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A GENERAL FORM OF THE LAW OF THE ITERATED LOGARITHM FOR THE WEIGHTED MULTIVARIATE EMPIRICAL PROCESS

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Abstract: A law of the iterated logarithm in “the middle” is established for weighted empirical processes based on a sequence of i.i.d. random vectors, uniformly distributed on the (multivariate) unit square. This result unifies several results in the literature.

Keywords: exponential probability inequalities, laws of the iterated logarithm, weighted empirical processes.

1. Introduction and main results

Let $X_1, X_2, \dots$ be a sequence of independent random vectors, each uniformly distributed on $[0, 1]^d$, $d \in \mathbb{N}$. The first n random vectors determine the empirical distribution function F_n in the usual way:

$$F_n(t) = n^{-1} \sum_{i=1}^n \prod_{j=1}^d 1_{[0, t_j]}(X_{ij}), \quad t \in [0, 1]^d, \quad (1.1)$$

where X_{ij} is the j -th component of X_i and t_j the j -th component of t . Writing $|t| = \prod_{j=1}^d t_j$ we define, for $\nu \in [0, \frac{1}{2}]$,

$$U_{n,\nu}(t) = n^\nu (F_n(t) - |t|), \quad t \in [0, 1]^d. \quad (1.2)$$

Much attention has been paid in the literature to the behaviour of

$$\limsup_{n \rightarrow \infty} \sup_{a_n \leq |t| \leq 1} \frac{|U_{n,\nu}(t)|}{(\log \log n)^\nu |t|^{1-\nu}} \quad (1.3)$$

($a_n = cn^{-1} \log \log n$, $c \in (0, \infty)$ arbitrary) for special values of ν and/or d . Csáki (1977) determined this behaviour for $d=1$ and $\nu = \frac{1}{2}$ and

Wellner (1978) for $d=1$ and $\nu=0$. Recently Alexander (1984) generalized Csáki's result to arbitrary dimension $d \in \mathbb{N}$. In the same paper also some partial results for $\nu=0$ and $d \in \mathbb{N}$ can be found. Also recently a related problem, namely the behaviour of (1.3) with a_n replaced by 0 (i.e. the supremum is taken over the whole (multivariate) unit square), is solved for any $\nu \in [0, \frac{1}{2}]$ and $d \in \mathbb{N}$ in Einmahl and Mason (1985).

In this paper we generalize and unify all the known results for (1.3). For this purpose let us define for $\sigma > 0$, β_σ by $\beta_\sigma > 1$ and

$$\beta_\sigma (\log \beta_\sigma - 1) + 1 = \sigma^{-1}; \quad (1.4)$$

for some properties of β_σ see Kiefer (1972, p. 232).

Theorem. Let $d \in \mathbb{N}$. For every $\nu \in [0, \frac{1}{2}]$ and $c \in (0, \infty)$ we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{a_n \leq |t| \leq 1} \frac{|U_{n,\nu}(t)|}{(\log \log n)^\nu |t|^{1-\nu}} &= \max \left(c^\nu (\beta_{c/d} - 1), (2(d+1))^{1/2} 1_{\{\frac{1}{2}\}}(\nu) \right) \\ &\text{a.s.} \end{aligned} \quad (1.5)$$

Note that there is a discontinuity in behaviour in $\nu = \frac{1}{2}$ if

$$c^{1/2} (\beta_{c/d} - 1) < (2(d+1))^{1/2}.$$

Research performed at the Catholic University Nijmegen.

Finally we present some results which are (almost) corollaries to our theorem.

Corollary 1.1. For every fixed $\alpha \in (0, 1)$ we have

$$\limsup_{n \rightarrow \infty} \sup_{a_n \leq |t| \leq \alpha} \frac{|U_{n,v}(t)|}{(\log \log n)^v (|t|(1-|t|))^{1-v}} \quad (1.6)$$

$$= \max \left(c^v (\beta_{c/d} - 1), (2(d+1))^{1/2} 1_{\{\frac{1}{2}\}}(\nu) \right) \quad a.s.$$

Corollary 1.2

$$\limsup_{n \rightarrow \infty} \sup_{a_n \leq |t| \leq 1} |F_n(t)/|t| - 1| = \beta_{c/d} - 1 \quad a.s. \quad (1.7)$$

Corollary 1.2 is the multidimensional version of Corollary 2(iii) in Wellner (1978).

Corollary 1.3

$$\limsup_{n \rightarrow \infty} \sup_{a_n \leq |t| \leq 1} F_n(t)/|t| = \beta_{c/d} \quad a.s. \quad (1.8)$$

Corollary 1.3 is the multidimensional version of Theorem 2(i) in Wellner (1978).

Remarks. A little reflection shows that the above results remain unaltered if $a_n = cn^{-1} \log \log n$ is replaced by $\lim_{n \rightarrow \infty} na_n / \log \log n = c$. Note that $\lim_{c \rightarrow 0} c^v (\beta_{c/d} - 1) = \infty$. Therefore the choice of a_n is rather natural: choosing $a_n = o(n^{-1} \log \log n)$ in (1.3) would yield ∞ for the limsup almost surely.

2. Inequalities

In this section we present some inequalities which we require for the proofs of our main results. Those proofs are deferred to the next section.

Let $\psi: (0, \infty) \rightarrow (0, \infty)$ be the decreasing function defined by

$$\psi(\lambda) = 2\lambda^{-2} \int_0^\lambda \log(1+\sigma) d\sigma$$

$$= 2\lambda^{-2} ((1+\lambda) \log(1+\lambda) - \lambda). \quad (2.1)$$

Also, for $s, t \in [0, 1]^d$ let $s \leq t$ mean $s_j \leq t_j$ for all $1 \leq j \leq d$.

Inequality 2.1. Let $t \in [0, 1]^d$ with $0 < |t| \leq \frac{1}{2}$. For any $\varepsilon \in (0, 1)$ we have

$$P \left(\sup_{s \leq t} |U_{n,1/2}(s)| \geq \lambda \right) \leq C_1 \exp \left(\frac{-(1-\varepsilon)\lambda^2}{2|t|} \psi \left(\frac{\lambda}{n^{1/2}|t|} \right) \right), \quad \lambda > 0, \quad (2.2)$$

where $C_1 \in (0, \infty)$ depends only on d and ε .

This inequality is a stronger version of Theorem 1.1 in Ruymgaart and Wellner (1984); for a detailed proof we refer to Einmahl (1987, Section 2.1). In fact Inequality 2.1 holds true in general, i.e. we do not need the restriction to the uniform distribution.

Inequality 2.1 is needed for proving our next inequality which is the substantial tool for the proof of our main result.

Inequality 2.2. Let $0 \leq \nu \leq \frac{1}{2}$, $0 < \alpha \leq \beta \leq \frac{1}{4}$ and $0 < \varepsilon \leq \frac{1}{2}$. Write $\gamma = 1 - \varepsilon$. Then we have, for all $\lambda > 0$,

$$P \left(\sup_{\alpha \leq |t| \leq \beta} |U_{n,v}(t)| / |t|^{1-\nu} \geq \lambda \right) \quad (2.3)$$

$$\leq C_2 \int_{\gamma\alpha}^{\beta/\gamma} \frac{(\log(1/\sigma))^{d-1}}{\sigma} \times \exp \left(-\frac{1}{2} \gamma^3 \lambda^2 (n\sigma)^{1-2\nu} \psi(\lambda(n\sigma)^{-\nu}) \right) d\sigma,$$

where $C_2 \in (0, \infty)$ depends only on d and ε .

Proof. The proof will be patterned on the proof of Inequality 2.1 in Ruymgaart and Wellner (1982). We use a special countably infinite partition of $(0, 1]^d$ that becomes arbitrarily fine near the lower boundary of this set. For $t \in [0, 1]^d$ write $t = \langle t_1, \dots, t_d \rangle = \langle t_j \rangle$; let the half-open rectangles $(s_1, t_1] \times \dots \times (s_d, t_d]$ be denoted by $R(s, t)$ or simply R . Fix $\varepsilon \in (0, \frac{1}{2}]$ and define $\theta > 0$ by $\theta^d = \gamma = 1 - \varepsilon$.

The partition is the collection of rectangles

$$\mathcal{P} = \{ R(\langle \theta^{k(j)} \rangle, \langle \theta^{k(j)-1} \rangle) : \langle k(j) \rangle \in \mathbb{N}^d \}. \quad (2.4)$$

Notice that we have, for any $R(a, b) \in \mathcal{P}$,

$$\frac{|a|}{|b|} = \theta^d = \gamma. \quad (2.5)$$

For any $0 < \alpha \leq \beta \leq \frac{1}{4}$ write

$$\mathcal{P}_{\alpha, \beta} = \{R(a, b) \in \mathcal{P}: |b| \geq \alpha, |a| < \beta\} \quad (2.6)$$

and observe that

$$\{\alpha \leq |t| \leq \beta\} \subset \bigcup_{R \in \mathcal{P}_{\alpha, \beta}} R \subset \{\gamma\alpha \leq |t| \leq \beta/\gamma\}. \quad (2.7)$$

Now we have, using (2.5) for the last inequality,

$$\begin{aligned} & P\left(\sup_{\alpha \leq |t| \leq \beta} \frac{|U_{n, \nu}(t)|}{|t|^{1-\nu}} \geq \lambda\right) \\ & \leq P\left(\max_{R(a, b) \in \mathcal{P}_{\alpha, \beta}} \sup_{t \in R(a, b)} \frac{|U_{n, \nu}(t)|}{|a|^{1-\nu}} \geq \lambda\right) \\ & \leq \sum_{R(a, b) \in \mathcal{P}_{\alpha, \beta}} P\left(\sup_{t \in R(a, b)} |U_{n, \nu}(t)| \geq \lambda |a|^{1-\nu}\right) \\ & \leq C_1 \sum_{R(a, b) \in \mathcal{P}_{\alpha, \beta}} \exp\left(\frac{-(1-\varepsilon)\lambda^2 n^{1-2\nu} |a|^{2-2\nu}}{2|b|}\right) \\ & \quad \times \psi\left(\frac{\lambda n^{1/2-\nu} |a|^{1-\nu}}{|b| n^{1/2}}\right) \\ & \leq C_1 \sum_{R(a, b) \in \mathcal{P}_{\alpha, \beta}} \exp\left(-\frac{1}{2}(1-\varepsilon)\gamma^{2-2\nu}\lambda^2\right) \\ & \quad \times (n|b|)^{1-2\nu} \psi(\lambda(n\alpha)^{-\nu}). \end{aligned} \quad (2.8)$$

Write $|dt|$ for integration w.r.t. Lebesgue measure on $[0, 1]^d$. When we use

$$\begin{aligned} 1 &= (1-\theta)^{-d} |b|^{-1} \int_{R(a, b)} |dt| \\ &\leq (1-\theta)^{-d} \int_{R(a, b)} |t|^{-1} |dt|, \\ &\text{for } R(a, b) \in \mathcal{P}, \end{aligned} \quad (2.9)$$

at the transition from summation to integration we find, by combining (2.7) and (2.8), that the

first expression in (2.8) is less than or equal to

$$C_1 (1-\theta)^{-d} \int_{\{\gamma\alpha \leq |t| \leq \beta/\gamma\}} \frac{1}{|t|} \exp\left(-\frac{1}{2}\gamma^3 \lambda^2 (n|t|)^{1-2\nu} \psi(\lambda(n\alpha)^{-\nu})\right) |dt|. \quad (2.10)$$

The change of variables $\sigma = u_1 = |t|$, $u_2 = t_2, \dots, u_d = t_d$ completes the proof. $\square$

We now present our maximal inequality, which is a slight generalization of Lemma 3(ii) in Shorack (1980); cf. also Lemma 2.3 in James (1975). Therefore its proof, which can be found in Einmahl (1987, Ineq. 2.10), will be omitted.

Inequality 2.3. Let $0 \leq \nu \leq \frac{1}{2}$, $0 < \alpha \leq \beta \leq 1$, $\varepsilon \in (0, 1)$ and write $n_k = [(1 + \varepsilon/12)^k]$, $k \in \mathbb{N}$. Then we have, for all $k \in \mathbb{N}$ and for $\lambda > (2/\varepsilon)^\nu$,

$$\begin{aligned} & P\left(\max_{n_k < n \leq n_{k+1}} \sup_{\alpha \leq |t| \leq \beta} |U_{n, \nu}(t)| / |t|^{1-\nu} \geq \lambda\right) \\ & \leq 2P\left(\sup_{\alpha \leq |t| \leq \beta} |U_{n_{k+1}, \nu}(t)| / |t|^{1-\nu} \geq (1-\varepsilon)\lambda\right). \end{aligned} \quad (2.11)$$

3. Proofs of the main results

Proof of the Theorem. The case $\nu = \frac{1}{2}$ is due to Alexander (1984, Corollary 3.5). So assume $0 \leq \nu < \frac{1}{2}$. (The fact that we do not give a proof for $\nu = \frac{1}{2}$ is not imposed by the limitations of our approach. The case $\nu = \frac{1}{2}$, however, behaves somewhat differently with regard to the other values of ν , so that in some parts of the proof we should have to distinguish between the cases $\nu = \frac{1}{2}$ and $0 \leq \nu < \frac{1}{2}$.)

We first consider the upper bound. We split the interval $[a_n, 1]$ in three subintervals: $[a_n, n^{-1} \log n]$, $[n^{-1} \log n, \frac{1}{4}]$ and $[\frac{1}{4}, 1]$ and prove first, with $A = c^\nu (\beta_{c/d} - 1)$,

$$\limsup_{n \rightarrow \infty} \sup_{a_n \leq |t| \leq n^{-1} \log n} \frac{|U_{n, \nu}(t)|}{(\log \log n)^\nu |t|^{1-\nu}} \leq A \quad \text{a.s.} \quad (3.1)$$

By the Borel-Cantelli lemma and Inequality 2.3 it suffices so show that for every small positive ε we

have $\sum PD_k < \infty$, where

$$D_k = \left\{ \sup_{a_{n_{k+1}} \leq |t| \leq n_k^{-1} \log n_k} \frac{|U_{n_{k+1}, \nu}(t)|}{|t|^{1-\nu}} \right. \quad (3.2)$$

$$\left. \geq (1+\varepsilon)(1-\varepsilon)^{-4} A (\log \log n_k)^\nu \right\}$$

and $n_k = [(1+\varepsilon/12)^k]$. (Note that when $\nu = 0$, Inequality 2.3 is not applicable for $\lambda \leq 1$; a slightly modified version of the inequality, however, does the work in this case.)

We now use Inequality 2.2, which yields

$$PD_k \leq C_2 \int_{\gamma a_{n_{k+1}}}^{(\gamma n_k)^{-1} \log n_k} \frac{(\log(1/\sigma))^{d-1}}{\sigma} d\sigma \quad (3.3)$$

$$\cdot \exp\left(-\frac{1}{2}(1+\varepsilon)^2(1-\varepsilon)^{-4} A^2 c^{1-2\nu}\right) \cdot \log \log n_k \psi\left((1+\varepsilon)(1-\varepsilon)^{-4} A c^{-\nu}\right).$$

Using $\zeta^d - \tau^d = (\zeta - \tau) \sum_{l=0}^{d-1} \zeta^l \tau^{d-l-1}$ we see that, for large k ,

$$\int_{\gamma a_{n_{k+1}}}^{(\gamma n_k)^{-1} \log n_k} \frac{(\log(1/\sigma))^{d-1}}{\sigma} d\sigma \quad (3.4)$$

$$= d^{-1} \left\{ \left(\log \left(\frac{n_{k+1}}{\gamma c \log \log n_{k+1}} \right) \right)^d - \left(\log \left(\frac{\gamma n_k}{\log n_k} \right) \right)^d \right\}$$

$$\leq (\log n_k)^{d-1} \log \log n_k.$$

Using (1.4) and the fact that the function $\lambda \mapsto \lambda \psi(\lambda)$ is increasing on $(0, \infty)$ we see that the exponential in (3.3) is less than or equal to

$$\exp(-(1+\varepsilon)d \log \log n_k). \quad (3.5)$$

Combining (3.3)–(3.5) yields as an upper bound for PD_k

$$C_2 \log \log n_k (\log n_k)^{-(1+\varepsilon d)}, \quad (3.6)$$

provided k is sufficiently large. Recalling the definition of n_k it is easy to see that (3.6) is summable in k . This completes the proof of (3.1).

A similar but somewhat easier proof can be

given for

$$\lim_{n \rightarrow \infty} \sup_{n^{-1} \log n \leq |t| \leq \frac{1}{4}} \frac{|U_{n, \nu}(t)|}{(\log \log n)^\nu |t|^{1-\nu}} = 0 \quad \text{a.s.} \quad (3.7)$$

Using the law of the iterated logarithm for the unweighted empirical process in Kiefer (1961) we immediately have

$$\lim_{n \rightarrow \infty} \sup_{1/4 \leq |t| \leq 1} \frac{|U_{n, \nu}(t)|}{(\log \log n)^\nu |t|^{1-\nu}} = 0 \quad \text{a.s.} \quad (3.8)$$

Combining (3.1), (3.7) and (3.8) completes the upper bound part of the proof.

We now consider the lower bound. It suffices to prove that

$$\limsup_{n \rightarrow \infty} \sup_{|t|=a_n} \frac{n F_n(t)}{\log \log n} \geq c \beta_{c/d} \quad \text{a.s.} \quad (3.9)$$

Let ε be a small positive number and define $\delta = 1 - (1-\varepsilon)^{1/(2d)}$. We now need again the partition $\mathcal{P}$ defined in (2.4) but with $\theta = \theta_n = (1 + o(1))\delta$ such that there exists $l = l_n \in \mathbb{N}$ with

$$\theta' = cn^{-1} \log \log n. \quad (3.10)$$

This implies $l = (1 + o(1))(\log 1/\delta)^{-1} \log n = (1 + o(1))c_1 \log n$, where $c_1 = c_1(\varepsilon) \in (0, \infty)$. Define

$$\mathcal{R}_n = \{R(a, b) \in \mathcal{P}: |b| = a_n\}. \quad (3.11)$$

For any $R(a, b) \in \mathcal{R}_n$ we have

$$|R(a, b)| = (1 - \theta_n)^d a_n$$

$$= (1 + o(1))(1 - \varepsilon)^{1/2} a_n, \quad (3.12)$$

where $|R|$ denotes the Lebesgue measure of R . Extending the definition of F_n to half-open rectangles by

$$F_n\{R\} = \frac{1}{n} \# \{1 \leq i \leq n: X_i \in R\},$$

we easily see that we have, for $\lambda > 0$,

$$P\left(\sup_{|t|=a_n} F_n(t) \geq \lambda\right)$$

$$\geq P\left(\max_{R(a, b) \in \mathcal{R}_n} F_n\{R(a, b)\} \geq \lambda\right) \quad (3.13)$$

$$\begin{aligned}
&= 1 - P\left(\max_{R(a, b) \in \mathcal{R}_n} F_n\{R(a, b)\} < \lambda\right) \\
&\geq 1 - \prod_{R(a, b) \in \mathcal{R}_n} (1 - P(F_n\{R(a, b)\} \geq \lambda)) \\
&= 1 - (1 - P(F_n\{R\} \geq \lambda))^{\#\mathcal{R}_n},
\end{aligned}$$

where the second inequality follows from a result in Mallows (1968) and where R is an arbitrary element of $\mathcal{R}_n$. A combinatorial argument shows

$$\#\mathcal{R}_n = \binom{l+d-1}{d-1} = \binom{(1+o(1))c_1 \log n + d-1}{d-1} \quad (3.14)$$

$$= (1 + o(1))c_2 (\log n)^{d-1},$$

where $c_2 = c_2(d, \varepsilon) \in (0, \infty)$.

Define $F_{n,m}$ for $m > n$ by

$$(m-n)F_{n,m} = mF_m - nF_n \quad (3.15)$$

and let $\{n_j\}_{j=1}^\infty$ be a subsequence of the natural numbers such that

$$n_j = (1 + o(1)) \left(1 / (1 - (1 - \varepsilon)^{1/2})\right)^j.$$

Arrived at this stage we can proceed as in Lemma 1, Lemma 2 and the "inner class" proof of Theorem 3 in Kiefer (1972), where (3.9) is proved for $d = 1$. With some minor modifications we obtain, for $R \in \mathcal{R}_{n_j}$ and $\lambda = (1 - \varepsilon)c\beta_{c/d} \log \log n_j$,

$$P((n_j - n_{j-1})F_{n_{j-1}, n_j}\{R\} \geq \lambda) = j^{-(1+o(1))(1-\varepsilon)d}. \quad (3.16)$$

Combining this with (3.13) and (3.14) yields

$$\begin{aligned}
&P\left(\sup_{|t|=a_{n_j}} (n_j - n_{j-1})F_{n_{j-1}, n_j}(t) \geq \lambda\right) \\
&\geq 1 - \{1 - j^{-(1+o(1))(1-\varepsilon)d}\}^{(1+o(1))c_3 j^{d-1}} \\
&= c_3 j^{-(1-d\varepsilon)(1+o(1))},
\end{aligned} \quad (3.17)$$

where $c_3 = c_3(d, \varepsilon) \in (0, \infty)$. Again like in Kiefer (1972) an application of the Borel-Cantelli lemma completes the proof of (3.9). $\square$

Proofs of the corollaries. Corollary 1.1 is in

Alexander (1984) for $\nu = \frac{1}{2}$; for $0 \leq \nu < \frac{1}{2}$ this corollary can easily be derived from the Theorem using the unweighted law of the iterated logarithm in Kiefer (1961) and (3.9). It is also easy to derive Corollary 1.1 for all $0 \leq \nu \leq \frac{1}{2}$ from the Theorem by use of Corollary 6.2 in Alexander (1982).

Corollary 1.2 is the Theorem for $\nu = 0$. Corollary 1.3 is a simple consequence of (3.9) and Corollary 1.2. $\square$

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