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Tijs, S.H.; Brânzei, R.

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Various Characterizations of Convex Fuzzy Games

Stef Tijs

*CentER and Department of Econometrics and Operations Research,
Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
E-mail: s.h.tijs@uvt.nl*

Rodica Brânzei

*Faculty of Computer Science
"Alexandru Ioan Cuza" University, Iasi, Romania
E-mail: branzeir@infoiasi.ro*

Abstract

This note enlarges the literature on convex fuzzy games with new characterizing properties of such games besides the increasing average marginal return property, namely: the monotonicity of the first partial derivatives, the directional convexity and for C^2 -functions the non-negativity of the second order partial derivatives.

Key Words: Cooperative game, fuzzy game, convex game.

AMS subject classification: 91A12.

1 Introduction

Convex fuzzy games are introduced by Brânzei et al. (2003) as games with fuzzy coalitions possessing both the supermodularity property and the coordinate-wise convexity property. A characterizing property for convex fuzzy games called the IAMR-property has been given there, too. In this paper we provide new characterizing properties for convex fuzzy games, among which the directional convexity property, introduced by Shaked and Shanthikumar (1990) and also studied by Meester and Shanthikumar (1999). Inspiring for us has been the important characterization of convex crisp games by Martínez-de-Albéniz and Rafels (2002). It is proved there that any n -person convex cooperative (crisp) game can be represented as a measure game by combining n measures, one for each player, by a directionally convex function. Convex games and convex measure games are introduced by Shapley (1971). A convex measure game is induced by a convex function

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on $\mathbb{R}$, acting over a measure on the coalitions of players. For a formal treatment of properties of convex (crisp) games we refer the reader to Shapley (1971), Ichiishi (1981), Driessen (1988) and Curiel (1997).

Convex fuzzy games possess appealing properties with respect to incentives for cooperation like increasing marginal contribution for players, increasing marginal contribution for coalitions, marginal vector stability, and their solutions behave nicely. A survey on properties and solution concepts for convex fuzzy games based on Brânzei et al. (2003), Brânzei et al. (2004a), Brânzei et al. (2004b) is given by Brânzei and Tijs (2004c). Preliminaries on convex fuzzy games required for our results in this note are presented in Section 2. In Section 3 we prove that the pair of defining properties for convex fuzzy games, the IAMR-property, the directional convexity and the set of monotonicity properties of partial derivatives are equivalent for continuous games. Also an easy check criterium is given for the convexity of fuzzy games when the function is twice differentiable.

2 Preliminaries on convex fuzzy games

Let $N = \{1, 2, \dots, n\}$ be a nonempty set of players considering possibilities of fuzzy cooperation, i.e. the players may be involved in cooperation with participation levels varying between 0 (non-cooperation) and 1 (full cooperation). Formally, a *fuzzy coalition* of players in N is a vector $s \in [0, 1]^N$, whose the i -th coordinate s_i is called the *participation level* of player i . Instead of $[0, 1]^N$ we will also write $\mathcal{F}^N$ for the set of fuzzy coalitions. Special cases of fuzzy coalitions are those corresponding to crisp coalitions $S \in 2^N$, which are denoted here by e^S , where $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N \setminus S$. Then $e^\emptyset = (0, \dots, 0)$ stands for the empty coalition in a fuzzy setting, $e^N = (1, \dots, 1)$ denotes the grand coalition, whereas e^i is the fuzzy coalition corresponding to the crisp coalition $S = \{i\}$ (and also the i -th standard basis vector in $\mathbb{R}^N$). One can identify a fuzzy coalition with a point in the hypercube $[0, 1]^N$; the fuzzy coalitions e^S , $S \in 2^N$, are the $2^{|N|}$ extreme points (vertices) of this hypercube. A cooperative *fuzzy game* with player set N is a function $v : \mathcal{F}^N \rightarrow \mathbb{R}$ with the property $v(e^\emptyset) = 0$, assigning to each fuzzy coalition s the value $v(s)$ achieved as the result of cooperation with participation levels s_i , $i \in N$. Here we denote the set of fuzzy games with player set N by FG^N .

The Aubin core $C(v)$ of $v \in FG^N$ (Aubin (1974)) is the set

$$C(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in \mathcal{F}^N \right\}.$$

Let $v \in FG^N$ and $t \in \mathcal{F}^N$. The t -restricted game of v is the game $v_t : \mathcal{F}^N \rightarrow \mathbb{R}$ given by $v_t(s) = v(t * s)$ for all $s \in \mathcal{F}^N$. Here $t * s$ is the coordinate-wise product of t and s , that is $(t * s)_i = t_i s_i$ for all $i \in N$.

A game $v \in FG^N$ is called totally balanced if $C(v_t) \neq \emptyset$ for all $t \in \mathcal{F}^N$.

A special class of totally balanced fuzzy games is the class of convex fuzzy games introduced by Brânzei et al. (2003).

Let N be a finite set and let $v : [0, 1]^N \rightarrow \mathbb{R}$ be a real-valued function on $[0, 1]^N$. Then

- (i) v is called a *supermodular function* on $[0, 1]^N$ if

$$v(s \vee t) + v(s \wedge t) \geq v(s) + v(t) \text{ for all } s, t \in [0, 1]^N,$$

where $s \vee t$ and $s \wedge t$ are those elements of $[0, 1]^N$ with the i -th coordinate equal to $\max\{s_i, t_i\}$ and $\min\{s_i, t_i\}$, respectively;

- (ii) v is called a *coordinate-wise convex function* if for each $i \in N$ and each $s^{-i} \in [0, 1]^{N \setminus \{i\}}$ the function $g_{s^{-i}} : [0, 1] \rightarrow \mathbb{R}$ with $g_{s^{-i}}(t) = v(s^{-i} || t)$ for each $t \in [0, 1]$ is a convex function. Here $(s^{-i} || t)$ is the element in $[0, 1]^N$ with $(s^{-i} || t)_j = s_j$ for each $j \in N \setminus \{i\}$ and $(s^{-i} || t)_i = t$.

Let $v \in FG^N$. The fuzzy game v is called a *convex fuzzy game* if the function $v : [0, 1]^N \rightarrow \mathbb{R}$ is supermodular and coordinate-wise convex on $[0, 1]^N$.

We denote the set of convex fuzzy games with player set N by CFG^N . If $v \in CFG^N$, then also $v_t \in CFG^N$ for each $t \in \mathcal{F}^N$. This is the fuzzy analogue of the fact that subgames of crisp convex games are convex.

Let $v \in CFG^N$. Then the following properties hold:

- (i) (*Increasing marginal contribution for players*). Let $i \in N, s^1, s^2 \in \mathcal{F}^N$

with $s^1 \leq s^2$ and let $\varepsilon \in \mathbb{R}_+$ with $0 \leq \varepsilon \leq 1 - s_i^2$. Then

$$v(s^1 + \varepsilon e^i) - v(s^1) \leq v(s^2 + \varepsilon e^i) - v(s^2).$$

- (ii) (*Increasing marginal contribution for coalitions*). Let $s, t \in \mathcal{F}^N$ and $z \in \mathbb{R}_+^N$ such that $s \leq t \leq t + z \leq e^N$. Then

$$v(s + z) - v(s) \leq v(t + z) - v(t).$$

- (iii) (*Stable marginal contribution property*). For each $\sigma \in \Pi(N)$ the marginal vector $m^\sigma(v)$ is a core element. Here $\Pi(N)$ stands for the set of linear orderings of N and $m^\sigma(v)$ for each $\sigma \in \Pi(N)$ has the i -th coordinate $m_\sigma^i(v)$ for $i = \sigma(k)$ given by

$$m_\sigma^i(v) = v\left(\sum_{r=1}^k e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} e^{\sigma(r)}\right).$$

These properties are established in Propositions 3 and 4, and Theorem 7 in the work of Brânzei et al. (2003).

Further, v is said to satisfy the *increasing average marginal return property* (IAMR-property) if for each $i \in N$, $s^1, s^2 \in \mathcal{F}^N$ with $s^1 \leq s^2$ and each $\varepsilon_1, \varepsilon_2 > 0$ with $s_i^1 + \varepsilon_1 \leq s_i^2 + \varepsilon_2 \leq 1$

$$\varepsilon_1^{-1}(v(s^1 + \varepsilon_1 e^i) - v(s^1)) \leq \varepsilon_2^{-1}(v(s^2 + \varepsilon_2 e^i) - v(s^2)).$$

The IAMR-property expresses the fact that an increase in participation level of any player in a smaller coalition yields per unit of participation level less than an increase in participation level in a bigger coalition under the condition that the reached level of participation in the first case is still not bigger than the reached participation level in the second case. The IAMR-property turns out to be crucial for convex fuzzy games as Theorem 6 in the work of Brânzei et al. (2003) states: If $v \in FG^N$ then $v \in CFG^N$ iff the increasing average marginal return property (IAMR-property) holds.

Convex fuzzy games form a convex cone, that is for all $v, w \in CFG^N$ and all $a, b \geq 0$, $av + bw \in CFG^N$. For examples of convex fuzzy games the reader is referred to Brânzei et al. (2003, 2004a, 2004b) and Tijs et al. (2004).

3 Characterizing properties for convex fuzzy games

First we introduce some new notation and definitions. Let $v : [0, 1]^N \rightarrow \mathbb{R}$, $i \in N$, $s \in [0, 1]^N$. We say that the left derivative $D_i^- v(s)$ of v in the i -th direction at s with $0 < s_i \leq 1$ exists if $\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \varepsilon^{-1}(v(s) - v(s - \varepsilon e^i))$ exists and is finite:

$$D_i^- v(s) := \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \varepsilon^{-1}(v(s) - v(s - \varepsilon e^i)).$$

Similarly, the right derivative of v in the i -th direction at s with $0 \leq s_i < 1$, denoted by $D_i^+ v(s)$, is $\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \varepsilon^{-1}(v(s + \varepsilon e^i) - v(s))$ if this limit exists and is finite.

Definition 3.1 (Monotonicity of the first partial derivatives (MOPAD) - property). We say that $v : [0, 1]^N \rightarrow \mathbb{R}$ has the MOPAD-property if for each $i \in N$ and each $s \in [0, 1]^N$ with $0 < s_i \leq 1$ the left derivative $D_i^- v(s)$ exists and for each $s \in [0, 1]^N$ with $0 \leq s_i < 1$ the right derivative $D_i^+ v(s)$ exists and they satisfy

$$D_i^- v(s) \leq D_i^+ v(s), \quad (\text{M1})$$

and if moreover, for each $s^1, s^2 \in [0, 1]^N$ with $s^1 \leq s^2$ and each $i \in N$ we have

$$D_i^- v(s^1) \leq D_i^- v(s^2), \quad D_i^+ v(s^1) \leq D_i^+ v(s^2), \quad (\text{M2})$$

where we put for convenience $D_i^- v(s) = -\infty$ if $s_i = 0$ and $D_i^+ v(s) = \infty$ if $s_i = 1$.

Definition 3.2 (Martínez-de-Albéniz and Rafels (2002), p. 3). The function $v : [0, 1]^N \rightarrow \mathbb{R}$ is *directionally convex* if for each $a, b \in [0, 1]^N$ with $a \leq b$ and each pair $c, d \in [a, b]$ with $c + d = a + b$ it follows that

$$v(a) + v(b) \geq v(c) + v(d).$$

Here $[a, b] = \{x \in [0, 1]^N \mid a_i \leq x_i \leq b_i \text{ for each } i \in N\}$.

Alternatively, we say that v has the directional convexity (DICOV)-property.

The following lemma will play a role in what follows.

Lemma 3.1. *Let $s \in [0, 1]^N$, $i \in N$ with $0 \leq s_i < 1$ and $0 < \varepsilon_1 \leq \varepsilon_2 < \varepsilon_3 \leq 1 - s_i$. If v satisfies the MOPAD-property then*

$$\varepsilon_1^{-1}(v(s + \varepsilon_1 e^i) - v(s)) \leq (\varepsilon_3 - \varepsilon_2)^{-1}(v(s + \varepsilon_3 e^i) - v(s + \varepsilon_2 e^i)).$$

Proof. From the definition of the left derivative and by (M2) it follows

$$\varepsilon_1^{-1}(v(s + \varepsilon_1 e^i) - v(s)) = \varepsilon_1^{-1} \int_0^{\varepsilon_1} D_i^- v(s + x e^i) dx \leq$$

$$\varepsilon_1^{-1} \int_0^{\varepsilon_1} D_i^- v(s + \varepsilon_1 e^i) dx = D_i^- v(s + \varepsilon_1 e^i) \leq$$

$$D_i^- v(s + \varepsilon_2 e^i) = (\varepsilon_3 - \varepsilon_2)^{-1} \int_{\varepsilon_2}^{\varepsilon_3} D_i^- v(s + \varepsilon_2 e^i) dx \leq$$

$$(\varepsilon_3 - \varepsilon_2)^{-1} \int_{\varepsilon_2}^{\varepsilon_3} D_i^- v(s + x e^i) dx =$$

$$(\varepsilon_3 - \varepsilon_2)^{-1}(v(s + \varepsilon_3 e^i) - v(s + \varepsilon_2 e^i)).$$

□

The main result of this note is

Theorem 3.1. *Let $v : [0, 1]^N \rightarrow \mathbb{R}$ be a continuous function. The following assertions are equivalent:*

- (i) *v has the supermodularity (SM) property and the coordinate-wise convexity (CwC) property;*
- (ii) *v has the MOPAD-property;*
- (iii) *v satisfies the IAMR-property;*
- (iv) *v has the DICOV-property.¹*

Proof. (i) $\rightarrow$ (ii): From the CwC-property the existence of the relevant left and right derivatives and also (M1) follow. To prove (M2) note first that

¹ We thank Professors Martínez-de-Albéniz and Rafels for discussions on directionally convex functions.

for s^1 with $s_i^1 = 0$ we have $D_i^- v(s^1) = -\infty \leq D_i^- v(s^2)$. If $s_i^1 > 0$, then the CwC-property implies (Proposition 3 in Brânzei et al. (2003)) that

$$v(s^1) - v(s^1 - \varepsilon e^i) \leq v(s^2) - v(s^2 - \varepsilon e^i)$$

for $\varepsilon > 0$ such that $s_i^1 - \varepsilon \geq 0$. By multiplying the left and right sides of the above inequality with ε^{-1} and then taking the limit for ε going to 0, we obtain $D_i^- v(s^1) \leq D_i^- v(s^2)$. The second inequality in (M2) can be proved in a similar way.

(ii) $\rightarrow$ (iii): Suppose that v satisfies the MOPAD-property. We have to prove that for each $a, b \in [0, 1]^N$ with $a \leq b$, each $i \in N$, $\delta \in (0, 1 - a_i]$, $\varepsilon \in (0, 1 - b_i]$ such that $a_i + \delta \leq b_i + \varepsilon \leq 1$ it follows that

$$\delta^{-1}(v(a + \delta e^i) - v(a)) \leq \varepsilon^{-1}(v(b + \varepsilon e^i) - v(b)). \quad (3.1)$$

Take $a, b, i, \delta, \varepsilon$ as above. Let $c = a + (b_i - a_i)e^i$ and $d = b + (a_i + \delta - b_i)e^i$. We consider two cases

a) $a_i + \delta \leq b_i$. Then

$$\begin{aligned} \delta^{-1}(v(a + \delta e^i) - v(a)) &= \delta^{-1} \int_0^\delta D_i^- v(a + x e^i) dx \leq \\ \delta^{-1} \int_0^\delta D_i^- v(a + \delta e^i) dx &= D_i^- v(a + \delta e^i) \leq D_i^- v(b) \leq \\ \varepsilon^{-1} \int_0^\varepsilon D_i^- v(b + x e^i) dx &= \varepsilon^{-1}(v(b + \varepsilon e^i) - v(b)), \end{aligned}$$

where the inequalities follow by (M2).

b) $a_i + \delta \in (b_i, b_i + \varepsilon]$. Then

$$\begin{aligned} \delta^{-1}(v(a + \delta e^i) - v(a)) &= \delta^{-1}(v(c) - v(a)) + \delta^{-1}(v(a + \delta e^i) - v(c)) \leq \\ \delta^{-1}(c_i - a_i)(a_i + \delta - c_i)^{-1}(v(a + \delta e^i) - v(c)) &+ \delta^{-1}(v(a + \delta e^i) - v(c)) = \\ (a_i + \delta - c_i)^{-1}(v(a + \delta e^i) - v(c)), \end{aligned}$$

where the inequality follows by Lemma 3.1, with $s = a$, $\varepsilon_1 = \varepsilon_2 = c_i - a_i < \delta = \varepsilon_3$.

Thus we have

$$\delta^{-1}(v(a + \delta e^i) - v(a)) \leq (a_i + \delta - c_i)^{-1}(v(a + \delta e^i) - v(c)). \quad (3.2)$$

Similarly, by applying Lemma 3.1 with $s = b$, $\varepsilon_1 = \varepsilon_2 = d_i - b_i < \varepsilon = \varepsilon_3$, it follows that

$$(d_i - b_i)^{-1}(v(d) - v(b)) \leq \varepsilon^{-1}(v(b + \varepsilon e^i) - v(b)). \quad (3.3)$$

Further, using (M2) with $(a_{-i}, t) = (a_1, \dots, a_{i-1}, t, a_{i+1}, \dots, a_n)$ and $(b_{-i}, t) = (b_1, \dots, b_{i-1}, t, b_{i+1}, \dots, b_n)$ in the role of s^1 and s^2 , respectively, and the equalities $a_i + \delta = d_i$ and $c_i = b_i$ we obtain

$$v(a + \delta e^i) - v(c) = \int_{c_i}^{a_i + \delta} D_i^- v(a_{-i}, t) dt \leq \int_{b_i}^{d_i} D_i^- v(b_{-i}, t) dt \leq v(d) - v(b).$$

Now, since $a_i + \delta - c_i = d_i - b_i$, we have

$$(a_i + \delta - c_i)^{-1}(v(a + \delta e^i) - v(c)) \leq (d_i - b_i)^{-1}(v(d) - v(b)) \quad (3.4)$$

Combining (3.2) and (3.3) via (3.4) (by using the transitivity property of the inequality relation), (3.1) follows.

(iii) $\rightarrow$ (iv): Assume that v has the IAMR-property. Take $a, b \in [0, 1]^N$ with $a \leq b$ and $c, d \in [a, b]$ with $c + d = a + b$. Define $h = b - c$. Then $b = c + h$ and $d = a + h$. We have

$$v(b) - v(c) = \sum_{r=1}^n (v(c + \sum_{i=1}^r h_i e^i) - v(c + \sum_{i=1}^{r-1} h_i e^i)) \geq \sum_{r=1}^n (v(a + \sum_{i=1}^r h_i e^i) - v(a + \sum_{i=1}^{r-1} h_i e^i)) = f(d) - f(a),$$

where the inequality follows by applying n times the IAMR-property.

(iv) $\rightarrow$ (i): Let v be a directionally convex function on $[0, 1]^N$. To prove the coordinate-wise convexity of v , note that for $s^{-i} \in [0, 1]^{N \setminus \{i\}}$ and $0 \leq p < \frac{1}{2}(p + q) < q \leq 1$, we have $(s^{-i} \| \frac{1}{2}(p + q)) \in [(s^{-i} \| p), (s^{-i} \| q)]$. So, (iv) with $a = (s^{-i} \| p)$, $b = (s^{-i} \| q)$, $c = d = (s^{-i} \| \frac{1}{2}(p + q))$ implies $v(s^{-i} \| p) + v(s^{-i} \| q) \geq 2v(s^{-i} \| \frac{1}{2}(p + q))$.

To prove the supermodularity property of v , let $c, d \in [0, 1]^N$. Then $c, d \in [c \wedge d, c \vee d]$. By (iv) one obtains

$$v(c \vee d) + v(c \wedge d) \geq v(c) + v(d).$$

□

Now we consider for $v \in C^2$ the property of non-negativity of second partial derivatives (NNSPAD)

$$(v) \quad \frac{\partial^2 v}{\partial s_i \partial s_j} \geq 0 \text{ for all } i, j \in N \text{ on } [0, 1]^N.$$

For C^2 -functions the equivalence (ii) $\Leftrightarrow$ (v), i.e. the equivalence of MOPAD and NNSPAD, is obvious.

Remark 3.1. The equivalence (i) $\Leftrightarrow$ (iii) is stated by Theorem 6 in the work of Brânzei et al. (2003).

That (iv) $\Rightarrow$ (v) if v is a C^2 -function is showed in Shaked and Shathikumar (1990).

Obvious consequences of Theorem 3.1 are:

- C1. Let $v \in FG^N$. Then $v \in CFG^N$ iff one of the properties in Theorem 3.1 holds.
- C2. When $v \in C^2$ the NNSPAD-property provides an easy criterium to check the convexity of the fuzzy game.
- C3. For each $r \in \{1, \dots, m\}$, let μ_r be defined for each $s \in [0, 1]^N$ by $\mu_r(s) = \sum_{i \in N} s_i \mu_r(\{i\})$, with $\mu_r(\{i\}) \geq 0$ for each $i \in N$ and $\sum_{i \in N} \mu_r(\{i\}) \leq 1$. If $f \in C^1$ is a directionally convex function, then v with $v(s) = f(\mu_1(s), \dots, \mu_m(s))$ for each $s \in [0, 1]^N$ is a convex game (cf. Martínez-de-Albéniz and Rafels (2002)).

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