each $g \in \mathcal{D}$ we have

$$\int_0^\infty g(x) dx = \int_0^\infty g^{-1}(y) dy.$$

Note that the two integrals express the area (Lebesgue measure) of the sets $(\text{hypo } g) \cap \mathbb{R}_+^2$ and $(\text{hypo}(g^{-1})) \cap \mathbb{R}_+^2$, respectively.

The following proposition gives answer to the first question stated in the introduction.

Proposition 1. Consider the holding situation $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$. Then, the following statements hold:

(i) For every $S \subset N \setminus \{0\}$, the aggregate benefit function B_S belongs to the class $\mathcal{R}$, and its generalized derivative g_S is given through the expression

$$g_S = \left(\sum_{i \in S} g_i^{-1} \right)^{-1},$$

where g_i is the generalized derivative of B_i , $i \in S$.

(ii) $z = (z_i)_{i \in S} \in \mathbb{R}_+^S$ is an optimal storage plan for $S \cup \{0\}$; i.e., $(z_i)_{i \in S} \in \arg \max B_S(h)$, if and only if $\sum_{i \in S} z_i = h$ and $\cap_{i \in S} g_i(z_i) \neq \emptyset$. Moreover

$$g_S(h) = \bigcap_{i \in S} g_i(z_i). \quad (2)$$

(iii) The generalized derivatives of the aggregate benefit functions g_S , $S \subset N \setminus \{0\}$, are increasingly monotonic with respect to the inclusion order in the family of subsets of $N \setminus \{0\}$; i.e., if $S \subset T \subset N \setminus \{0\}$ we have, at each $a \in \mathbb{R}_+$,

$$\lambda \geq \mu \quad \text{for every pair } \mu \in g_S(a) \text{ and } \lambda \in g_T(a).$$

Proof

(i) From the point of view of convex analysis, the aggregate benefit function is the sup-convolution of the individual benefit functions; i.e., $B_S = \square_{i \in S} B_i$ (see, for instance, Rockafellar, 1970, p. 34). Because $\arg \max B_S(a)$ is non-empty, it is said that this sup-convolution is *exact* (see, for instance, Hiriart-Urruty and Lemaréchal, 1993, p. 163).

Let us denote by B_S^* , B_i^* , $i \in S$, the corresponding Fenchel conjugates. Then, by Theorem 16.4 in Rockafellar (1970),

$$B_S^* = (\square_{i \in S} B_i)^* = \sum_{i \in S} B_i^*. \quad (3)$$

The continuity of the individual benefit functions implies that all of them are closed, and we can apply Theorem 23.5 in Rockafellar (1970) asserting that g_S^{-1} and g_i^{-1} , $i \in S$, are the generalized derivatives of B_S^* and B_i^* , $i \in S$, respectively.

(D.5) yields $B_i^* \in \mathcal{R}$, $i \in N \setminus \{0\}$, and (3) allows us to write

$$g_S^{-1} = \partial \left(\sum_{i \in S} B_i^* \right) = \sum_{i \in S} \partial B_i^* = \sum_{i \in S} g_i^{-1}. \quad (4)$$

Thus, (D.4) together with (D.5) lead to the formula we were looking for. The second equality in (4) is a consequence of Theorem 23.8 in Rockafellar (1970).

(ii) If $h = 0$, the first statement is trivial, with $z_i = 0$, $i \in S$, being the only optimal storage plan. If $h > 0$, we are dealing with a concave optimization problem, with the objective function $\sum_{i \in S} B_i(x_i)$, which has to be maximized on the subset of $\mathbb{R}_+^S$ of points $\mathbf{x} = (x_i)_{i \in S}$ for which the equality $f(\mathbf{x}) := \sum_{i \in S} x_i - h = 0$ holds. Then, by Corollary 28.3.1 in Rockafellar (1970), we conclude that $\mathbf{z} = (z_i)_{i \in S}$ is an optimal solution for this problem if and only if there exists $\lambda > 0$ such that

$$\mathbf{0}_S \in \partial \left(\sum_{i \in S} B_i(z_i) \right) - \lambda \partial f(z) = \prod_{i \in S} g_i(z_i) - \lambda \{\mathbf{1}_S\}, \quad (5)$$

where $\mathbf{0}_S$ and $\mathbf{1}_S$ are vectors in $\mathbb{R}^S$ with all the coordinates equal to zero and one, respectively, whereas $\prod$ stands for the Cartesian product. Finally, (5) is equivalent to $\lambda \in \bigcap_{i \in S} g_i(z_i) \neq \emptyset$.

We finish by pointing out that (2) is a straightforward consequence of the associativity of the sup-convolution and Corollary 4.5.5 in Hiriart-Urruty and Lemaréchal (1993, p. 274). Concerning the assumptions of the applied corollary, observe that, due to (R.2), the functions B_i , $i \in N \setminus \{0\}$, are majorized by a common affine function (actually, by a constant function).

(iii) Suppose that $(z_i)_{i \in S} \in \arg \max B_S(a)$ and $(y_i)_{i \in T} \in \arg \max B_T(a)$. Thus $\sum_{i \in S} z_i = \sum_{i \in T} y_i = a$, $\mu \in g_i(z_i)$, $i \in S$, and $\lambda \in g_i(y_i)$, $i \in T$. If we had $\lambda < \mu$, property (D.1) would imply $z_i < y_i$, $i \in S$, and we would get a contradiction: $a = \sum_{i \in S} z_i < \sum_{i \in S} y_i \leq \sum_{i \in T} y_i = a$. $\square$

Remark 2. Once we have established in (i) that $B_S \in \mathcal{R}$, the relation

$$B_S(h) = \int_0^h g_S(x) dx$$

is an immediate consequence of the properties (D.6) and (R.4).

Example 3. Consider the holding situation

$$\mathcal{H} = \langle \{0, 1, 2, 3\}, 3/2, B_1, B_2, B_3 \rangle,$$

where player 0 has holding capacity $3/2$, and for each $i \in \{1, 2, 3\}$

$$B_i(x) = \begin{cases} (1+i)x, & \text{for } 0 \leq x \leq 1, \\ 1+i, & \text{for } 1 < x < \infty. \end{cases}$$

Here, player i , $i \in \{1, 2, 3\}$, owns one unit to be stored, with storage marginal value constant and equal to $1+i$.

For this holding situation the plan $(0, 1/2, 1)$ is an optimal storage plan for the grand coalition because

$$g_1(0) \cap g_2(1/2) \cap g_3(1) = [2, \infty) \cap \{3\} \cap [0, 4] = \{3\} \neq \emptyset,$$

and

$$v_{\mathcal{H}}(N) = B_1(0) + B_2(1/2) + B_3(1) = 0 + \frac{3}{2} + 4 = \frac{11}{2}.$$

In this optimal plan the highest quality good (with the highest marginal benefit), possessed by player 3, is stored, and the rest of the holding capacity is used for half of the second best good, owned by player 2. The holding game $(N, v_{\mathcal{H}})$ associated with $\mathcal{H}$ is given by $N = \{0, 1, 2, 3\}$ and the characteristic function

$$v_{\mathcal{H}}(S) = 0, \quad \text{for all } S \subset \{1, 2, 3\},$$

$$v_{\mathcal{H}}(\{0, 1, 2, 3\}) = v_{\mathcal{H}}(\{0, 2, 3\}) = 11/2,$$

$$v_{\mathcal{H}}(\{0, 1, 2\}) = 4, \quad v_{\mathcal{H}}(\{0, 1, 3\}) = 5,$$

$$v_{\mathcal{H}}(\{0, 1\}) = 2, \quad v_{\mathcal{H}}(\{0, 2\}) = 3,$$

$$v_{\mathcal{H}}(\{0, 3\}) = 4, \quad v_{\mathcal{H}}(\{0\}) = 0.$$

Moreover, we find, for instance,

$$g_{\{1,2\}}(x) = (g_1^{-1} + g_2^{-1})^{-1}(x) = \begin{cases} 3, & x \in (0, 1), \\ 2, & x \in (1, 2), \\ 0, & x \in (2, \infty), \end{cases}$$

where the obvious values of $g_{\{1,2\}}$ in 0, 1, and 2 are omitted. So

$$v_{\mathcal{H}}(\{0, 1, 2\}) = B_{\{1,2\}}(3/2) = \int_0^{3/2} g_{\{1,2\}}(x) dx = \int_0^1 3 dx + \int_1^{3/2} 2 dx = 4,$$

a result that we know already. On the other hand, $g_{\{1,2,3\}}$ is single valued in the intervals $(0, 1)$, $(1, 2)$, $(2, 3)$, and $(3, \infty)$ with values 4, 3, 2, and 0, respectively. So,

$$v_{\mathcal{H}}(N) = B_{\{1,2,3\}}(3/2) = \int_0^{3/2} g_{\{1,2,3\}}(x) dx = \int_0^1 4 dx + \int_1^{3/2} 3 dx = 11/2.$$

Example 4. Consider the holding situations

$$\mathcal{H} = \langle \{0, 1, 2\}, h, B_1, B_2 \rangle,$$

$$\mathcal{H}' = \langle \{0, 1, 2\}, h', B_1, B_2 \rangle,$$

with $h = 3/2$ and $h' = 4$, and

$$B_1(x) = \begin{cases} 2x - \frac{1}{2}x^2, & x \in [0, 2], \\ 2, & x \in (2, \infty), \end{cases}$$

$$B_2(x) = \begin{cases} x, & x \in [0, 1], \\ 1, & x \in (1, \infty). \end{cases}$$

Next, we analyze the associated games:

(i) $\arg \max_{N \setminus \{0\}}(h) = \{(1, 1/2)\}$ because one has $z_1 + z_2 = 3/2 = h$, together with $g_1(z_1) \cap g_2(z_2) \neq \emptyset$, if and only if $z_1 = 1, z_2 = 1/2$. The holding game $(\{0, 1, 2\}, v_{\mathcal{H}})$ is given by

$$v_{\mathcal{H}}(\{1\}) = v_{\mathcal{H}}(\{2\}) = v_{\mathcal{H}}(\{0\}) = v_{\mathcal{H}}(\{1, 2\}) = 0,$$

$$v_{\mathcal{H}}(\{0, 1, 2\}) = B_1(1) + B_2(1/2) = \frac{3}{2} + \frac{1}{2} = 2,$$

$$v_{\mathcal{H}}(\{0, 1\}) = B_1(3/2) = 15/8, \quad v_{\mathcal{H}}(\{0, 2\}) = B_2(3/2) = 1.$$

Moreover

$$g_1(x) = \begin{cases} [2, +\infty), & x = 0, \\ 2 - x, & x \in (0, 2), \\ 0, & x \in [2, \infty), \end{cases}$$

$$g_2(x) = \begin{cases} [1, +\infty), & x = 0, \\ 1, & x \in (0, 1), \\ [0, 1], & x = 1, \\ 0, & x \in (1, \infty), \end{cases}$$

and

$$g_{\{1,2\}}(x) = (g_1^{-1} + g_2^{-1})^{-1}(x) = \begin{cases} [2, \infty), & x = 0, \\ 2 - x, & x \in (0, 1), \\ 1, & x \in [1, 2), \\ 3 - x, & x \in [2, 3), \\ 0, & x \in [3, \infty). \end{cases}$$

Then

$$B_{\{1,2\}}(x) = \int_0^x g_{\{1,2\}}(y) dy = \begin{cases} 2x - \frac{1}{2}x^2, & x \in [0, 1), \\ x + \frac{1}{2}, & x \in [1, 2), \\ -\frac{3}{2} + 3x - \frac{1}{2}x^2, & x \in [2, 3), \\ 3, & x \in [3, \infty), \end{cases}$$

and $v_{\mathcal{H}}(\{0, 1, 2\}) = B_{\{1,2\}}(3/2) = \frac{3}{2} + \frac{1}{2} = 2$.

(ii) Each $\mathbf{z} = (z_1, z_2) \in \mathbb{R}_+^2$ with $z_1 + z_2 = 4, z_1 \geq 2$ and $z_2 \geq 1$, is an element of $\arg \max_{N \setminus \{0\}}(h')$ because, for such $\mathbf{z}$, one has $0 \in g_1(z_1) \cap g_2(z_2)$. The associated holding game $(\{0, 1, 2\}, v_{\mathcal{H}'})$ is given by

$$v_{\mathcal{H}'}(\{1\}) = v_{\mathcal{H}'}(\{2\}) = v_{\mathcal{H}'}(\{0\}) = v_{\mathcal{H}'}(\{1, 2\}) = 0,$$

$$v_{\mathcal{H}'}(\{0, 1\}) = B_1(4) = 2, \quad v_{\mathcal{H}'}(\{0, 2\}) = B_2(4) = 1,$$

$$v_{\mathcal{H}'}(\{0, 1, 2\}) = B_1(z_1) + B_2(z_2) = 2 + 1 = 3.$$

Let us give an economic interpretation of the optimality criterion established above, motivating a possible benefit allocation for the grand coalition.

Suppose that μ is a uniform price that agents $1, 2, \dots, n$ have to pay per unit of storage capacity. As a reaction, this price will reasonably lead to a storage demand, of agent i , $x_i \in [\underline{g}_i^{-1}(\mu), \bar{g}_i^{-1}(\mu)]$, where the only goods that are offered to storage are those having storage quality (marginal benefit) at least μ . In general $\sum_{i=1}^n x_i$ will not coincide with h .

We say that μ is an *equilibrium price* if there is a demand profile $\mathbf{x} := (x_i)_{i \in N \setminus \{0\}}$ corresponding to μ which fits with the supply h ; i.e. $x_i \in g_i^{-1}(\mu)$, for each $i \in N \setminus \{0\}$, and $\sum_{i=1}^n x_i = h$. Such a pair $(\mu, \mathbf{x})$ will be called an *equilibrium pair* for the holding situation. Now, our optimality criterion, when applied to $S = N$, can be read as follows:

A feasible holding profile $\mathbf{x} = (x_i)_{i \in N \setminus \{0\}}$ is optimal if and only if there is a price μ such that $(\mu, \mathbf{x})$ is an equilibrium pair for $\mathcal{H}$.

For an equilibrium pair $(\mu, \mathbf{x})$, the vector

$$E(\mu, \mathbf{x}) := (h\mu, B_1(x_1) - x_1\mu, \dots, B_n(x_n) - x_n\mu)$$

is called the *equilibrium allocation* for $(\mu, \mathbf{x})$. Note that $\sum_{i=0}^n E_i(\mu, \mathbf{x}) = v_{\mathcal{H}}(N)$, and that the allocation corresponds to a situation where player 0 obtains $h\mu$ and player i pays a price μ per unit of stored good.

3. Holding games are strong big boss games

The class of big boss games, introduced by Muto et al. (1988), includes many games modelling economic problems. The corresponding versions of the main solutions in cooperative games theory have already been studied for this class, as well as the relationship among them. It will turn out that we can use these results for our holding games, because we will show, in this section, that holding games are big boss games.

The cooperative game (N, v) , with $N = \{0, 1, 2, \dots, n\}$, is a *strong big boss game*, with 0 as *big boss*, if the following conditions hold:

(B.B.1) *Monotonicity*: $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subset T$.

(B.B.2) *Big boss property*: $v(S) = 0$ if $0 \notin S$.

(B.B.3) *Strong union property*: $v(T) - v(S) \geq \sum_{i \in T \setminus S} M_i(v)$, for all $0 \in S \subset T \subset N$, where $M_i(v) := v(N) - v(N \setminus \{i\})$ is the *marginal contribution* of player i to the grand coalition N .

(N, v) will be an ordinary *big boss game* if (B.B.1) and (B.B.2) are satisfied together with

(B.B.3*) *Union property*: $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(v)$, for all $S \subset N$ such that $0 \in S$.

Notice that a strong big boss game is a big boss game.

In Muto et al. (1988) attention is also given to convex big boss games. Recall that a game (N, v) is *convex* (Shapley, 1971) if for all S, T , and k , with $S \subset T \subset N \setminus \{k\}$, we have

$$v(S \cup \{k\}) - v(S) \leq v(T \cup \{k\}) - v(T).$$

In the following lemma we gather some results (Proposition 3.4, p. 307, and Theorem 4.5, p. 311, in Muto et al. (1988)), which are used in the last section.

Lemma 5. *Let (N, v) be a big boss game, with 0 as big boss, and $v(0) = 0$. Then, the following statements are equivalent:*

- (i) (N, v) is convex,
- (ii) $v(N) - v(S) = \sum_{i \in N \setminus S} M_i(v)$, for each $S \subset N$ with $0 \in S$,
- (iii) $v(S) = \sum_{i \in S \setminus \{0\}} M_i(v)$, for each $S \subset N$ with $0 \in S$,
- (iv) the Shapley value coincides with the τ -value.

The following lemma is crucial to prove that holding games are big boss games, namely in the proof of (B.B.3). The lemma states that, in a holding game, the marginal contribution of a player $k \in \{1, 2, \dots, n\}$ to a larger coalition is smaller than that to a smaller coalition.

Lemma 6. *Let $(N, v_{\mathcal{H}})$ be a holding game associated with $\mathcal{H} = \langle N, h, B_1, B_2, \dots, B_n \rangle$. Then, if $0 \in S \subset T \subset N \setminus \{k\}$, the following inequality is satisfied*

$$v_{\mathcal{H}}(S \cup \{k\}) - v_{\mathcal{H}}(S) \geq v_{\mathcal{H}}(T \cup \{k\}) - v_{\mathcal{H}}(T).$$

Proof. Suppose that $(z_i)_{i \in S \cup \{k\} \setminus \{0\}}$ and $(y_i)_{i \in T \cup \{k\} \setminus \{0\}}$ are optimal storage plans for $S \cup \{k\}$ and $T \cup \{k\}$, respectively. Thus, we can write

$$v_{\mathcal{H}}(S \cup \{k\}) = B_k(z_k) + B_{S \setminus \{0\}}(h - z_k),$$

and

$$v_{\mathcal{H}}(T \cup \{k\}) = B_k(y_k) + B_{T \setminus \{0\}}(h - y_k).$$

These expressions, together with (R.4), yield

$$v_{\mathcal{H}}(S \cup \{k\}) - v_{\mathcal{H}}(S) = \int_0^{z_k} g_k(x) dx - \int_{h-z_k}^h g_{S \setminus \{0\}}(x) dx, \quad (6)$$

and

$$v_{\mathcal{H}}(T \cup \{k\}) - v_{\mathcal{H}}(T) = \int_0^{y_k} g_k(x) dx - \int_{h-y_k}^h g_{T \setminus \{0\}}(x) dx. \quad (7)$$

Now, let us take an arbitrary pair $\mu \in g_{S \cup \{k\} \setminus \{0\}}(h)$ and $\lambda \in g_{T \cup \{k\} \setminus \{0\}}(h)$. By Proposition 1(iii), we know that $\mu \leq \lambda$. Since $\mu \in \cap_{i \in S \setminus \{0\}} g_i(z_i)$, as a consequence of (2), we have in addition $\mu \in g_{S \setminus \{0\}}(h - z_k)$ and, by monotonicity,

$$g_{S \setminus \{0\}}(x) \subset [0, \mu] \quad \text{for } x \in (h - z_k, \infty). \quad (8)$$

Also due to the monotonicity of g_k

$$g_k(x) \subset [\mu, \infty) \quad \text{for } x \in [0, z_k). \quad (9)$$

Since $\mu \leq \lambda$, $z_k \in g_k^{-1}(\mu)$, $y_k \in g_k^{-1}(\lambda)$, and once again by (D.1), applied to g_k^{-1} , we get $z_k \geq y_k$. So, from (6) and (7) we obtain

$$\begin{aligned} & [v_{\mathcal{H}}(S \cup \{k\}) - v_{\mathcal{H}}(S)] - [v_{\mathcal{H}}(T \cup \{k\}) - v_{\mathcal{H}}(T)] \\ &= \int_{y_k}^{z_k} g_k(x) dx - \int_{h-z_k}^{h-y_k} g_{S \setminus \{0\}}(x) dx + \int_{h-y_k}^h g_{T \setminus \{0\}}(x) dx - \int_{h-y_k}^h g_{S \setminus \{0\}}(x) dx. \end{aligned} \quad (10)$$

Next, we prove that

$$\int_{y_k}^{z_k} g_k(x) dx - \int_{h-z_k}^{h-y_k} g_{S \setminus \{0\}}(x) dx \geq 0, \quad (11)$$

and that

$$\int_{h-y_k}^h g_{T \setminus \{0\}}(x) dx - \int_{h-y_k}^h g_{S \setminus \{0\}}(x) dx \geq 0, \quad (12)$$

which finishes the proof. Actually, by (8) and (9), we can write

$$\int_{y_k}^{z_k} g_k(x) dx - \int_{h-z_k}^{h-y_k} g_{S \setminus \{0\}}(x) dx \geq \mu(z_k - y_k) - \mu(h - y_k - h + z_k) = 0,$$

and, so, (11) holds. Moreover, since $S \subset T$, we conclude that (12) also holds, by assertion (iii) in Proposition 1. $\square$

The main result in this section comes next.

Proposition 7. *The holding game $(N, v_{\mathcal{H}})$, corresponding to a holding situation $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$, is a strong big boss game with player 0 as big boss. Moreover, each subgame of $(N, v_{\mathcal{H}})$ is a strong big boss game.*

Proof

(B.B.1) Consider $S \subset T$. If $0 \notin S$, then $v_{\mathcal{H}}(T) \geq 0 = v_{\mathcal{H}}(S)$. Thus, we shall suppose $0 \in S \subset T$.

Let $(z_i)_{i \in S \setminus \{0\}}$ be an optimal storage plan for S . Then $(z'_i)_{i \in T \setminus \{0\}}$, with $z'_i = z_i$ for $i \in S \setminus \{0\}$, and $z'_i = 0$ for $i \in T \setminus S$, is a feasible storage plan for T . So,

$$v_{\mathcal{H}}(S) = \sum_{i \in S \setminus \{0\}} B_i(z_i) = \sum_{i \in T \setminus \{0\}} B_i(z'_i) \leq v_{\mathcal{H}}(T),$$

and $v_{\mathcal{H}}$ satisfies the monotonicity condition.

(B.B.2) Obviously, $v_{\mathcal{H}}(S) = 0$ if $0 \notin S$.

(B.B.3) Take $S, T \subset N$, such that $0 \in S \subset T$. Let $T \setminus S = \{1, 2, \dots, k\}$. Then

$$\begin{aligned} v_{\mathcal{H}}(T) - v_{\mathcal{H}}(S) &= \sum_{r=1}^k [v_{\mathcal{H}}(S \cup \{1, 2, \dots, r\}) - v_{\mathcal{H}}(S \cup \{1, 2, \dots, r-1\})] \geq \sum_{r=1}^k [v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{r\})] \\ &= \sum_{r=1}^k M_r(v_{\mathcal{H}}) = \sum_{i \in T \setminus S} M_i(v_{\mathcal{H}}), \end{aligned}$$

where the inequality follows from Lemma 6, which is applied k times. So, $v_{\mathcal{H}}$ has the strong union property.

Similarly, it can be proved that for each $S \subset N$, $(v_{\mathcal{H}})_S$ satisfies the conditions (B.B.1), (B.B.2) and (B.B.3). (Actually, if $0 \notin S$, the game $(S, (v_{\mathcal{H}})_S)$ is trivial, since its characteristic function is the zero function.) $\square$

4. Solutions for holding games

Since holding games are strong big boss games, we can apply the results given in Muto et al. (1988), Potters et al. (1989), and Tijs (1990), all these papers dealing with the theoretical solutions of this class of games. Interesting results related to the *set solutions* and *values* for big boss games, are given in the following. The core and the bargaining set coincide, both being a parallelotope, which can be easily described. About the stability of the core, it is showed that it is a stable set if and only if the game is convex. Moreover, the τ -value and the nucleolus coincide and they are located in the center of the core. The kernel consists of one point, which coincides with the nucleolus. The Shapley value, the nucleolus and the τ -value coincide for convex games.

Let us start with the core. For an arbitrary cooperative game (N, v) , an element $\mathbf{x} \in \mathbb{R}^N$ belongs to the core, denoted by $C(v)$, if and only if $\sum_{i \in N} x_i = v(N)$ and $\sum_{i \in S} x_i \geq v(S)$ for each $S \subset N$. For a strong big

boss game many of these inequalities are redundant, and only the values $v_{\mathcal{H}}(N)$ and $v_{\mathcal{H}}(N \setminus \{i\})$, for $i \in N \setminus \{0\}$, are involved in the core. In fact, the core turns out to be the intersection of the hypercube $\prod_{i=1}^n [0, M_i(v_{\mathcal{H}})]$ with the hyperplane $\{\mathbf{x} \in \mathbb{R}^N \mid \sum_{i=0}^n x_i = v_{\mathcal{H}}(N)\}$.

The τ -value and the nucleolus are in the center of the core and they are obtained by splitting the marginal contributions $M_i(v_{\mathcal{H}})$, for $i \in N \setminus \{0\}$, in two equal parts, one part being given to player i and the other part to the holding facility owner 0. The Shapley value, denoted by Φ , is less favorable to player 0 than the τ -value, except for the case of convexity. As just mentioned, in this convex case, the Shapley value, the nucleolus and the τ -value coincide. The kernel, $\mathcal{K}(v_{\mathcal{H}})$, consists of one point, which coincides with the τ -value, whereas the bargaining set, $\mathcal{M}(v_{\mathcal{H}})$, coincides with the core. We summarize these results in the following proposition.

Proposition 8. *Let $(N, v_{\mathcal{H}})$ be the holding game associated with the holding situation $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$. Then,*

(i) *The core $C(v_{\mathcal{H}})$ is the parallelotope*

$$\left\{ \mathbf{x} \in \mathbb{R}^N \mid \sum_{i=0}^n x_i = v_{\mathcal{H}}(N), 0 \leq x_i \leq M_i(v_{\mathcal{H}}), \text{ for all } i \in N \setminus \{0\} \right\}.$$

(ii) $\mathcal{M}(v_{\mathcal{H}}) = C(v_{\mathcal{H}})$.

(iii) *The nucleolus, denoted by $\mathcal{N}(v_{\mathcal{H}})$, and the τ -value, $\tau(v_{\mathcal{H}})$, coincide, both being equal to*

$$\left(v_{\mathcal{H}}(N) - \frac{1}{2} \sum_{i=1}^n M_i(v_{\mathcal{H}}), \frac{1}{2} M_1(v_{\mathcal{H}}), \frac{1}{2} M_2(v_{\mathcal{H}}), \dots, \frac{1}{2} M_n(v_{\mathcal{H}}) \right).$$

(iv) $\mathcal{K}(v_{\mathcal{H}}) = \{\mathcal{N}(v_{\mathcal{H}})\}$.

(v) *Concerning the first coordinate of the Shapley value, $\Phi(v_{\mathcal{H}})$, we have $\Phi_0(v_{\mathcal{H}}) \leq \mathcal{N}_0(v_{\mathcal{H}})$.*

Proof. For (i), (iii) and (v) see Theorems 3.2, 4.2 and 4.3 in Muto et al. (1988), respectively. For (ii) and (iv) see Theorems 2.1 and 3.1 in Potters et al. (1989), respectively. $\square$

The following example illustrates the above solution concepts.

Example 9. Let $\mathcal{H}$ be the holding situation described in Example 3. Then, $M_1(v_{\mathcal{H}}) = v_{\mathcal{H}}(\{0, 1, 2\}) - v_{\mathcal{H}}(\{0, 2\}) = 1$, $M_2(v_{\mathcal{H}}) = v_{\mathcal{H}}(\{0, 1, 2\}) - v_{\mathcal{H}}(\{0, 1\}) = 1/8$. So, $C(v_{\mathcal{H}}) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_0 + x_1 + x_2 = 2, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1/8\}$, which is a parallelogram with vertices $(2, 0, 0)$, $(1, 1, 0)$, $(7/8, 1, 1/8)$ and $(15/8, 0, 1/8)$. Moreover, $\mathcal{N}(v_{\mathcal{H}}) = \tau(v_{\mathcal{H}}) = (23/16, 1/2, 1/16)$. The Shapley value is $\Phi(v_{\mathcal{H}}) = (55/48, 31/48, 10/48)$, and it is not a core element of this non-convex game.

Next, we characterize those holding games which are, in addition, convex. We shall refer to the numbers

$$s_i := \min \{t \in \mathbb{R}_+ \mid B_i \text{ restricted to } [t, \infty) \text{ is constant}\}, \quad i = 1, 2, \dots, n.$$

This minimum exists, in view of property (R.2), for $B_i \in \mathcal{R}$. Then $\text{supp } g_i := \text{cl}\{x \mid g_i(x) \neq \{0\}\} = [0, s_i]$, from which it follows that $0 \in g_i(s_i)$ and $g_i(x_i) = \{0\}$ for all $x_i \in (s_i, \infty)$. Furthermore, for each $S \subset N \setminus \{0\}$, $\text{supp } g_S = [0, \sum_{i \in S} s_i]$.

Proposition 10. *Let $(N, v_{\mathcal{H}})$ be a holding game corresponding to a holding situation $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$. Suppose that $h > 0$ and that the set $\{i \in N \setminus \{0\} \mid s_i > 0\}$ contains at least two elements. Then*

- (i) $(N, v_{\mathcal{H}})$ is convex if and only if $\sum_{i \in N \setminus \{0\}} s_i \leq h$.
(ii) If $(N, v_{\mathcal{H}})$ is convex, we have

$$\mathcal{N}(v_{\mathcal{H}}) = \tau(v_{\mathcal{H}}) = \Phi(v_{\mathcal{H}}) = \left(\frac{1}{2} \sum_{i=1}^n B_i(s_i), \frac{1}{2} B_1(s_1), \dots, \frac{1}{2} B_n(s_n) \right).$$

- (iii) $C(v_{\mathcal{H}})$ is a stable set if and only if $(N, v_{\mathcal{H}})$ is convex.

Proof. First we prove the ‘only if’ part of (i). If we suppose that $\sum_{i \in N \setminus \{0\}} s_i > h$, then taking $k \in N \setminus \{0\}$ with $s_k > 0$, we shall prove

$$v_{\mathcal{H}}(\{0, k\}) - v_{\mathcal{H}}(\{0\}) > v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{k\}), \quad (13)$$

which contradicts the assumed convexity of $(N, v_{\mathcal{H}})$.

Now, note that

$$v_{\mathcal{H}}(\{0, k\}) - v_{\mathcal{H}}(\{0\}) = \int_0^{s_k} g_k(x) dx. \quad (14)$$

Take an optimal storage plan $(z_i)_{i \in N \setminus \{0\}}$ satisfying $z_k \leq s_k$ (if $z_k > s_k$, and since $B(z_k) = B(s_k)$, we would redistribute the surplus of storage capacity $z_k - s_k$ among the other players). Applying formula (6) in Lemma 6 (with $N \setminus \{k\}$ in the role of S), we know that

$$v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{k\}) = \int_0^{z_k} g_k(x) dx - \int_{h-z_k}^h g_{N \setminus \{0, k\}}(x) dx. \quad (15)$$

We shall discuss two possibilities. If $z_k < s_k$, then

$$\int_0^{z_k} g_k(x) dx - \int_{h-z_k}^h g_{N \setminus \{0, k\}}(x) dx \leq \int_0^{z_k} g_k(x) dx < \int_0^{s_k} g_k(x) dx,$$

and (13) follows from (14) and (15).

If $z_k = s_k$, then $\text{supp } g_{N \setminus \{0, k\}} = [0, \sum_{i \in N \setminus \{0, k\}} s_i]$, with $\sum_{i \in N \setminus \{0, k\}} s_i > 0$, because of the assumption asserting that the set $\{i \in N \setminus \{0\} | s_i > 0\}$ contains at least two elements. Moreover $\sum_{i \in N \setminus \{0\}} s_i > h$ entails

$$\sum_{i \in N \setminus \{0, k\}} s_i > h - s_k = h - z_k.$$

Thus

$$\int_{h-z_k}^h g_{N \setminus \{0, k\}}(x) dx > 0.$$

So, (14) and (15) jointly imply

$$v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{k\}) = \int_0^{s_k} g_k(x) dx - \int_{h-z_k}^h g_{N \setminus \{0, k\}}(x) dx < \int_0^{s_k} g_k(x) dx = v_{\mathcal{H}}(\{0, k\}) - v_{\mathcal{H}}(\{0\}),$$

and (13) holds.

Now we prove the ‘if’ part of (i), starting from the assumption $\sum_{i \in N \setminus \{0\}} s_i \leq h$. In this case, it is obvious that

$$\arg \max_{B_{N \setminus \{0\}}(h)} = \left\{ (z_i)_{i \in N \setminus \{0\}} \left| s_i \leq z_i, i \in N \setminus \{0\}, \text{ and } \sum_{i \in N \setminus \{0\}} z_i = h \right. \right\}.$$

For a fixed $k \in N \setminus \{0\}$, let us take an optimal storage plan $(z_i)_{i \in N \setminus \{0\}}$ with $z_k = s_k$. Then,

$$\int_{h-z_k}^h g_{N \setminus \{0,k\}}(x) dx = 0,$$

because $\text{supp } g_{N \setminus \{0,k\}} = [0, \sum_{N \setminus \{0,k\}} s_i]$, with $\sum_{N \setminus \{0,k\}} s_i \leq h - s_k = h - z_k$. Then, by (15),

$$M_k(v_{\mathcal{H}}) = v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{k\}) = \int_0^{s_k} g_k(x) dx = B_k(s_k). \quad (16)$$

Thus, we conclude that, for each $k \in N \setminus \{0\}$, we have $M_k(v_{\mathcal{H}}) = B_k(s_k)$.

Following the same reasoning, one gets, for each $S \subset N$ with $0 \in S$ and $k \in S$,

$$v_{\mathcal{H}}(S) - v_{\mathcal{H}}(S \setminus \{k\}) = B_k(s_k). \quad (17)$$

So, for all $S \subset N$ with $0 \in S$,

$$v_{\mathcal{H}}(S) = \sum_{i \in S \setminus \{0\}} B_i(s_i) = \sum_{i \in S \setminus \{0\}} M_i(v_{\mathcal{H}}),$$

and Lemma 5 yields the conclusion that $(N, v_{\mathcal{H}})$ is a convex game. Moreover, we observe that

$$v_{\mathcal{H}} = \sum_{i=1}^n B_i(s_i) u_{\{0,i\}},$$

where $(N, u_{\{0,i\}})$ is the corresponding convex unanimity game. Further, from the additivity of the Shapley value, it follows

$$\Phi(v_{\mathcal{H}}) = \sum_{i=1}^n \Phi(B_i(s_i) u_{\{0,i\}}) = \left(\frac{1}{2} \sum_{i=1}^n B_i(s_i), \frac{1}{2} B_1(s_1), \dots, \frac{1}{2} B_n(s_n) \right),$$

and, taking into account that $B_i(s_i) = M_i(v_{\mathcal{H}})$, we also have $\Phi(v_{\mathcal{H}}) = \mathcal{N}(v_{\mathcal{H}}) = \tau(v_{\mathcal{H}})$. Finally, for (iii) see Theorem 3.5 in Muto et al. (1988). $\square$

Remark 11. The following holding situations lead to convex holding games.

- (i) $\langle N, h, B_1, \dots, B_n \rangle$ with $h = 0$. This is a trivial convex holding game, with all the coalitions having value zero.
- (ii) $\langle N, h, B_1, \dots, B_n \rangle$ with $h = 1$, $s_1 = 1$, and $s_i = 0$, for $i \in \{2, \dots, n\}$.

Example 12. Consider the holding situation $\mathcal{H}'$ in Example 3, where $s_1 + s_2 = 2 + 1 < 4 = h$. For this problem

$$\mathcal{N}(v_{\mathcal{H}'}) = \tau(v_{\mathcal{H}'}) = \Phi(v_{\mathcal{H}'}) = (3/2, 1, 1/2).$$

Finally we study the geometric structure of the set of equilibrium allocations for a holding situation $\mathcal{H}$, which are introduced in Section 2. Given a holding situation $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$, we consider equilibrium pairs of the form $(\mu, \mathbf{x})$, and the associated equilibrium allocations $E(\mu, \mathbf{x})$. Denoting by $E(\mathcal{H})$ the set of all possible equilibrium allocations, we investigate the geometric structure of this set, which is also compared with the core of the associated game $(N, v_{\mathcal{H}})$.

Proposition 13. The equilibrium allocation set $E(\mathcal{H})$, associated with the holding problem $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$, is either a singleton or a line segment in $\mathbb{R}^N$.

Proof

(i) First we show that, for equilibrium pairs $(\mu, \mathbf{x})$ and $(\mu, \mathbf{z})$ with the same price μ , the equilibrium allocations $E(\mu, \mathbf{x})$ and $E(\mu, \mathbf{z})$ are equal. Let us start by noting that $E_0(\mu, \mathbf{x}) = E_0(\mu, \mathbf{z}) = h\mu$.

Take $k \in N \setminus \{0\}$. If $x_k = z_k$, then $E_k(\mu, \mathbf{x}) = E_k(\mu, \mathbf{z}) = B_k(x_k) - x_k h$. Then, suppose that $x_k \neq z_k$ and, without loss of generality, $x_k < z_k$. Thus, $[x_k, z_k] \subset g_k^{-1}(\mu)$ and

$$\begin{aligned} E_k(\mu, \mathbf{z}) &= \int_0^{z_k} g_k(x) dx - \mu z_k = \int_0^{x_k} g_k(x) dx + \int_{x_k}^{z_k} g_k(x) dx - \mu z_k = \int_0^{x_k} g_k(x) dx + \mu(z_k - x_k) - \mu z_k \\ &= B_k(x_k) - \mu x_k = E_k(\mu, \mathbf{x}). \end{aligned}$$

So, $E(\mu, \mathbf{x}) = E(\mu, \mathbf{z})$.

(ii) Now we consider the case in which $g_N(h) = [\underline{\mu}, \bar{\mu}]$ consists of one point; i.e., $\underline{\mu} = \bar{\mu}$. Then, by (i),

$$E(\mathcal{H}) = \{E(\underline{\mu}, \mathbf{x}) | (\underline{\mu}, \mathbf{x}) \text{ is an equilibrium pair}\}$$

is an one-point set.

(iii) Finally, consider the case that $g_{N \setminus \{0\}}(h) = [\underline{\mu}, \bar{\mu}]$ with $\bar{\mu} > \underline{\mu}$. Let $(\underline{\mu}, \mathbf{x})$ and $(\mu, \mathbf{z})$ be equilibrium pairs. We claim that $\mathbf{x} = \mathbf{z}$. To prove this equality, let us observe that $x_i \in g_i^{-1}(\underline{\mu})$, $z_i \in g_i^{-1}(\mu)$, and $\underline{\mu} \leq \mu$ together imply $x_i \geq z_i$, for all $i \in N \setminus \{0\}$. From $\sum_{i=1}^n z_i = \sum_{i=1}^n x_i = h$ it follows that $\mathbf{x} = \mathbf{z}$, entailing

$$E(\mathcal{H}) = \{E(\mu, \mathbf{x}) | \underline{\mu} \leq \mu \leq \bar{\mu}, \text{ and } (\mu, \mathbf{x}) \text{ is an equilibrium pair}\}.$$

Now

$$\begin{aligned} E(\mu, \mathbf{x}) &= (h\mu, B_1(x_1) - \mu x_1, \dots, B_n(x_n) - \mu x_n) \\ &= E(\underline{\mu}, \mathbf{x}) + (\mu - \underline{\mu})(h, -x_1, \dots, -x_n), \end{aligned}$$

and $E(\mathcal{H})$ will be the line segment

$$\left\{ E(\underline{\mu}, \mathbf{x}) + \lambda(h, -x_1, \dots, -x_n) | 0 \leq \lambda \leq \bar{\mu} - \underline{\mu} \right\}. \quad \square$$

Now we show that the equilibrium allocation set is a refinement of the core.

Proposition 14. Let $(N, v_{\mathcal{H}})$ be the holding game corresponding to the holding situation $\mathcal{H} = \langle N, h, B_1, \dots, B_n \rangle$. Then $E(\mathcal{H}) \subset C(v_{\mathcal{H}})$.

Proof. Take an equilibrium pair $(\mu, \mathbf{x})$ for $\mathcal{H}$. In view of Proposition 7, we have to prove that $\sum_{i=0}^n E_i(\mu, \mathbf{x}) = v_{\mathcal{H}}(N)$, that $E_k(\mu, \mathbf{x}) \geq 0$ and that $E_k(\mu, \mathbf{x}) \leq v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{k\})$, for each $k \in N \setminus \{0\}$.

We have already seen, in Section 2, that the equality holds. The first inequality follows from $B_k(x_k) = \int_0^{x_k} g_k(x) dx \geq \mu x_k$ since $g_k(x) \subset [\mu, \infty)$ for $x \in [0, x_k]$. For the second inequality note that, adapting the formulas (6) and (8) in Lemma 6, with $N \setminus \{k\}$ in the role of S , it follows

$$v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{k\}) = \int_0^{x_k} g_k(x) dx - \int_{h-x_k}^h g_{N \setminus \{0,k\}}(x) dx,$$

and

$$g_{N \setminus \{0\}}(x) \subset [0, \mu], \quad \text{for every } x \in (h - x_k, \infty).$$

Thus,

$$v_{\mathcal{H}}(N) - v_{\mathcal{H}}(N \setminus \{k\}) \geq B_k(x_k) - \int_{h-x_k}^h \mu dx = B_k(x_k) - \mu x_k = E_k(\mu, \mathbf{x}). \quad \square$$

Finally, we give an example illustrating the results obtained in this section.

Example 15. Consider the holding situations

$$\mathcal{H} = \langle \{0, 1, 2\}, h, B_1, B_2 \rangle,$$

$$\mathcal{H}' = \langle \{0, 1, 2\}, h', B_1, B_2 \rangle,$$

with $h = 1$, $h' = 4$ respectively, and

$$B_1(x) = B_2(x) = \begin{cases} 5x, & x \in [0, 2], \\ 10, & x \in (2, \infty). \end{cases}$$

For the holding situation $\mathcal{H}$ the set of equilibrium pairs is

$$\{(\mu, x_1, x_2) \in \mathbb{R}_+^3 \mid \mu = 5, x_1 + x_2 = 1\}.$$

Accordingly, $E(\mathcal{H}) = \{(5, 0, 0)\} = C(v_{\mathcal{H}})$.

For the holding situation $\mathcal{H}'$ the set of equilibrium pairs is equal to

$$\{(\mu, x_1, x_2) \in \mathbb{R}_+^3 \mid 0 \leq \mu \leq 5, x_1 = x_2 = 2\};$$

so that $E(\mathcal{H}')$ is the line segment

$$\{(0, 10, 10) + \lambda(4, -2, -2) \mid 0 \leq \lambda \leq 5\},$$

which is a proper subset of the core

$$C(v_{\mathcal{H}'}) = \text{conv}\{(0, 10, 10), (20, 0, 0), (10, 10, 0), (10, 0, 10)\}.$$

Note that the game $(N, v_{\mathcal{H}})$ is a convex game, and that $\mathcal{N}(v_{\mathcal{H}'}) = \tau(v_{\mathcal{H}'}) = \Phi(v_{\mathcal{H}'}) = (10, 5, 5)$.

5. Algorithmic approaches

Section 2 provides optimality conditions for storage plans. However, how to find an optimal plan requires to apply an algorithm to solve the non-smooth concave optimization problem

$$\begin{aligned} (P) \quad & \text{Max} \quad \sum_{i=1}^n B_i(x_i), \\ & \text{s.t.} \quad \sum_{i=1}^n x_i = h, \\ & \quad x_i \geq 0, \quad \text{for } i = 1, 2, \dots, n. \end{aligned} \tag{18}$$

Note that (P) only makes sense in the non-trivial case $\sum_{i=1}^n s_i > h$.

Parallel to what is standard in smooth optimization codes, we assume throughout this section that we dispose of a subroutine (*oracle* or *black box*) which computes the function $\sum_{i=1}^n B_i(x_i) \equiv \sum_{i=1}^n B_i(\mathbf{x})$, with $\mathbf{x} := (x_i)_{i=1}^n \in \mathbb{R}^n$, and a subgradient of this function at $\mathbf{x}$

$$\mathbf{u} \in \partial \left(\sum_{i=1}^n B_i(\mathbf{x}) \right) = \prod_{i=1}^n g_i(x_i).$$

Among the first methods dealing with (18) are the so-called *subgradient methods* (see, eg. Shor (1985)), which try to imitate typical gradient methods for smooth optimization. Since for non-smooth problems the norm of the subgradient, $\|\mathbf{u}\|$, is not very informative, the subgradient scheme will use *normalized* directions $\mathbf{u}/\|\mathbf{u}\|$:

0. Choose $\mathbf{x}^0 \in F$, where F is the feasible set of (P) ; i.e.,

$$F := \left\{ \mathbf{x} = (x_i)_{i=1}^n \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = h \right\},$$

and a sequence $\{\lambda_k\}_{k=0}^\infty$ such that $\lambda_k \geq 0$ and $\lambda_k \rightarrow 0$.

1. k th iteration ($k \geq 0$):

Compute $\sum_{i=1}^n B_i(x_i^k)$, $\mathbf{u}^k \in \partial(\sum_{i=1}^n B_i(x_i^k))$ and set

$$\mathbf{x}^{k+1} = \pi_F(\mathbf{x}^k + \lambda_k \mathbf{u}^k / \|\mathbf{u}^k\|), \quad (19)$$

where π_F denotes the projection operator onto F .

By applying Theorem 27.4 in Rockafellar (1970), the reader might verify that, in our case, $\mathbf{x}^{k+1}$ is the only point satisfying the following condition:

$$x_i^{k+1} > 0 \Rightarrow x_i^{k+1} - x_i^k - \lambda_k u_i^k / \|\mathbf{u}^k\| \leq x_j^{k+1} - x_j^k - \lambda_k u_j^k / \|\mathbf{u}^k\|, \quad \text{for all } j$$

(see, also, Bertsekas, 1995, p. 178).

Observe that, according to the previous scheme, $\mathbf{x}^{k+1} = \mathbf{x}^k$ will imply $\mathbf{x}^k = \mathbf{x}^*$. This is obvious, if $\mathbf{u}^k$ is the zero vector. Otherwise, since the projection $\mathbf{x}^{k+1}$ of $\mathbf{x}^k + \lambda_k \mathbf{u}^k / \|\mathbf{u}^k\|$ on F is characterized by the inequality

$$\{(\mathbf{x}^k + \lambda_k \mathbf{u}^k / \|\mathbf{u}^k\|) - \mathbf{x}^{k+1}\}'(\mathbf{z} - \mathbf{x}^{k+1}) \leq 0, \quad \text{for every } \mathbf{z} \in F,$$

we get, in the case $\mathbf{x}^{k+1} = \mathbf{x}^k$,

$$(\mathbf{u}^k)'(\mathbf{z} - \mathbf{x}^k) \leq 0, \quad \text{for every } \mathbf{z} \in F,$$

and this implies, once again, $\mathbf{x}^k = \mathbf{x}^*$. (We have applied, again, Theorem 27.4 in Rockafellar (1970).)

If never $\mathbf{x}^{k+1} = \mathbf{x}^k$, it can be proved that all the limit points of the sequence $\{\mathbf{x}^k\}_{k=0}^\infty$ are optimal solutions of (P) , provided that $\sum_{k=0}^\infty \lambda_k = \infty$.

As a consequence of the simplicity of iteration (19) (in particular, no line-search is needed), these methods do not guarantee a descent at each step and one can expect a poor convergence speed to these limit points (*less than geometric* or *sublinear*). If we are in the very special situation in which the optimal value of (P) , $B_N(h)$, is known and there is a unique optimal solution, $\mathbf{x}^*$, which is, in addition, *strongly unique* or *sharp* (i.e., for some $\rho > 0$, $\sum_{i=1}^n B_i(x_i) \leq B_N(h) - \rho \|\mathbf{x} - \mathbf{x}^*\|$, for all $\mathbf{x} \in F$), then the sequence $\{\mathbf{x}^k\}_{k=0}^\infty$ generated by (19) with the stepsize rule

$$\alpha \in]0, 2[\quad \text{and} \quad \lambda_k := \alpha \frac{B_N(h) - \sum_{i=1}^n B_i(x_i^k)}{\|\mathbf{u}^k\|}, \quad k = 0, 1, 2, \dots \quad (20)$$

will converge to $\mathbf{x}^*$ with *geometric* convergence rate (also called *R-linear convergence*). Observe that, assumed that the optimal solution is unique, the strong uniqueness is held when the functions B_i are polyhedral.

In case we do not know the optimal value $B_N(h)$, the expression in (20) can be replaced by

$$\lambda_k := \alpha \frac{\sum_{i=1}^n \{B_i(s_i^k) - B_i(x_i^k)\}}{\|\mathbf{u}^k\|}, \quad k = 0, 1, 2, \dots,$$

but, if the coefficient α is held constant, the sequence of $\{\sum_{i=1}^n B_i(x_i^k)\}_{k=0}^\infty$ will in general not converge to the optimal value $B_N(h)$. Held et al. (1974) have then suggested to let α go to 0, and have proposed a number of heuristic rules for the sequence of the α_k giving satisfactory results in practice.

In general, if we look for more efficiency, we need to give up the Markov nature of (19) by considering information obtained in previous steps. The methods based on the *space extension* (due to Shor (1985), and

closely related to the *ellipsoid methods*), provide geometric convergence of the sequence values $\max_{j=0,1,\dots,k} \sum_{i=1}^n B_i(x_i^j)$ to $B_N(h)$. For instance, a geometric convergence with approximate ratio $1 - 1/(2n^2)$, for large n , is achieved in the method discussed in Lemaréchal (1989, Theorem 5.3). This ratio is not bad, mainly because the theoretical *optimal ratio* ($1 - c/n$, for a certain $c > 0$ and large n , according to Nemirovsky and Yudin (1983)) only is attained for *conceptual* algorithms like the *method of centers of gravity* in which $\mathbf{x}^{k+1}$ is the center of gravity of the polytope, containing $\mathbf{x}^*$,

$$\mathbb{P}_k := \{\mathbf{x} \in F | (\mathbf{u}^j)'(\mathbf{x} - \mathbf{x}^j) \geq 0, k = 0, 1, \dots, k\}.$$

(Unfortunately, there is no implementable way to compute the center of gravity of a polyhedron.)

A step forward in non-smooth optimization are the *bundle methods*, introduced in Wolfe (1985) and Lemaréchal (1989). These algorithms, at the current iterate $\mathbf{x}^k$, make use of the *bundle* of information relative to subgradients and objective function values gathered so far, and they even mobilize more subgradient information close to $\mathbf{x}^k$ (when the objective function is rather kinky).

Conceptually, the bundle methods use the ε -subgradient of minimal norm at $\mathbf{x}^k$ as the search direction, which maximizes the ε -directional derivative at this iterate and guarantees an increase of at least ε (with a convenient stepsize). So, the bundle methods are ascent algorithms that, if the objective function is bounded from above, lead to an ε -optimal point in finitely many steps. All these ε -concepts can be found in any standard book of convex analysis, but for a nice description of the bundle methods philosophy, as well as information about implementation we recommend the survey papers Lemaréchal (1989) and Zowe (1985). Bundle methods appear as a stabilization of cutting plane algorithms, and also can be viewed as extended versions of gradient type algorithms in the non-differentiable setting. Some of them have good implementations, but their convergence rates are still far from achieving the performance of their homologues in differentiable optimization. The goal of obtaining faster convergence requires either to use second order information, as the ellipsoid method does, or to develop proper second order concepts (like the *second order directional derivative* in Lemaréchal and Zowe (1983)).

One can expect a nicer performance of the algorithms presented in this section when they are applied to a more narrow class of functions, which would allow us to exploit the particular structure of the problem. Actually our model really has a peculiar structure: a separable concave objective function and a canonical simplex as feasible set. In fact the variable change $z_i := x_i - (h/n)$ and $\bar{B}_i(z_i) := B_i(z_i + (h/n))$ permit to classify our problem as a *monotropic programming* problem (see Rockafellar, 1984) and to apply efficient specific algorithms based on the ε -relaxation methods for convex-cost network flow (see, for instance, Tseng (2001) where an ε -out-of-kilter method is presented, together with a satisfactory complexity analysis).

Coming back to subgradient methods, but exploiting separability, Kao and Meyer (1981) suggest an original approach that, rather than an upper approximation of the objective function based on its subgradients, provides a lower approximation built via convex combinations of computed sample points of the graph of the objective function. Some other apparently successful algorithms which can be applied to our problem are the *incremental subgradient algorithms* (e.g., Nedić and Bersekas, 2000). The underlying idea is to perform the subgradient iteration by sequentially taking steps along the subgradients of the component functions (functions B_i , in our problem).

6. Concluding remarks

In this paper the benefit sharing problem arising from our holding situation is treated with the aid of cooperative game theory. The games corresponding to holding situations turn out to have a nice structure: they are big boss games. It makes them specially suitable for the study of benefit sharing solutions. Appealing benefit sharing solutions are:

(i) The equilibrium allocations (especially if the number of agents is large), which turn out to be core elements of the introduced holding game (Proposition 14).

(ii) The allocation resulting when agents $1, 2, \dots, n$ keep for themselves half of their marginal contributions, whereas the other half-part is given to the holding house keeper, player 0. This allocation coincides with the τ -value and with the nucleolus of the holding game, and lies in the center of the core of this game (Proposition 8).

In our analysis of the holding problem the assumptions (R.1)–(R.3) play a crucial role, because they lead to interesting cones of functions and of generalized derivatives. It is a topic for further research to weaken these assumptions.

Optimal storage and benefit sharing is just one of the many possible interpretations of the formal model presented here. Different other resource allocation problems can be caught by our model. Among them, we point out the following one: Agent 0 owns an input or factor of production. Every agent $i \geq 1$ owns a technology described by the production function B_i . The problem is to share the aggregate output. Alternatively, one can assume that agent i generates the profit $B_i(x_i)$ when using the input x_i . Then one has a profit sharing problem.

On the other hand, a resemblance with ‘core inclusion’ results for games corresponding to exchange economies can also be observed. It would allow to give an alternative proof for Proposition 14, based on the possibility of associating a two-good pure exchange economy with our holding model.

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