Tilburg University

The core and related solution concepts for infinite assignment games

Llorca, N.; Sanchez, S.M.; Tijs, S.H.; Timmer, J.B.

Published in:
Top

Publication date:
2004

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Llorca, N., Sanchez, S. M., Tijs, S. H., & Timmer, J. B. (2004). The core and related solution concepts for infinite assignment games. *Top*, 12, 331-350.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

The Core and Related Solution Concepts for Infinite Assignment Games

Natividad Llorca¹, Joaquín Sánchez-Soriano¹

*CIO and Department of Statistics and Applied Mathematics, Miguel Hernández University
Elche Campus, Avda. de la Universidad, s/n, 03202 Elche, Spain
E-mail: nllorca@umh.es joaquin@umh.es*

Stef Tijs

*CentER and Department of Econometrics and Operations Research,
Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
E-mail: s.h.tijs@uvt.nl*

Judith Timmer

*Department of Applied Mathematics, University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
E-mail: j.b.timmer@ewi.utwente.nl*

Abstract

Assignment problems where both sets of agents that have to be matched are countably infinite, the so-called infinite assignment problems, are studied as well as the related cooperative assignment games. Further, several solution concepts for these assignment games are studied. The first one is the utopia payoff for games with an infinite value. In this solution each player receives the maximal amount he can think of with respect to the underlying assignment problem. This solution is contained in the core of the game.

Second, we study two solutions for assignment games with a finite value. Our main result is the existence of core-elements of these games, although they are hard to calculate. Therefore another solution, the f -strong ε -core is studied. This particular solution takes into account that due to organisational limitations it seems reasonable that only finite groups of agents will eventually protest against unfair proposals of profit distributions. The f -strong ε -core is shown to be nonempty.

Key Words: Cooperative games, assignment problems, infinite programs, solutions.

AMS subject classification: 91A12.

¹ These authors' research is partially supported by the Generalitat Valenciana (Grant number GV-CTIDIA-2002-32) and by the Government of Spain (through a joint research grant Universidad Miguel Hernández – Università degli Studi di Genova HI2002-0032).

We thank two anonymous referees and an associate editor for their helpful comments and suggestions.

Manuscript received: January 2002. Final version accepted: June 2004.

1 Introduction

An assignment problem is a problem in which agents of two types have to be matched (or assigned) to each other. Assigning agent i from one type to agent j from the other type generates a nonnegative profit a_{ij} . The goal is to match the agents in such a way that the total profit is as large as possible. Examples of assignment problems in real-life are the marriage market where men and women are matched, and a factory where workers have to be assigned to tasks or machines.

Shapley and Shubik (1972) studied this assignment problem in a game-theoretic setting for finite sets of agents and introduced cooperative assignment games. In these games the value of a coalition of players equals the maximal profit that can be obtained from matching agents within this coalition. Assignment problems are one of the classical problems in Operations Research. In the last decade many of these problems have been studied in a cooperative setting with the help of game theory. A survey on such operations research games can be found in Borm et al. (2001).

Sánchez-Soriano et al. (2001) and Llorca et al. (2003) considered semi-infinite assignment problems and their related cooperative games, following the study in Fragnelli et al. (1999). These are assignment problems in which one set of agents is countably infinite while the second one is finite. Hence, they are an extension of the finite assignment problems as studied by Shapley et al. (1972).

Infinite assignment problems are an extension of semi-infinite assignment problems since there are countably infinitely many agents of both types. These problems arise, for example, as approximations of finite assignment problems with a huge number of agents of both types. In case of this very large number of agents, it is impossible to coordinate all of them. Then an approximation by means of an infinite assignment problem may be appropriate. This way, insights for infinite assignment problems can be used in finite problems with large numbers of agents.

This paper studies infinite assignment problems and corresponding cooperative games. The main question we address is whether there exist allocations of the total profit that all (groups of) agents find “reasonable”. For this end, two cases are considered.

First, if the optimal total profit is infinite then there are many candi-

dates for nice allocations. We propose a particular allocation, the so-called *utopia payoff* as a solution of the game. This solution has the nice property that it fulfills the ‘wishes’ of all the players. Moreover, this solution is also an element of the core of the game.

Second, if the optimal total profit is finite then it is hard to analyse the assignment game. Nevertheless, it is shown that the assignment problem has no duality gap and that the *core* is nonempty, although elements in this set are not easy to calculate in general. Therefore, the *f-strong ε -core* is introduced. This is a set of allocations of (almost) the total profit that all *finite* groups of agents find reasonable since on their own they can improve by at most a small amount ε . It turns out that the f-strong ε -core is also nonempty.

The outline of this paper is as follows. In the next section infinite assignment problems and their corresponding assignment games are introduced. The utopia payoff is introduced in section 3 as a solution for assignment games with an infinite value. Section 4 studies the core for assignment games with a finite value and the f-strong ε -core is studied in section 5. Finally, section 6 concludes.

2 Infinite assignment problems and games

An assignment problem describes a situation in which two types of agents have to be matched. In case of an infinite assignment problem each type of agents is available in a countable, infinite amount. (Infinite assignment problems are an extension of semi-infinite assignment problems in which there is a finite amount available of one type.) Matching agent i from one type to agent j of the other type generates a profit $a_{ij} \geq 0$. The goal is to match the agents in such a way that the total profit is as large as possible.

An infinite assignment problem $\mathcal{A}$ is modelled by a tuple $(M, W, \mathbf{A})$. M and W represent the two countably infinite sets of agents and $\mathbf{A}$ is an $M \times W$ -matrix containing the nonnegative profit a_{ij} in cell (i, j) . For ease of notation we assume that both M and W are represented by $\mathbb{N} = \{1, 2, 3, \dots\}$. Let $\bar{\mathbb{R}}_+ = [0, +\infty]$.

The largest total profit or, more precise, the smallest upper bound of the total profit that can be achieved from assigning agents in M to agents

in W equals

$$\begin{aligned}
 v_p(\mathcal{A}) = \sup_X \quad & \sum_{i \in M} \sum_{j \in W} a_{ij} x_{ij} \\
 \text{s.t.} \quad & \sum_{i \in M} x_{ij} \leq 1, \\
 & \sum_{j \in W} x_{ij} \leq 1, \\
 & x_{ij} \in \{0, 1\} \text{ for all } i \in M, j \in W.
 \end{aligned} \tag{2.1}$$

Here, $X = [x_{ij}]_{i \in M, j \in W}$ is an assignment matrix with $x_{ij} = 1$ if and only if agent $i \in M$ is assigned to agent $j \in W$. The first two conditions of the optimization program in (2.1) ensure that each agent is assigned to at most one other agent. Because of the infinite character of this problem the supremum over all matchings is taken instead of the maximum, although this is not easily done.

An assignment is a map $\pi : M \rightarrow W$ that assigns each agent in M to at most one agent in W and ensures that an agent in W is assigned to at most one agent in M . For each assignment matrix X there is a corresponding assignment π_X with $\pi_X(i) = j$ if $x_{ij} = 1$. Conversely, for each assignment π there is a corresponding assignment matrix X with $x_{ij} = 1$ if $\pi(i) = j$ and $x_{ij} = 0$ otherwise. An assignment π for $\mathcal{A}$ is called an optimal assignment if it attains the value of $\mathcal{A}$, $\sum_{i \in M} a_{i\pi(i)} = v_p(\mathcal{A})$. Note that optimal assignments need not exist, as the example below shows.

Example 2.1. Consider the infinite assignment problem $\mathcal{A} = (M, W, \mathbf{A})$ with

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The value of this problem is $v_p(\mathcal{A}) = 2$ and there exists no optimal assignment. This is due to agent 1 in M because for no matching $\pi(1) \in W$ the value $v_p(\mathcal{A})$ is achieved, $\sum_{i \in M} a_{i\pi(i)} < v_p(\mathcal{A})$.

According to theorem 4.3 in Cross et al. (1998) the condition $x_{ij} \in \{0, 1\}$ in (2.1) may be replaced by the nonnegativity condition $x_{ij} \geq 0$. After

replacement, the program

$$\begin{aligned} \inf \quad & \sum_{i \in M} u_i + \sum_{j \in W} v_j \\ \text{s.t.} \quad & u_i + v_j \geq a_{ij} \\ & u_i, v_j \geq 0, \text{ for all } i \in M, j \in W \end{aligned}$$

with value $v_d(\mathcal{A})$, is the dual program corresponding to the so-called primal program in (2.1). This dual program has feasible solutions, which are solutions that satisfy all the constraints. An example of such a feasible solution is (u^*, v^*) where $u_i^* = \sup_{j \in W} a_{ij}$ and $v_j^* = \sup_{i \in M} a_{ij}$. Since the original primal problem also has feasible solutions (for example $x_{ij} = 0$ for all i and j), the inequality $v_p(\mathcal{A}) \leq v_d(\mathcal{A})$ holds according to theorem 3.1 in Anderson and Nash (1987). This property is called weak duality. If $v_p(\mathcal{A}) < v_d(\mathcal{A})$ then the difference $v_d(\mathcal{A}) - v_p(\mathcal{A})$ is called the duality gap.

Notice that if $v_p(\mathcal{A}) < +\infty$ then $u_i^* < +\infty$ for all i and $v_j^* < +\infty$ for all j . Hence, the condition $v_p(\mathcal{A}) < +\infty$ is a sufficient but not necessary condition for obtaining finite values for u_i^* and v_j^* . This shows a difference with semi-infinite assignment problems $(M, W, \mathbf{A})$, in which M is finite, because then it is both a sufficient and a necessary condition. In fact, for semi-infinite assignment problems the three properties

- (i) $\sum_{i \in M} u_i^* < +\infty$ or $\sum_{j \in W} v_j^* < +\infty$,
- (ii) $v_p(\mathcal{A}) < +\infty$,
- (iii) $u_i^* < +\infty$ for all i and $v_j^* < +\infty$ for all j ,

are equivalent. On the contrary, for infinite assignment problems (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) while the inverse implications are not true. These differences indicate that analysing infinite assignment problems is more difficult than analysing semi-infinite assignment problems as was done in Llorca et al. (2003) and Sánchez-Soriano et al. (2001).

The value $v_p(\mathcal{A})$ of $\mathcal{A}$ can be infinite or finite. A necessary condition for a finite value is that the matrix $\mathbf{A}$ is bounded. Of course, if $\sum_{i \in M} u_i^* < +\infty$ or $\sum_{j \in W} v_j^* < +\infty$ then also $v_p(\mathcal{A}) < +\infty$. This sufficient condition is not necessary as example 2.2 below shows. In this example finite subproblems of an infinite assignment problem are mentioned. Given an assignment problem $\mathcal{A}$ the (finite) subproblem $\mathcal{A}_n$ is defined as the restriction of $\mathcal{A}$ to the first n agents of both types: $\mathcal{A}_n = (M_n, W_n, \mathbf{A}_n)$ where $M_n = \{1, \dots, n\} = W_n$ and $\mathbf{A}_n = [a_{ij}]_{i \in M_n, j \in W_n}$.

Example 2.2. Let $\mathcal{A}$ be an infinite assignment problem $(M, W, \mathbf{A})$ with $a_{ij} = 2^{1-\min\{i,j\}}$, that is,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \cdots \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Here $u_i^* = 1$ for all i and $v_j^* = 1$ for all j so $\sum_{i \in M} u_i^* = +\infty$ and $\sum_{j \in W} v_j^* = +\infty$. Nevertheless, $v_p(\mathcal{A})$ is finite. This can be seen as follows. For all $k \in \mathbb{N}$ the value of the subproblem $\mathcal{A}_{2k}$ equals $v_p(\mathcal{A}_{2k}) = 4(1 - 2^{-k})$, where the optimal assignment $\pi^{2k}(i) = 2k + 1 - i$ for $\mathcal{A}_{2k}$ is used. The value $v_p(\mathcal{A}_{2k})$ tends to $v_p(\mathcal{A}) = 4$ as k goes to infinity.

Furthermore, the optimal dual value equals $v_d(\mathcal{A}) = 4 = v_p(\mathcal{A})$. Hence, there is no duality gap here. The optimal dual value is obtained via the optimal solution (u, v) with $u = v = (1, \frac{1}{2}, \frac{1}{4}, \dots)$. Notice that $u_i + v_j > a_{ij}$ for all i and j , while for (semi-in)finite problems $u_i + v_j = a_{ij}$ if the agents i and j are matched. Hence, infinite assignment problems are more difficult to analyse than semi-infinite problems.

Given an infinite assignment problem $\mathcal{A} = (M, W, \mathbf{A})$ we define a corresponding game, the assignment game, so that we are able to study the benefits of cooperation between the agents. This game is a cooperative game (N, w) with transferable utility. The set N is the player set and w is a function that assigns to each coalition S the payoff $w(S)$ and, by convention, $w(\emptyset) = 0$. The assignment game has an infinite player set $N = M \cup W$, where any player corresponds to either an agent in M or in W and vice versa, any agent in M or in W corresponds to a player. A coalition S of players is a nonempty subset of N . Let $M_S = M \cap S$ and $W_S = W \cap S$. Then a coalition S receives $w(S) = 0$ if $S = M_S$ or $S = W_S$ because in these cases S consists of agents of a single type and there are no agents from the other type to be assigned to. Otherwise, $w(S) = v_p(\mathcal{A}_S)$ where $\mathcal{A}_S$ is the subproblem $(M_S, W_S, [a_{ij}]_{i \in M_S, j \in W_S})$.

An important question is how the total value $w(N) = v_p(\mathcal{A})$ should be distributed among the players. In the sections hereafter we study solutions for games with $w(N) = +\infty$ and with a finite value $w(N)$, respectively.

3 The utopia payoff

In this section we consider assignment games (N, w) with the value $w(N) = +\infty$ for the total player set. Due to this infinite value there should be no problem in distributing $w(N)$ among the players since there is enough for all of them. Any player can receive the largest amount he can think of.

A natural distribution of $w(N)$ is the solution in which a player receives $u_i^* = \sup_{j \in W} a_{ij}$ if he corresponds to an agent $i \in M$ and $v_j^* = \sup_{i \in M} a_{ij}$ if he corresponds to an agent $j \in W$. One may think of the numbers u_i^* and v_j^* as utopia values of the players. Notice that such a utopia value may equal $+\infty$. Putting these values together results in the utopia payoff (u^*, v^*) . This payoff is a solution that is particularly suitable for assignment games with $w(N) = +\infty$ because for these games the utopia payoff is always feasible, which means that the total amount distributed is at most equal to the amount $w(N)$ available for distribution.

Theorem 3.1. *The utopia payoff (u^*, v^*) for assignment games with $w(N) = +\infty$ satisfies $\sum_{i \in M} u_i^* + \sum_{j \in W} v_j^* = w(N)$. Hence, it is a feasible solution.*

Proof. Let $\mathcal{A}$ be an assignment problem with value $v_p(\mathcal{A}) = +\infty$ and with corresponding assignment game (N, w) . Applying the weak-duality property $v_p(\mathcal{A}) \leq v_d(\mathcal{A})$ results in

$$v_d(\mathcal{A}) = v_p(\mathcal{A}) = +\infty \quad (3.1)$$

Further,

$$\sum_{i \in M} u_i^* = \sum_{i \in M} \sup_{j \in W} a_{ij} \geq v_p(\mathcal{A}),$$

and therefore $\sum_{i \in M} u_i^* = +\infty$. Similarly it can be shown that $\sum_{j \in W} v_j^* = +\infty$. We conclude that $\sum_{i \in M} u_i^* + \sum_{j \in W} v_j^* = +\infty = w(N)$. $\square$

The value $w(N) = +\infty$ does not indicate whether the individual utopia values u_i^* and v_j^* are finite or infinite. Both may occur as is shown in the example below.

Example 3.1. Let $\mathcal{A}^b = (M, W, \mathbf{A}^b)$ for $b = 1, 2$ be an infinite assignment

problem where

$$\mathbf{A}^1 = \begin{bmatrix} 1 & 1 & 1 & \dots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } \mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 3 & \dots \\ 2 & 3 & 4 & \dots \\ 3 & 4 & 5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Denote by (N, w^b) the assignment game corresponding to $\mathcal{A}^b$. Then $w^1(N) = +\infty$ (take $\pi(i) = i$ for all i), while the utopia values are finite: $u_i^* = 1/i$ and $v_j^* = 1$ for all i and j . For the second problem $w^2(N) = +\infty$ and $u_i^* = v_j^* = +\infty$ for all i and j . Hence, $w(N) = +\infty$ does not tell whether the utopia values are finite or infinite.

The utopia payoff of an assignment game with $w(N) = +\infty$ is not only feasible but it is also an element of the set

$$C(w) = \left\{ (u, v) \in \bar{\mathbb{R}}_+^M \times \bar{\mathbb{R}}_+^W \left| \begin{array}{l} \sum_{i \in M} u_i + \sum_{j \in W} v_j = w(N), \\ \sum_{i \in M_S} u_i + \sum_{j \in W_S} v_j \geq w(S) \text{ for all } S \subset N \end{array} \right. \right\},$$

which is the *core* of the corresponding assignment game (N, w) . An element of the core describes a distribution of the total value $w(N)$ among the players that is efficient, since it distributes exactly $w(N)$, and stable, because any coalition S cannot do better on its own.

Theorem 3.2. *The utopia payoff (u^*, v^*) belongs to the core $C(w)$ of the assignment game (N, w) if $w(N) = +\infty$.*

Proof. Efficiency of the utopia payoff (u^*, v^*) is shown in theorem 3.1. Thus, we only have to show stability. Let S be a coalition. Then the restriction of (u^*, v^*) to coalition S is a feasible solution of the program

$$\begin{array}{ll} \inf & \sum_{i \in M_S} u_i + \sum_{j \in W_S} v_j \\ \text{s.t.} & u_i + v_j \geq a_{ij} \\ & u_i, v_j \geq 0, \text{ for all } i \in M_S, j \in W_S \end{array}$$

with value $v_d(\mathcal{A}_S)$. Thus $\sum_{i \in M_S} u_i^* + \sum_{j \in W_S} v_j^* \geq v_d(\mathcal{A}_S) \geq v_p(\mathcal{A}_S) = w(S)$. We conclude that the utopia payoff is an element of the core $C(w)$. $\square$

If (N, w) is an assignment game with $w(N) = +\infty$ then any player can receive any amount that he likes from $w(N)$ because there is enough. Thus it is not really surprising that the utopia payoff is an element of the core of the assignment game. In fact, any optimal dual solution (u, v) is an element of the core, and vice versa.

Theorem 3.3. $(u, v) \in C(w) \Leftrightarrow (u, v)$ is an optimal dual solution of the corresponding infinite assignment problem $\mathcal{A}$, for any assignment game with $w(N) = +\infty$.

Proof. First, notice that $w(N) = +\infty$ and weak duality imply $v_p(\mathcal{A}) = v_d(\mathcal{A}) = +\infty$. Now, let $(u, v) \in C(w)$. Then u and v are nonnegative. Let $i \in M$ and $j \in W$ and let $S = \{i, j\}$ be a coalition. According to stability

$$u_i + v_j \geq w(S) = v_p(\mathcal{A}_S) = a_{ij}.$$

Thus (u, v) is a feasible dual solution. By efficiency, $\sum_{i \in M} u_i + \sum_{j \in W} v_j = +\infty = v_d(\mathcal{A})$. Hence, (u, v) is an optimal dual solution.

Next, let (u, v) be an optimal dual solution. Then u and v are nonnegative. Besides, $v_d(\mathcal{A}) = w(N)$ implies efficiency. Let S be a coalition. Then, by using weak duality,

$$\sum_{i \in M_S} u_i + \sum_{j \in W_S} v_j \geq v_d(\mathcal{A}_S) \geq v_p(\mathcal{A}_S) = w(S).$$

We conclude that $(u, v) \in C(w)$. □

Hence, there are many core-elements. The utopia payoff is a special core-element because it has a nice interpretation. Nevertheless, one may have objections against individual (utopia) values that equal $+\infty$. One can overcome this if the amount $+\infty$ is replaced by a certain finite amount, say k . This k is such that the players are very content if they receive k and they consider it to be “incredibly” large. Define the k -bounded utopia payoff $(u^*(k), v^*(k))$ by $u_i^*(k) = \min\{u_i^*, k\}$ and $v_j^*(k) = \min\{v_j^*, k\}$ for all $i \in M$ and $j \in W$. We present the following result without proof.

Theorem 3.4. The k -bounded utopia $(u^*(k), v^*(k))$ is the utopia payoff for the game (N, w^k) corresponding to the assignment problem $\mathcal{A}^k = (M, W, \mathbf{A}^k)$ where $\mathbf{A}^k = [a_{ij}(k)]_{ij \in M \times W}$ and $a_{ij}(k) = \min\{a_{ij}, k\}$. Fur-

thermore, $\sum_{i \in M_S} u_i^*(k) + \sum_{j \in W_S} v_j^*(k) \geq w^k(S)$ for all coalitions S and if $w^k(N) = +\infty$ then $(u^*(k), v^*(k)) \in C(w^k)$.

4 The core

From now on we only consider infinite assignment games with a finite value, $w(N) < +\infty$. For these games it is very likely that the utopia payoff is not a feasible solution, that is, the total amount that is distributed among the players is larger than $w(N)$. In example 2.2 the utopia values are $u_i^* = v_j^* = 1$ for all i and j . Thus $\sum_{i \in M} u_i^* + \sum_{j \in W} v_j^* = +\infty > v_p(\mathcal{A}) = w(N) = 4$, which shows that the utopia payoff (u^*, v^*) is not feasible.

Therefore, another solution should be used for assignment games with $w(N) < +\infty$. One possible solution for such games is the core $C(w)$. The definition of the core of an infinite assignment game is given in section 3. If an element of the core is proposed as a distribution of the total value $w(N)$ then no coalition has an incentive to object because they all receive as much as they can obtain on their own.

The theorem below is the main result of this paper. It shows that there exist core-elements for infinite assignment games with a finite value and that the corresponding assignment problem has no duality gap.

Theorem 4.1. *Assignment problems with a finite value $v_p(\mathcal{A})$ have no duality gap, $v_p(\mathcal{A}) = v_d(\mathcal{A})$, and the core $C(w)$ of the corresponding game (N, w) is nonempty.*

Proof. Let $\mathcal{A} = (M, W, \mathbf{A})$ be an infinite assignment problem with finite value. Since $v_p(\mathcal{A}) < +\infty$ we know that $u_i^* < +\infty$ for all $i \in M$ and $v_j^* < +\infty$ for all $j \in W$. Define

$$C_{\mathcal{A}} = \prod_{i \in M} [0, u_i^*] \times \prod_{j \in W} [0, v_j^*].$$

This set is compact with respect to the product topology according to the Tychonoff theorem (see for example theorem 2.57 in Aliprantis and Border (1999)). On the other hand, $\mathbb{R}$ is a complete metric space and therefore $\mathbb{R}^{\mathbb{N}} (= \mathbb{R}^M = \mathbb{R}^W)$ is also a complete metric space (see page 28 in Köthe (1983)). In complete metric spaces a sequence in a compact set has

a convergent subsequence in the topological product sense, which means pointwise convergence.

Now consider the related infinite assignment problems $\mathcal{A}_n^0 = (M, W, \mathbf{A}_n^0)$ for $n = 1, 2, \dots$ with $\mathbf{A}_n^0 = [a_{ij}^n]_{i \in M, j \in W}$ where

$$a_{ij}^n = \begin{cases} a_{ij} & \text{if } i \leq n \text{ and } j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

The dual problem of $\mathcal{A}_n^0$ is

$$\begin{aligned} \inf \quad & \sum_{i \in M} u_i + \sum_{j \in W} v_j \\ \text{s.t.} \quad & u_i + v_j \geq a_{ij}^n, \\ & u_i, v_j \geq 0, \text{ for all } i \in M, j \in W. \end{aligned}$$

These problems satisfy $v_d(\mathcal{A}_n^0) = v_d(\mathcal{A}_n)$ for all n , the dual values of the infinite assignment problem $\mathcal{A}_n^0$ and the finite assignment problem $\mathcal{A}_n$ are equal. Because $\mathcal{A}_n$ is a finite assignment problem it has an optimal dual solution, let's say (u^n, v^n) . Let $(\bar{u}^n, \bar{v}^n) \in \mathbb{R}^M \times \mathbb{R}^W$ be such that

$$\bar{u}_i^n = \begin{cases} u_i^n & \text{if } i \leq n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{v}_j^n = \begin{cases} v_j^n & \text{if } j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\bar{u}^n, \bar{v}^n)$ is an optimal dual solution for $\mathcal{A}_n^0$ since it is a feasible dual solution and

$$\sum_{i \in M} \bar{u}_i^n + \sum_{j \in W} \bar{v}_j^n = \sum_{i=1}^n u_i^n + \sum_{j=1}^n v_j^n = v_d(\mathcal{A}_n) = v_d(\mathcal{A}_n^0).$$

The sequence $\{(\bar{u}^n, \bar{v}^n)\}_{n \in \mathbb{N}}$ lies in $C_{\mathcal{A}}$, which is a compact set — with respect to the product topology — in the complete metric space $\mathbb{R}^M \times \mathbb{R}^W$. Therefore, there exists a subsequence $\{(\bar{u}^{n_k}, \bar{v}^{n_k})\}_{k \in \mathbb{N}}$ that converges to, say, $(\bar{u}, \bar{v}) \in C_{\mathcal{A}}$.

Suppose that this limit $(\bar{u}, \bar{v})$ is not a feasible dual solution for $\mathcal{A}$. Then there exist $l \in M$ and $m \in W$ such that $\bar{u}_l + \bar{v}_m < a_{lm}$. Take $\bar{n}_k > \max\{l, m\}$ then $\bar{u}_l^{n_k} + \bar{v}_m^{n_k} \geq a_{lm}$ for all $n_k > \bar{n}_k$. Thus $\bar{u}_l + \bar{v}_m \geq a_{lm}$. Contradiction. Hence, $(\bar{u}, \bar{v})$ is a feasible dual solution for $\mathcal{A}$. This implies

$$\sum_{i \in M} \bar{u}_i + \sum_{j \in W} \bar{v}_j \geq v_d(\mathcal{A}) \geq v_p(\mathcal{A}) \quad (4.1)$$

where the last inequality follows from weak duality. By lemma 5.1 and the finiteness of $\mathcal{A}_{n_k}$ we deduce

$$v_p(\mathcal{A}) = \lim_{k \rightarrow \infty} v_p(\mathcal{A}_{n_k}) = \lim_{k \rightarrow \infty} v_d(\mathcal{A}_{n_k}). \quad (4.2)$$

Also, using the definitions of $\mathcal{A}_{n_k}^0$ and $(\bar{u}, \bar{v})$ we obtain

$$\lim_{k \rightarrow \infty} v_d(\mathcal{A}_{n_k}) = \lim_{k \rightarrow \infty} v_d(\mathcal{A}_{n_k}^0) = \sum_{i \in M} \bar{u}_i + \sum_{j \in W} \bar{v}_j. \quad (4.3)$$

Combining (4.1), (4.2) and (4.3) results in

$$\sum_{i \in M} \bar{u}_i + \sum_{j \in W} \bar{v}_j = v_d(\mathcal{A}) = v_p(\mathcal{A}),$$

which states that $(\bar{u}, \bar{v})$ is an optimal dual solution for the infinite assignment problem $\mathcal{A}$ and that $\mathcal{A}$ has no duality gap. Finally, one can show along the same lines as the proof of theorem 3.3 that $(\bar{u}, \bar{v}) \in C(w)$ if and only if $(\bar{u}, \bar{v})$ is an optimal dual solution for $\mathcal{A}$. $\square$

This proof shows that it is difficult to find a core-element. Therefore, another solution, the so-called f-strong ε -core, for infinite assignment problems is considered.

5 The f-strong ε -core

In the previous section it turned out that elements of the core are hard to calculate. Hence, a modification of the core, the so-called f-strong ε -core $F_\varepsilon(w)$, is introduced as an alternative and easier-to-compute solution for assignment games with a finite value $w(N)$. It is shown that $F_\varepsilon(w)$ is also a nonempty set.

The f-strong ε -core is based upon the strong ε -core as introduced by Shapley and Shubik (1966) for games with a finite number of players, and also on the ideas behind the f-core by Kaneko and Wooders (1986). The strong ε -core is the set of efficient allocations such that no coalition can improve if a fixed cost (or tax) of size $\varepsilon > 0$ is imposed. Unfortunately, efficient allocations make no sense in infinite assignment games because optimal assignments need not exist in the corresponding assignment problem.

Therefore, we consider ε -efficiency of an allocation $z \in \mathbb{R}_+^N$,

$$w(N) \geq \sum_{i \in N} z_i \geq w(N) - \varepsilon, \quad (5.1)$$

as well as ε -stability for finite coalitions S ,

$$\sum_{i \in S} z_i \geq w(S) - \varepsilon. \quad (5.2)$$

Now define the f-strong ε -core $F_\varepsilon(w)$ of an assignment game (N, w) to be the set of allocations $z \in \mathbb{R}_+^N$ that satisfy both the constraints (5.1) and (5.2). The 'f' in the name refers to the stability condition (5.2) for *finite* coalitions only (as in Kaneko and Wooders (1986)). That condition is based upon the idea that coordination of agents is expensive (in terms of money and time) and that in real-life it seems impossible to coordinate a group (different from the grand coalition N) of infinite size. Besides, the restriction of ε -stability to finite coalitions is not so restrictive as it seems, because the only relevant coalitions in the game are of size two. This is due to the matchings among the players.

Our analysis of the f-strong ε -core $F_\varepsilon(w)$ starts with finite subproblems $\mathcal{A}_n = (M_n, W_n, \mathbf{A}_n)$ and its relation to the original problem $\mathcal{A}$. Recall that in example 2.2 an infinite assignment problem is shown where the value of the finite subproblems converges to the value of the original problem: $\lim_{n \rightarrow \infty} v_p(\mathcal{A}_n) = v_p(\mathcal{A}) < +\infty$. This relation holds in general.

Lemma 5.1. $\lim_{n \rightarrow \infty} v_p(\mathcal{A}_n) = v_p(\mathcal{A})$ for all infinite assignment problems $\mathcal{A}$.

Proof. It is easy to check that the sequence $\{v_p(\mathcal{A}_n)\}_{n \in \mathbb{N}}$ is nondecreasing and that $v_p(\mathcal{A}_n) \leq v_p(\mathcal{A})$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then there exists an assignment π for $\mathcal{A}$ such that

$$\sum_{i \in M} a_{i\pi(i)} \geq v_p(\mathcal{A}) - \varepsilon/2.$$

Further, there exists an $l \in \mathbb{N}$ such that

$$\sum_{i=1}^l a_{i\pi(i)} \geq v_p(\mathcal{A}) - \varepsilon.$$

Take $m = \max\{\pi(1), \dots, \pi(l)\}$, then for all $n \geq m$

$$v_p(\mathcal{A}_n) \geq \sum_{i=1}^m a_{i\pi(i)} \geq v_p(\mathcal{A}) - \varepsilon.$$

We conclude that $\lim_{n \rightarrow \infty} v_p(\mathcal{A}_n) = v_p(\mathcal{A})$. $\square$

For finite assignment problems like $\mathcal{A}_n$ it has been shown, in e.g. Birkhoff (1946), that if the integer condition $x_{ij} \in \{0, 1\}$ in the primal program is replaced by nonnegativity, $x_{ij} \geq 0$, then there still exists an optimal solution with integer values for the x_{ij} . After substituting $x_{ij} \geq 0$ it is easy to see that the dual problem for $\mathcal{A}_n$ is

$$\begin{aligned} \min \quad & \sum_{i \in M_n} u_i + \sum_{j \in W_n} v_j \\ \text{s.t.} \quad & u_i + v_j \geq a_{ij}, \\ & u_i, v_j \geq 0, \text{ for all } i \in M_n, j \in W_n, \end{aligned}$$

with optimal value $v_d(\mathcal{A}_n)$. This value is equal to the optimal primal value, $v_p(\mathcal{A}_n) = v_d(\mathcal{A}_n)$. Furthermore, there exists an optimal primal and dual solution for $\mathcal{A}_n$. In the sequel we investigate whether the f-strong ε -core is nonempty by using these properties of finite subproblems.

Lemma 5.1 implies that for a given $\varepsilon > 0$ there exists an integer $m(\varepsilon)$ such that

$$v_p(\mathcal{A}) \geq v_p(\mathcal{A}_n) \geq v_p(\mathcal{A}) - \varepsilon \quad (5.3)$$

for all $n \geq m(\varepsilon)$. Let (u^l, v^l) be an optimal dual solution for $\mathcal{A}_{m(\varepsilon)+l}$ for $l = 0, 1, 2, \dots$. Define the shortening map s^l by

$$s^l(u_1, \dots, u_{m(\varepsilon)+l}; v_1, \dots, v_{m(\varepsilon)+l}) = (u_1, \dots, u_{m(\varepsilon)}; v_1, \dots, v_{m(\varepsilon)}).$$

Hence, the map s^l shortens the pair (u^l, v^l) by deleting the last l coordinates of both u and v . Now each element of the sequence $\{s^l(u^l, v^l)\}_{l=0}^\infty$ is a feasible dual solution for $\mathcal{A}_{m(\varepsilon)}$. Notice that this sequence lies in a compact set (recall that a finite value $v_p(\mathcal{A})$ implies that the matrix $\mathbf{A}$ is bounded). Without loss of generality, assume that the limit of this sequence exists (otherwise take a subsequence) and denote it by

$$(\bar{u}, \bar{v}) = \lim_{l \rightarrow \infty} s^l(u^l, v^l). \quad (5.4)$$

Using this limit point as an allocation results in a total amount distributed that is close to the value $v_p(\mathcal{A})$ of $\mathcal{A}$.

Lemma 5.2. $v_p(\mathcal{A}) \geq \sum_{i=1}^{m(\varepsilon)} \bar{u}_i + \sum_{j=1}^{m(\varepsilon)} \bar{v}_j \geq v_p(\mathcal{A}) - \varepsilon$ for all $\varepsilon > 0$.

*Proof.*¹ Recall that the sequence $\{s^l(u^l, v^l)\}_{l=0}^\infty$ lies in a compact set and that $s^l(u^l, v^l)$ is a feasible dual solution for $\mathcal{A}_{m(\varepsilon)}$ for all l . Therefore

$$\sum_{i=1}^{m(\varepsilon)} \bar{u}_i + \sum_{j=1}^{m(\varepsilon)} \bar{v}_j \geq v_d(\mathcal{A}_{m(\varepsilon)}) = v_p(\mathcal{A}_{m(\varepsilon)}) \geq v_p(\mathcal{A}) - \varepsilon,$$

where the last inequality follows from (5.3).

Second, using the fact that (u^l, v^l) is an optimal dual solution of $\mathcal{A}_{m(\varepsilon)+l}$ results in

$$v_p(\mathcal{A}) \geq v_p(\mathcal{A}_{m(\varepsilon)+l}) = v_d(\mathcal{A}_{m(\varepsilon)+l}) = \sum_{i=1}^{m(\varepsilon)+l} u_i^l + \sum_{j=1}^{m(\varepsilon)+l} v_j^l \geq \sum_{i=1}^{m(\varepsilon)} u_i^l + \sum_{j=1}^{m(\varepsilon)} v_j^l$$

for all l . From lemma 5.1 we conclude that $v_p(\mathcal{A}) \geq \lim_{l \rightarrow \infty} (\sum_{i=1}^{m(\varepsilon)} u_i^l + \sum_{j=1}^{m(\varepsilon)} v_j^l)$. $\square$

For $\varepsilon > 0$ let $(\bar{u}, \bar{v})$ be the limit point as defined in (5.4). Let $(\hat{u}(\varepsilon), \hat{v}(\varepsilon)) \in \mathbb{R}^M \times \mathbb{R}^W$ be an allocation defined by

$$\hat{u}(\varepsilon) = (\bar{u}_1, \dots, \bar{u}_{m(\varepsilon)}, 0, 0, \dots) \quad \text{and} \quad \hat{v}(\varepsilon) = (\bar{v}_1, \dots, \bar{v}_{m(\varepsilon)}, 0, 0, \dots).$$

So, $\hat{u}(\varepsilon)$ and $\hat{v}(\varepsilon)$ are obtained from $\bar{u}$ and $\bar{v}$, respectively, by adding an infinite number of zeros, meaning that only the first $m(\varepsilon)$ agents of each type receive a nonzero amount. This allocation belongs to the f -strong ε -core $F_\varepsilon(w)$ of the corresponding assignment game, which shows that $F_\varepsilon(w)$ is nonempty.

Theorem 5.1. *The f -strong ε -core $F_\varepsilon(w)$ is a nonempty set because $(\hat{u}(\varepsilon), \hat{v}(\varepsilon)) \in F_\varepsilon(w)$, for all $\varepsilon > 0$ and all assignment games (N, w) .*

¹ We thank an anonymous referee for providing a shorter proof of the latter part.

Proof. Let $\varepsilon > 0$ and let $m(\varepsilon)$ be as defined for (5.3). Using lemma 5.2 and the equality

$$\sum_{i=1}^{m(\varepsilon)} \bar{u}_i + \sum_{j=1}^{m(\varepsilon)} \bar{v}_j = \sum_{i \in M} \hat{u}_i(\varepsilon) + \sum_{j \in W} \hat{v}_j(\varepsilon)$$

we derive

$$\begin{aligned} w(N) = v_p(\mathcal{A}) &\geq \sum_{i \in M} \hat{u}_i(\varepsilon) + \sum_{j \in W} \hat{v}_j(\varepsilon) \\ &\geq v_p(\mathcal{A}) - \varepsilon = w(N) - \varepsilon. \end{aligned}$$

Hence, $(\hat{u}(\varepsilon), \hat{v}(\varepsilon))$ satisfies ε -efficiency.

To show ε -stability, let S be a finite coalition. If $\sum_{i \in M_S} \hat{u}_i(\varepsilon) + \sum_{j \in W_S} \hat{v}_j(\varepsilon) \geq w(S)$ then we are done since $\varepsilon > 0$. Otherwise,

$$w(S) - \sum_{i \in M_S} \hat{u}_i(\varepsilon) - \sum_{j \in W_S} \hat{v}_j(\varepsilon) > 0. \quad (5.5)$$

Now, let l be an integer such that all agents in S are among the first $m(\varepsilon/2) + l$ of their type,

$$M_S, W_S \subset \{1, 2, \dots, m(\varepsilon/2) + l\},$$

and such that the optimal dual solution (u^l, v^l) of $\mathcal{A}_{m(\varepsilon/2)+l}$ is close to the pair $(\bar{u}, \bar{v})$ on which the solution $(\hat{u}(\varepsilon), \hat{v}(\varepsilon))$ is based,

$$\max_{i,j \leq m(\varepsilon/2)} \left\{ |u_i^l - \bar{u}_i|, |v_j^l - \bar{v}_j| \right\} \leq \frac{\varepsilon}{4m(\varepsilon/2)}, \quad (5.6)$$

where $|x|$ denotes the absolute value of x . Notice that the restriction of (u^l, v^l) to S is a feasible dual solution of $\mathcal{A}_S$,

$$\sum_{i \in M_S} u_i^l + \sum_{j \in W_S} v_j^l \geq v_d(\mathcal{A}_S) = v_p(\mathcal{A}_S) = w(S). \quad (5.7)$$

Let $M_S^{m(\varepsilon/2)} = \{i \in M_S \mid i \leq m(\varepsilon/2)\}$ be the set of those agents in M_S that are among the first $m(\varepsilon/2)$ of their type, and let $W_S^{m(\varepsilon/2)}$ be defined

similarly. Using (5.7), (5.5) and $m(\varepsilon) < m(\varepsilon/2)$ we obtain

$$\begin{aligned}
 w(S) - \sum_{i \in M_S} \hat{u}_i(\varepsilon) - \sum_{j \in W_S} \hat{v}_j(\varepsilon) & \\
 & \leq \sum_{i \in M_S} u_i^l + \sum_{j \in W_S} v_j^l - \sum_{i \in M_S} \hat{u}_i(\varepsilon) - \sum_{j \in W_S} \hat{v}_j(\varepsilon) \\
 & = \left| \sum_{i \in M_S} u_i^l + \sum_{j \in W_S} v_j^l - \sum_{i \in M_S^{m(\varepsilon/2)}} \bar{u}_i - \sum_{j \in W_S^{m(\varepsilon/2)}} \bar{v}_j \right| \\
 & \leq \sum_{i \in M_S^{m(\varepsilon/2)}} |u_i^l - \bar{u}_i| + \sum_{j \in W_S^{m(\varepsilon/2)}} |v_j^l - \bar{v}_j| \\
 & \quad + \left| \sum_{i \in M_S \setminus M_S^{m(\varepsilon/2)}} u_i^l + \sum_{j \in W_S \setminus W_S^{m(\varepsilon/2)}} v_j^l \right|, \tag{5.8}
 \end{aligned}$$

where the last inequality is due to the triangle inequality. We proceed by showing that these last three sums have very small values. Observe first that

$$v_p(\mathcal{A}) \geq v_p(\mathcal{A}_{m(\varepsilon/2)+l}) = v_d(\mathcal{A}_{m(\varepsilon/2)+l}) = \sum_{i=1}^{m(\varepsilon/2)+l} u_i^l + \sum_{j=1}^{m(\varepsilon/2)+l} v_j^l. \tag{5.9}$$

Furthermore, the restriction of (u^l, v^l) to the first $m(\varepsilon/2)$ agents of each type is a feasible dual solution of $\mathcal{A}_{m(\varepsilon/2)}$. This implies

$$\sum_{i=1}^{m(\varepsilon/2)} u_i^l + \sum_{j=1}^{m(\varepsilon/2)} v_j^l \geq v_d(\mathcal{A}_{m(\varepsilon/2)}) = v_p(\mathcal{A}_{m(\varepsilon/2)}) \geq v_p(\mathcal{A}) - \varepsilon/2, \tag{5.10}$$

where the last inequality follows from (5.3). Combining (5.9) and (5.10) leads to

$$\begin{aligned}
 \varepsilon/2 & = v_p(\mathcal{A}) - (v_p(\mathcal{A}) - \varepsilon/2) \\
 & \geq \sum_{i=1}^{m(\varepsilon/2)+l} u_i^l + \sum_{j=1}^{m(\varepsilon/2)+l} v_j^l - \sum_{i=1}^{m(\varepsilon/2)} u_i^l - \sum_{j=1}^{m(\varepsilon/2)} v_j^l
 \end{aligned}$$

$$\geq \sum_{i \in M_S \setminus M_S^{m(\varepsilon/2)}} u_i^l + \sum_{j \in W_S \setminus W_S^{m(\varepsilon/2)}} v_j^l. \quad (5.11)$$

Hence, starting from (5.8) and using (5.6) and (5.11) results in

$$\begin{aligned} & \sum_{i \in M_S^{m(\varepsilon/2)}} |u_i^l - \bar{u}_i| + \sum_{i \in W_S^{m(\varepsilon/2)}} |v_j^l - \bar{v}_j| + \left| \sum_{i \in M_S \setminus M_S^{m(\varepsilon/2)}} u_i^l + \sum_{j \in W_S \setminus W_S^{m(\varepsilon/2)}} v_j^l \right| \\ & \leq \sum_{i \in M_S^{m(\varepsilon/2)}} \frac{\varepsilon}{4m(\varepsilon/2)} + \sum_{i \in W_S^{m(\varepsilon/2)}} \frac{\varepsilon}{4m(\varepsilon/2)} + \varepsilon/2 \\ & \leq \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Upon combining this with (5.8), we conclude that $w(S) - \sum_{i \in M_S} \hat{u}_i(\varepsilon) - \sum_{i \in W_S} \hat{v}_j(\varepsilon) \leq \varepsilon$ and hence, $(\hat{u}(\varepsilon), \hat{v}(\varepsilon))$ satisfies ε -stability. $\square$

The solution $F_\varepsilon(w)$ for infinite assignment games with $w(N) < +\infty$ was introduced because elements of the core $C(w)$ are hard to find. The f-strong ε -core $F_\varepsilon(w)$ is always nonempty, as is shown in theorem 5.1 above. To prove this result, we showed that a certain allocation, $(\hat{u}(\varepsilon), \hat{v}(\varepsilon))$, is always contained in $F_\varepsilon(w)$. In the allocation $(\hat{u}(\varepsilon), \hat{v}(\varepsilon))$ an infinite number of agents is assigned a zero payoff but in total these agents can only achieve an amount of at most ε .

6 Concluding remarks

In this paper we studied infinite assignment problems and solutions for the related cooperative games. In particular we studied three solutions for the corresponding assignment games that depend upon the value $w(N)$ of the game.

If the value $w(N)$ is infinite, then the utopia payoff (u^*, v^*) is proposed as a solution. This solution assigns to each player its utopia value of the infinite assignment problem. Hence, all the players are content.

In case $w(N)$ is finite, two solutions are considered. First, we studied the core $C(w)$. It is shown to be nonempty but it is hard to find a single core-element. Therefore, the f-strong ε -core $F_\varepsilon(w)$ is introduced. This set-

solution guarantees ε -efficiency and ε -stability for finite coalitions. It is also shown to be nonempty.

With respect to the f-strong ε -core, one might want to take into account that coalitions of larger size are more expensive to form. This can be done by extending the definition of ε -stability to include a cost that is increasing in the coalition size. Shapley and Shubik (1966) already did this in the definition of their weak ε -core, where another kind of ε -stability, namely

$$\sum_{i \in S} z_i \geq w(S) - \varepsilon|S|, \quad (6.1)$$

is required for all (finite) coalitions S . Such a modification is also possible for infinite assignment games, but we decided to focus in the paper on the f-strong ε -core. Nevertheless, a modification of theorem 5.1 holds if we use (6.1) instead of (5.2) and, moreover this remains true when using a general cost function without assuming that the size of coalitions has to be finite. Then, the corresponding weak ε -cores will be nonempty. Furthermore, we also could have used an allocation that is a modification of $(\hat{u}(\varepsilon), \hat{v}(\varepsilon))$, namely where $\hat{u}_{m(\varepsilon)+l} = \varepsilon/2^{l+1} = \hat{v}_{m(\varepsilon)+l}$ for $l = 1, 2, \dots$. In such an allocation all players receive a positive, although perhaps very small, amount. We decided not to do so and concentrate on the f-strong ε -core (with ε -efficiency) because optimal assignments may not exist. Therefore it seems more realistic to distribute $v_p(\mathcal{A}) - \varepsilon$ instead of $v_p(\mathcal{A})$, also because the value $v_p(\mathcal{A}) - \varepsilon$ can be approximated via finite subproblems $\mathcal{A}_n$.

References

- Aliprantis C.D. and Border K.C. (1999). *Infinite Dimensional Analysis*. Springer.
- Anderson E.J. and Nash P. (1987). *Linear Programming in Infinite Dimensional Spaces*. Wiley.
- Birkhoff G. (1946). Tres Observaciones sobre el Algebra Lineal. *Revista de la Universidad Nacional de Tucumán* A 5, 147-151.
- Borm P., Hamers H. and Hendrickx R. (2001). Operations Research Games: A Survey. *TOP* 9, 139-198.
- Cross W.P., Romeijn H.E. and Smith R.L. (1998). Approximating Extreme Points of Infinite Dimensional Convex Sets. *Mathematics of Operations Research* 23, 433-442.

- Fagnelli V., Patrone F., Sideri E. and Tijs S. (1999). Balanced Games Arising from Infinite Linear Models. *Mathematical Methods of Operations Research* 50, 385-397.
- Kaneko M. and Wooders M. (1986). The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results. *Mathematical Social Sciences* 12, 105-137.
- Köthe G. (1983). *Topological Vector Spaces I*. Springer.
- Llorca N., Tijs S. and Timmer J. (2003). Semi-Infinite Assignment Problems and Related Games. *Mathematical Methods of Operations Research* 57, 67-78.
- Sánchez-Soriano J., Llorca N., Tijs S. and Timmer J. (2001). Semi-Infinite Assignment and Transportation Games. In: Goberna M.A. and López M.A. (eds.), *Semi-Infinite Programming: Recent Advances*. Kluwer Academic Publishers, 349-363.
- Shapley L.S. and Shubik M. (1966). Quasi-Cores in a Monetary Economy with Nonconvex Preferences. *Econometrica* 34, 805-827.
- Shapley L.S. and Shubik M. (1972). The Assignment Game I: The Core. *International Journal of Game Theory* 1, 111-130.